Spectral theory of Substitutive systems:
Combinatorial conditions for Pure spectrum and tilings

Anne Siegel

IRISA-CNRS, Rennes, France
Let $\sigma$ be a of Pisot type and $(X_\sigma, S)$ be its symbolic system.

- The Fibonacci substitution $(1 \mapsto 12, 2 \mapsto 1)$ provides the best representation for the addition of the golden ratio.
- The Morse substitution $(1 \mapsto 12, 2 \mapsto 21)$ is a coding of a two-point extension of the dyadic odometer.
- The Tribonacci substitution $(1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1)$ codes a domain exchange in $\mathbb{R}^2$ as well as a two-dimensional toral rotation.

Which substitive systems are isomorphic to a rotation on a compact group? What is their maximal equicontinuous factor?
Substitution of constant length

Caracterization of the maximal equicontinuous factor.

the Morse substitution has height 1.

Example with height 3.

Pure discrete spectrum if $h = 1$: the coincidence condition.

Pure discrete spectrum if $h \neq 1$: recoding in a pure base.
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Caracterization of the maximal equicontinuous factor.

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Pure discrete spectrum if $h \neq 1$: recoding in a pure base.

Theorem (Dekking)

Let $\sigma$ be non-periodic of constant length $n$. Let $u = \sigma^k(u)$.

The height of $\sigma$ is the greatest $m$, $(m, n) = 1$, that divides every $i > 0$, $u_i = u_0$.

The maximal equicontinuous factor of $(X_\sigma, S)$ is the addition of $(1, 1)$ on $\mathbb{Z}_n \times \mathbb{Z}/m\mathbb{Z}$. 

- p.3/21
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Morse substitution

$$\begin{align*}
1 & \mapsto 12 \\
2 & \mapsto 21
\end{align*}$$

$u = 122121122112122121121121 \cdots$

$u_0 = 1$ appears at rank 3 and 5

$\implies h = 1.$

The maximal equicontinuous factor is $\mathbb{Z}_2$. 
Substitution of constant length

Caracterization of the maximal equicontinuous factor.

The Morse substitution has height 1.

Example with height 3.

Pure discrete spectrum if \( h = 1 \): the coincidence condition.

Pure discrete spectrum if \( h \neq 1 \): recoding in a pure base.

\[
1 \leftrightarrow 121 \quad 2 \leftrightarrow 312 \quad 3 \leftrightarrow 213
\]

\( u = 12131212121312131213121312 \)

\( u_0 = 1 \) appears at rank 2, 4, 6 etc.

\[ \quad \implies \quad h = 2. \]

The maximal equicontinuous factor is \( \mathbb{Z}_3 \times \mathbb{Z}/2\mathbb{Z} \).
Substitution of constant length

Caracterization of the maximal equicontinuous factor.

the Morse substitution has height 1.

Example with height 3.

Pure discrete spectrum if $h = 1$: the coincidence condition.

Pure discrete spectrum if $h \neq 1$: recoding in a pure base.

coincidence condition

All $\sigma^k(a)$’s have the same $n$-th letter.

1 $\leftrightarrow$ 12 \hspace{1cm} 2 $\leftrightarrow$ 23 \hspace{1cm} 3 $\leftrightarrow$ 13

122323123131213231312131

231312131223121312232313

122323132313121312232313

Morse has no coincidences: it is a dyadic odometer.
Substitution of constant length

- Characterization of the maximal equicontinuous factor.
- The Morse substitution has height 1.
- Example with height 3.
- Pure discrete spectrum if $h = 1$: the coincidence condition.
- Pure discrete spectrum if $h \neq 1$: recoding in a pure base.

Non trivial height

$1 \leftrightarrow 121 \quad 2 \leftrightarrow 312 \quad 3 \leftrightarrow 213$

has a pure discrete spectrum.
Non constant-length: Host

- If $\sigma$ is primitive, measure-theoretic isomorphism = topological conjugacy.
- **Structure** of the spectrum: coboundaries.
- **Arithmetic** spectrum: incidence matrix.
- **Combinatorial** spectrum: return words.
If $\sigma$ is primitive, measure-theoretic isomorphism = topological conjugacy.

Structure of the spectrum: coboundaries.

Arithmetic spectrum: incidence matrix.

Combinatorial spectrum: return words.

Theorem (Host). All eigenfunctions of primitive substitutive systems are continuous.
If $\sigma$ is primitive, measure-theoretic isomorphism = topological conjugacy.

Structure of the spectrum: coboundaries.

Arithmetic spectrum: incidence matrix.

Combinatorial spectrum: return words.

**Coboundary**: map $h : \mathcal{A} \to \mathbb{U}$ such that $f : \mathcal{A} \to \mathbb{U}$ with $f(b) = f(a)h(a) \forall ab \in \mathcal{L}$.

**Spectrum (Host)**: $\lambda \subset \mathbb{U}$ is an eigenvalue of $(X_\sigma, \mathcal{S})$ iff $\exists p > 0$ such that $\forall a \in \mathcal{A}$, the limit

$$h(a) = \lim_{n \to \infty} \lambda^{|\sigma^{pn}(a)|}$$

is a coboundary.
If \( \sigma \) is primitive, measure-theoretic isomorphism = topological conjugacy.

Structure of the spectrum: coboundaries.

Arithmetic spectrum: incidence matrix.

Combinatorial spectrum: return words.

Arithmetic spectrum

The eigenvalues for the trivial coboundary are computable and depends only on the incidence matrix of the substitution.

\[
\lambda^{\sigma^p_n(a)} \mapsto 1.
\]
**Non constant-length : Host**

Combinatorial spectrum

The eigenvalues for non-trivial coboundaries depend on return words, playing the role of the height.

A return word is a word \( W = a_1 \ldots a_k \) such that \( W a_1 \) is in the language and \( a_i \neq a_1 \) (Livshits, Durand...).

1 \( \mapsto \) 1231, 2 \( \mapsto \) 232, 3 \( \mapsto \) 3123 has a non-trivial coboundary.

If \( \sigma \) is primitive, measure-theoretic isomorphism = topological conjugacy.

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Example of weakly mixing substitution

Livshits: conditions for purely discrete spectrum or partially continuous spectrum.

If the incidence polynomial is irreducible: the existence of discrete spectrum depends on expanding eigenvalues of the matrix (Solomyak).

1 ↦ 12121, 2 ↦ 112

is weakly mixing.

(1 is the only eigenvalue)
Example of weakly mixing substitution

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Mix of coincidences and return words

$1 \mapsto 23, \ 2 \mapsto 12, \ 3 \mapsto 23$ as a continuous spectral component but is not weakly mixing (constant length).
Example of weakly mixing substitution

Livshits: conditions for purely discrete spectrum or partially continuous spectrum.

If the incidence polynomial is irreducible: the existence of discrete spectrum depends on expanding eigenvalues of the matrix (Solomyak).

If \( P(\alpha) = C \) for every expanding eigenvalue \( \alpha \) of the matrix, then \( \exp(2\pi i C) \) is an eigenvalue of \( (X_\sigma, S) \) (Solomyak).

Partial converse by Ferenczi, Mauduit, Nogueira.

Application: The spectrum of
\[
1 \leftrightarrow 1244, \ 2 \leftrightarrow 23, \ 3 \leftrightarrow 4, \ 4 \leftrightarrow 1
\]
is \( \exp(2\pi i \mathbb{Z} \sqrt{2}) \).
Substitutions of Pisot type

Pisot type: the incidence matrix has only one expanding eigenvalue $\alpha$.

- They are never weakly mixing
- Their arithmetical spectrum can be computed
- The combinatorial spectrum is empty

Non empty spectrum

Every $P(\alpha)$ is an eigenvalue
(Solomyak)

The spectrum contains $\mathbb{Z}[\alpha]$
Substitutions of Pisot type

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Arithmetical spectrum

In the unimodular case, the arithmetic spectrum is generated by the frequencies of the letters in the fixed point.

In the non-unimodular case, additional rational eigenvalues have to be computed.
Substitutions of Pisot type

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- They are never weakly mixing
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Pure discrete spectrum $\iff$ there exists a metric conjugacy with an abelian translation defined by the arithmetic spectrum.
Pisot on two letters

Strong coincidences condition: \( \forall b_1, b_2, \exists a \sigma^n(b_1) = P_1 a S_1, \sigma^n(b_2) = P_2 a S_2, P_1 \) and \( P_2 \) contain the same letters.

- Pisot type and strong coincidences on two-letters implies pure discrete spectrum (Host, Hollander-Solomyak)

- Pisot type on two letters implies strong coincidences (Barge-Diamond).
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- Pisot type and strong coincidences on two-letters implies pure discrete spectrum (Host, Hollander-Solomyak)

- Pisot type on two letters implies strong coincidences (Barge-Diamond).

Theorem. Every substitutive system of Pisot type on two letters has a pure discrete spectrum.
Super/Geometrical coincidences

Combinatorial condition for tilings in the unimodular case.

Def : Two unit integral segment that cross the contracting plane have to share a common third segment after a finite iteration of the substitution.

Theorem (Barge-Kwapisz, Ito-Rao) : In the Pisot type and unimodular case, pure discrete spectrum $\iff$ Super coincidences.

Effectivity: Balanced pair algorithm checks super coincidences (but does not check the converse).
The Tribonacci substitution

1 $\leftrightarrow$ 12  \hspace{0.5cm} 2 $\leftrightarrow$ 13  \hspace{0.5cm} 3 $\leftrightarrow$ 1

- Studied by Rauzy in 1981.
- The fixed point provides a $\mathbb{R}^3$ stair.
- A projection: the Rauzy fractal.
- Ergodic properties.
- Periodic tiling.
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Theorem (Rauzy)
The substitutive system is measure-theoretically isomorphic with a domain exchange on the Rauzy fractal.

It is isomorphic with a two-dimensional toral translation.

It has an explicit pure discrete spectrum.
The Tribonacci substitution

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- Studied by Rauzy in 1981.
- The fixed point provides a $\mathbb{R}^3$ stair.
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Theorem (Rauzy, Arnoux-Ito, Canterini, S.)

Strong coincidences and Pisot type

\((X_\sigma, S)\) is isomorphic with a domain exchange in a compact set.

.exchange of domains
Theorem (Rauzy, Arnoux-Ito, Canterini, S.)

Strong coincidences and Pisot type

\((X_\sigma, S)\) is isomorphic with a domain exchange in a compact set.

\[ (X_\sigma, S) \quad \text{is isomorphic with a domain exchange in a compact set.} \]

\(\text{Rauzy fractals} \)

- Pisot substitution with coincidences
- self-similar compact
- Belong to an Euclidean space cross finite extensions of \(p\)-adic spaces.
- Non-zero Haar measure.
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Geometric representation

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1 $\mapsto$ 11223 2 $\mapsto$ 231 3 $\mapsto$ 2
Theorem (Rauzy, Arnoux-Ito, Canterini, S.)

Strong coincidences and Pisot type

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Rauzy fractals

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1 \(\leftrightarrow\) 112 2 \(\leftrightarrow\) 31 3 \(\leftrightarrow\) 1
Geometric representation

Theorem (Rauzy, Arnoux-Ito, Canterini, S.)

Strong coincidences and Pisot type

\[(X_\sigma, S)\] is isomorphic with a domain exchange in a compact set.

Rauzy fractals

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non connected example
Theorem (Rauzy, Arnoux-Ito, Canterini, S.)
Strong coincidences and Pisot type
\[(X_\sigma, S)\text{ is isomorphic with a domain exchange in a compact set.}\]

Rauzy fractals
- Pisot substitution with coincidences
- Self-similar compact
- Belong to an Euclidean space cross finite extensions of \(p\)-adic spaces.
- Non-zero Haar measure.

\[1 \mapsto 1112 \quad 2 \mapsto 12 \text{ (in } \mathbb{R}^2 \times \mathbb{Z}_2)\]
Theorem (Rauzy, Arnoux-Ito, Canterini, S.)

Strong coincidences and Pisot type

$(X_\sigma, S)$ is isomorphic with a domain exchange in a compact set.

All known substitutions of Pisot type have strong coincidences!

**Rauzy fractals**

- Pisot substitution with coincidences
- Self-similar compact
- Belong to an Euclidean space cross finite extensions of $p$-adic spaces.
- Non-zero Haar measure.
Pure discrete spectrum from numeration systems

Quotient by the translations vectors implied in the exchange of domains

⇒ translation on a compact abelian group

The quotient map is finite to one (Host). Is it one-to-one?
**Pure discrete spectrum from numeration systems**

Quotient by the translations vectors implied in the exchange of domains

\[\implies \text{translation on a compact abelian group}\]

The quotient map is finite to one (Host). Is it one-to-one?

- Desubstitution = combinatorial “division” by $\sigma$ over $\sigma$.
- Combinatorial expansions are recognized by an automaton or a Brattelli diagram.
- Non simple fibers of the quotient map correspond to non-proper expansions.
- An automaton recognize non-proper expansions.
Theorem (S., Thuswaldner): An explicit algorithm checks whether a Pisot substitutive system has a pure discrete spectrum. The condition is a CNS if $\sigma$ is unimodular.
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No counter-example is known: Each computed substitution of Pisot type has a pure discrete spectrum.
$d-1$-dimensional periodic tilings

- Pisot unimodular: the abelian group translation is generated by the frequency of letters.
- Discrete spectrum is equivalent with the Rauzy fractal to be a fundamental domain for the associated lattice.
$d-1$-dimensional periodic tilings
$d - 1$-dimensional periodic tilings
Embed the Rauzy fractal in the contracting plane.

Add a transverse component along the expanding direction.

Translate this $\mathbb{R}^d$ volume along $\mathbb{Z}^d$.

Prop. $d - 1$-dimensional periodic tiling $\implies d$-dimensional periodic tiling.
Embed the Rauzy fractal in the contracting plane.

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Embed the Rauzy fractal in the contracting plane.

Add a transverse component along the expanding direction.

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Prop. $d$ – 1-dimensional periodic tiling $\Rightarrow$ $d$-dimensional periodic tiling.
Embed the Rauzy fractal in the contracting plane.

Add a transverse component along the expanding direction.

Translate this $\mathbb{R}^d$ volume along $\mathbb{Z}^d$.

Prop. $d$-1-dimensional periodic tiling $\implies$ $d$-dimensional periodic tiling.
Prop. The $d - 1$-dim periodic tiling is obtained as the projection on the contracting plane of $R + \{ x \in \mathbb{Z}^d, \sum x_i = 0 \}$ (copies of the $d$-dim Rauzy fractal on the discrete anti-diagonal plane).

Prop. $d - 1$-dimensional periodic tiling $\iff$ $d$-dimensional periodic tiling.

$\iff$ Each piece of the Rauzy fractal has the measure of the projection of the corresponding unit cube face on the contracting plane.
Prop. The \( d - 1 \)-dim periodic tiling is obtained as the projection on the contracting plane of \( \mathcal{R} + \{ x \in \mathbb{Z}^d, \sum x_i = 0 \} \) (copies of the \( d \)-dim Rauzy fractal on the discrete anti-diagonal plane).

Prop. \( d - 1 \)-dimensional periodic tiling \( \iff \) \( d \)-dimensional periodic tiling.

\( \iff \) Each piece of the Rauzy fractal has the measure of the projection of the corresponding unit cube face on the contracting plane.
Prop. A self-similar aperiodic tiling is obtained as the intersection of the contracting hyperplane with the $d$-dimensional tiling.

Th. (Ito-Rao) \( d \neq 1 \) - dimensional APERIODIC tiling \( \iff \) \( d \)-dimensional periodic tiling.
Self-similar aperiodic tiling
Alternative explicit construction

- Discrete contracting plane.
- Regular lattice: vertices projected in $x + y + z = 0$.
- Replace each rhombus by a piece of the fractal.
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Alternative explicit construction

- Discrete contracting plane.
- Regular lattice: vertices projected in \( x + y + z = 0 \).
- Replace each rhombus by a piece of the fractal.

Then you get the aperiodic tiling if it exists.

Provides combinatorial conditions for tilings in terms of multi-dimensional substitutions and local rules (Arnoux-Berthé-S., Ito)
In the unimodular Pisot case, all the conditions are equivalent.

- Discrete spectrum.
- Periodic tiling.
- Aperiodic tiling.
- $d$-dimensional tiling.
- “Good” measures of pieces.
- Arithmetic automaton condition.
- Super coincidences.
- Balanced pairs.

Extra sufficient conditions

- “Ring” condition on discrete plane.
- 0 inner point and coincidences.
- (F)-property for $\beta$-substitutions.

Which condition can we use to prove that every Pisot substitution satisfy it?