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**Fractal-Topology**

# 2010 JULY WORKSHOP ON SUBSTITUTION TILING AND FRACTAL GEOMETRY

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Abstract. The *neighboring degree* of a compact space $Y$ is the minimum value of $n$ such that every open cover of $Y$ has an open refinement each element of which intersects at most $n$ other elements. Let $\mathcal{D}eg(X) = \max\{\mathcal{D}eg(Y)\}$ with $Y$ running through compact subsets of $X$. Then $\mathcal{D}eg(\cdot)$ defines a simple topological invariant. If $X$ is a one-dimensional continuum then $2 \leq \mathcal{D}eg(X) \leq 3$. For Euclidean spaces, we have $\mathcal{D}eg(\mathbb{R}) = 2, \mathcal{D}eg(\mathbb{R}^2) = 6$ and $\mathcal{D}eg(\mathbb{R}^n) \leq K(\sqrt{2}+1)^n$ for all $n$.

## 1. Introduction

Lebesgue covering dimension, or topological dimension, of a topological space $X$ is defined to be the minimum value of $n$, such that every open cover of $X$ has an open refinement in which no point is included in more than $n+1$ elements. The *neighboring degree* of a compact space $Y$ is defined to be the minimum value of $n$, such that every open cover of $Y$ has an open refinement each of whose elements intersects at most $n$ other elements. The neighboring degree $\mathcal{D}eg(X)$ of $X$ is then defined as the maximum of $\mathcal{D}eg(Y)$ for $Y$ running through compact subsets of $X$. Clearly, we have the following observations:

(1) $\mathcal{D}eg(X) \geq \dim(X) + 1$;
(2) $\mathcal{D}eg(Y) \leq \mathcal{D}eg(X)$ whenever $Y \subset X$ (monotonicity);
(3) $\mathcal{D}eg(Y) = \mathcal{D}eg(X)$ if $f : X \to Y$ is homeomorphism;
(4) $\mathcal{D}eg(\mathbb{R}) = 2$ for the real line $\mathbb{R}$, and $\mathcal{D}eg(T) = 3$ for the union $T$ of two line segments intersecting at a single point;
(5) $\mathcal{D}eg(M) \geq 2$ for all continua (connected compact set in a metric space), while $\mathcal{D}eg(M) = 2$ if and only if $M$ is either chainable or circularly chainable.

Here, a continuum is chainable if it admits an $\varepsilon$-mapping $f$ onto the interval $[0,1]$ for every number $\varepsilon > 0$, in the sense that each $f^{-1}(t)$ with $t \in [0,1]$ is of diameter less than $\varepsilon$. Equivalently, a (metric)

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continuum is defined to be chainable provided that every open cover has an open refinement $\{U_k : k\}$ such that $U_i \cap U_j \neq \emptyset$ if and only if $|i-j| \leq 1$. Similarly, a continuum is circularly chainable if it admits an $\varepsilon$-mapping onto the circle $\mathbb{S}^1$.

For continua $M$ with covering dimension $\dim(X) = 1$, we can infer a nice estimation of $\mathcal{D}eg(M)$.

**Theorem 1.1.** *If $M$ is a one-dimensional continuum then $2 \leq \mathcal{D}eg(M) \leq 3$.*

Examples of continua $M$ with $\mathcal{D}eg(M) = 3$ include Sierpinski's triangle and carpet. The above theorem provides typical cases that the neighboring degree $\mathcal{D}eg(X)$ is closely related to covering dimension $\dim(X)$, which can be seen from the definition, too. However, this connection is not clear when $\dim(X) > 1$, even for cases $X = \mathbb{R}^n$ with $n \geq 2$.

The pattern of brick walls from ancient times, depicted in the following figure, suggests an open cover of $\mathbb{R}^2$ each of whose elements intersects exactly 6 other elements.

Figure 1. Brick Wall Pattern

Since the minimal distance between two disjoint bricks is no less than a positive number $\delta$, we may assign to each brick its $\frac{\delta}{3}$-neighborhood, *i.e.* an open set consisting of points within $\frac{1}{3}\delta$ to the brick. Therefore, we obtain an open cover of the plane each of whose elements intersects exactly 6 other elements. Choosing a small enough size of the bricks, we can easily verify that $\mathcal{D}eg(\mathbb{R}^2) \leq 6$. On the other hand, we will use an argument based on isodiametric inequality on the plane to show $\mathcal{D}eg(\mathbb{R}^2) \geq 6$, and hence obtain the exact value of $\mathcal{D}eg(\mathbb{R}^2)$.

**Theorem 1.2.** $\mathcal{D}eg(\mathbb{R}^2) = 6$.

## 2. Motivation from Topology of $\mathbb{Z}^2$-Tiles

Our study in this paper is partly motivated by recent study on topology of integral self-affine tiles in the plane. Such are tiles $T := T(A, \mathcal{D})$ satisfying

$$A(T) = T + \mathcal{D} := \bigcup_{d \in \mathcal{D}} \{x + d : x \in T\}, \tag{2.1}$$

where $A$ is an expanding $2 \times 2$ matrix with integral entries and $\mathcal{D} \subset \mathbb{Z}^2$ with $|\mathcal{D}| = |detA|$ is a complete set of coset representatives for $\mathbb{Z}^2/A\mathbb{Z}^2$. If further $T + \mathbb{Z}^2$ is a tiling of $\mathbb{R}^2$, one can infer that there are at least 6 elements $d_1, \cdots, d_6 \in \mathbb{Z}^2 \setminus \{0\}$ with $T \cap (T + d_i) \neq \emptyset$. See, for instance, [2, Proposition 3.2]. We shortly call $T + d$ with $T \cap (T + d) \neq \emptyset$ a *neighboring tile* of $T$. In the case that $T$ has exactly 6 neighboring tiles, Bandt and Wang [1] prove that $T$ is homeomorphic with a closed disk if and only if $T$ is connected. For all the known tilings of the plane, whose tiles are uniformly bounded from above, some tile has $\geq 6$ neighboring tiles. Then, one may wonder whether this property holds for all tilings of $\mathbb{R}^2$. Here, a tiling of a space $X$ is a covering $\mathcal{T}$ by compact subsets with nonempty interior that satisfies the following conditions:

(1) $T_i^o \cap T_j^o = \emptyset$ for $T_i \neq T_j \in \mathcal{T}$;
(2) there is some constant number $K > 0$ such that every point $x \in X$ is contained by at most $K$ distinct elements of $\mathcal{T}$.

For any fixed tiling $\mathcal{T}$, we may assign to each tile $T \in \mathcal{T}$ an $\varepsilon-$neighborhood $U_T$ of $T$. If we require $3\varepsilon$ to be smaller than the distance between $T$ and the union of all the tiles $T' \in \mathcal{T}$ with $T \cap T' = \emptyset$, we obtain an open cover $\{U_T : T \in \mathcal{T}\}$ such that $U_T \cap U_{T'} \neq \emptyset$ if and only if $T \cap T' \neq \emptyset$. Therefore, we can discuss the notion of *neighboring degree* via open covers of a compact set, while the original motivation arises from tiling theory.

In the study of topology of three dimensional self-affine tiles, a currently open question of interest is to find reasonable conditions for such a tile to be homeomorphic with a solid ball in $\mathbb{R}^3$, or, for its boundary to be homeomorphic with the sphere $\mathbb{S}^2$.

Suppose that $T := T(A, \mathcal{D})$ is an integral self-affine tile satisfying

$$A(T) = T + \mathcal{D} := \bigcup_{d \in \mathcal{D}} \{x + d : x \in T\}, \tag{2.2}$$

where $A$ is an expanding $3 \times 3$ matrix with integral entries and $\mathcal{D} \subset \mathbb{Z}^3$ with $|\mathcal{D}| = |detA|$ is a complete set of coset representatives for $\mathbb{Z}^3/A\mathbb{Z}^3$. We further assume that $T + \mathbb{Z}^3$ is a tiling of $\mathbb{R}^3$. Then, one may consider those homeomorphy questions in the following particular case.

## 3. $\alpha_k$ and $\beta_k$

For any tiling $\mathcal{T}$ of $\mathbb{R}^k$ with a finite set of prototiles (finitely many shapes are used and equivalent shapes differ by an isometry), let $\mathcal{G}_\mathcal{T}$ be the *neighbor graph* is defined as the infinite graph $\mathcal{G}_\mathcal{T}$ with



vertex set $\mathcal{T}$ for which two vertices $T_1$ and $T_2$ are incident if and only if $T_1 \cap T_2 \neq \emptyset$. Let $\mathcal{D}eg_\mathcal{T}$ as the degree of neighbor graph $\mathcal{G}_\mathcal{T}$.

**Definition 3.1.** $\alpha_k := \inf\{\mathcal{D}eg_\mathcal{T} : \mathcal{T}\}$, where the infimum is taken over all the lattice tilings $\mathcal{T}$ of $\mathbb{R}^k$.

$\beta_k := \inf\{\mathcal{D}eg_\mathcal{T} : \mathcal{T}\}$, where the infimum is taken over all the tilings $\mathcal{T}$ of $\mathbb{R}^k$ with a finite set of prototiles.

*Remark* 3.2. As we assume a finite set of prototiles, the above mentioned tilings $\mathcal{T}$ necessarily have a *locally finite* property: there exists an integer $N_\mathcal{T}$ so that any point can be contained by at most $N$ tiles of the tiling.

$\alpha_1 = \beta_1 = 2$, $\alpha_2 = \beta_2 = 6$, $\beta_3 = 14$ and $\alpha_3$ is unknown for the moment.

**Lemma 3.3. (Product Lemma)** *$\alpha_k$ and $\beta_k$ are* sub-multiplicative*:*

(1) $\alpha_{m+n} + 1 \leq (\alpha_m + 1)(\alpha_n + 1)$.
(2) $(\beta_{n+m} + 1) \leq (\beta_m + 1)(\beta_n + 1)$.

**Theorem 3.4.** *The limits $\alpha = \lim_{n\to\infty} \frac{1}{n} \ln \alpha_n$ and $\beta = \lim_{n\to\infty} \frac{1}{n} \ln \beta_n$ both exist.*

Then, a very interesting problem is to estimate the two numbers $\alpha$ and $\beta$.

## References

[1] C. Bandt and Y. Wang, Disk-like self-affine tiles in $\mathbb{R}^2$, Discrete Comput. Geom. 26(2001): 591-601.
[2] B. Loridant and J. Luo, On a theorem of Bandt and Wang and its extension to $p2$-tiles, Discrete Comput. Geom. 41(2009): 616-642.
[3] Whyburn, Dynamic Topology.

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