An adaptive sparse grid method for elliptic PDEs with stochastic coefficients

J. Erhel[a], Z. Mghazli[b], M. Oumouni[a,b,*]

[a]*INRIA Rennes-Bretagne Atlantique, campus de Beaulieu, 35042 Rennes, France*
[b]*LIRNE-EIMA, Ibn Tofaïl University, B.P. 133, 14 000 Kenitra, Morroco*

Abstract

The stochastic collocation method based on an anisotropic sparse grid is nowadays a significant tool to solve partial differential equations with random input data. This method is based on a level of interpolation and weights of anisotropy. The objective of some adaptive approaches is to select cleverly these parameters, in order to reduce the computational cost.

In this work, we propose such an adaptive approach, based on an approximation of the *inverse* diffusion coefficient. We introduce an error indicator which is an upper bound of the error in the solution and use this indicator as a reliable and cheap tool for choosing the level of interpolation. We also propose a new error estimation in one dimension, for unbounded random variables, and use it to compute suitable weights.

Numerical examples show the efficiency of our methodology, since the cost is considerably reduced, without loss of accuracy.

Keywords: Elliptic PDEs with random input data, Stochastic collocation method, Anisotropic sparse grid, Adaptive method.

1. Introduction

The Monte Carlo method [11] is the most standard approach used to compute statistical quantities of interest depending on the solution of partial differential

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[*]Corresponding author
Email addresses: `jocelyne.erhel@inria.fr` (J. Erhel), `mghazli_zoubida@yahoo.com` (Z. Mghazli), `oumouni.mustapha@yahoo.fr` (M. Oumouni)

equation with stochastic inputs. It consists of solving M deterministic problems, where M is the number of independent and identically distributed simulations (iid) of the parameters. The main disadvantage of this approach is its slow convergence given by the order $O(\frac{1}{\sqrt{M}})$, hence the method requires in general a large computational effort.

Recently, spectral methods have been developed, such as stochastic finite element and collocation methods. They offer a robust tool for solving problems of stochastic PDE ([1, 2, 3, 15, 17]). They approach the response of the model as a stochastic function by a polynomial interpolation in the stochastic space, and provide an exponentially convergent approximation when the solution of the problem is a smooth function with respect to the random variables. However, stochastic finite element methods require solving a large problem combining physical and probability spaces, whereas collocation methods require solving many deterministic problems in the physical space. These methods suffer from a curse of dimensionality, since the computational cost increases exponentially with the stochastic dimension. This often limits their application to problems with a small stochastic dimension. When the stochastic dimension is moderately large, it is necessary to minimize the computational cost.

Reduced basis methods [5, 16, 24, 25] are a first approach for problems with linear random coefficients. A first offline-online step and a posteriori error estimates are used to find the most representative samples of the solution and to build a reduced basis. The physical deterministic problem is projected into this reduced space to get an approximation at a lower cost. The stochastic collocation method can be combined with this approach [16, 23].

Sparse and anisotropic polynomial interpolation aims at minimizing the total number of collocation points. The anisotropic method proposed in [21] uses a priori and a posteriori information based on the regularity of the solution, while the adaptive method developed in [14] and applied to the problem of PDEs in [12] defines the level of the approximation by successive enrichments. The approach developed in [4, 27] uses sparse wavelet chaos subspaces for linear elliptic PDEs. These approximations are also anisotropic and exploit the

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decay of the eigenvalues in the expansion of the diffusion coefficient to define an approximation in the chaos polynomial space.

In this work, we focus our attention to collocation with an anisotropic sparse grid. We develop an adaptive approach to estimate the weights and the level of interpolation in an anisotropic sparse grid, in the same spirit as [3, 4, 21]. In [10], we introduced the Kahrunen-Loève expansion of the inverse of the diffusion coefficient, and used the basis of this expansion to compute the average solution of a one dimensional elliptic problem. Motivated by this work, we now use this inverse of the diffusion to define an error indicator. We prove that this indicator satisfies an upper bound of the interpolation error in the solution and we provide estimations of the constant in this bound. This error indicator can be used to select a level of interpolation. Then we prove a new error bound for problems with one-dimensional unbounded random variables. We use our indicator and one-dimensional error bounds to compute the weights of anisotropy. This process requires only a sequence of cheap interpolation problems, without solving expensive deterministic problems in the physical space.

The paper is organized as follows. In the first section, we introduce the mathematical problem setting with a finite noise assumption. In section 2, we recall the stochastic collocation methods based either on a full tensor product or a sparse grid. In Section 3, we develop our methodology and our theoretical results. Finally, in section 4, we present numerical examples to illustrate the efficiency of our approach.

2. Problem formulation

Let $(\Omega, \mathcal{F}, d\mathbb{P})$ be a complete probability space, where Ω is the space of elementary event, $\mathcal{F} \subset 2^{\Omega}$ is the σ-algebra of events and $\mathbb{P}$ is the probability measure. Also, we consider a bounded domain $D \subset \mathbb{R}^d$, with a smooth boundary ∂D. The input data k and f are two random fields on $\Omega \times D$. We focus on the following linear elliptic boundary value problem: find a stochastic function,

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$u : \Omega \times D \longrightarrow \mathbb{R}$ such that the following equation holds (a.s) in Ω:

$$(\mathcal{P}_s) \begin{cases} -\mathrm{div}(k\nabla u) = f & \text{in } \Omega \times D, \\ u \mid_{\partial D} = 0. \end{cases} \tag{1}$$

We make the following assumptions on the random input data k and f to preserve the well-posedness of the problem $(\mathcal{P}_s)$.

Assumption 1:

- **H1**: f belongs to $L^p(\Omega) \otimes L^2(D)$, for all $p \in [1, \infty[$.

- **H2**: There exist k_{min}, k_{max} such that, for each $\omega \in \Omega$ (a.s) we have

  $0 < k_{min}(\omega) \le k(\omega, .) \le k_{max}(\omega)$ and $k_{min}^{-1} \in L^p(\Omega)$.

Let $H_0^1(D)$ be the subspace of $H^1(D)$ consisting of the functions with vanishing trace on ∂D, then we define the space $L^p(\Omega) \otimes H_0^1(D)$,

$$L^p(\Omega) \otimes H_0^1(D) := \left\{ v : \Omega \longrightarrow H_0^1(D); \int_\Omega \|v\|_{H_0^1(D)}^p d\mathbb{P} < \infty \right\}.$$

To introduce the stochastic discretization, we first state a finite dimensional assumption for the functions k and f (see [3, 20, 21]).

2.1. Finite dimensional noise

In many practical problems, random coefficients are parameterized by a finite number of uncorrelated random variables, sometimes independent, as in the case of a truncated Karhunen-Loève or Fourier expansion [18, 19, 2]. This motivates us to assume that k and f are parameterized by N random variables $\{Y_n\}_{n=1}^N$. The number N is called the stochastic dimension.

Assumption 2 (Finite dimensional noise)

The stochastic fields k and f are parameterized by a random vector $\mathbf{Y} = (Y_1, ..., Y_N)$ with N components, so that $k(\omega, x) = k(\mathbf{Y}(\omega), x)$ and $f(\omega, x) = f(\mathbf{Y}(\omega), x)$.

We define $\Gamma_n := Y_n(\Omega)$ for each $n = 1, \ldots, N$ and $\Gamma := \mathbf{Y}(\Omega)$. We assume that the components of $\mathbf{Y}$ are independent. If ρ_n is the density function of each

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Y_n, then $\mathbf{Y}$ has a joint probability density $\varrho(y) = \prod_{n=1}^{N} \rho_n(y_n), \forall y \in \Gamma$ and $\Gamma = \prod_{n=1}^{N} \Gamma_n$.

Example 1 (Karhunen-Loève (K-L) expansion)

The Karhunen-Loève expansion of a given random field $k \in L^2(\Omega) \otimes L^2(D)$ is a series of products of deterministic functions and random variables [18, 19]:

$$k(\omega, x) = \mathbb{E}[k](x) + \sum_{n=1}^{\infty} \sqrt{\lambda_n} \phi_n(x) Y_n(\omega), \tag{2}$$

where $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\phi_n\}_{n=1}^{\infty}$ are respectively eigenvalues and eigenfunctions of the following eigenvalue problem:

$$\int_D \text{cov}[k](x, z)\phi(z)dz = \lambda\phi(x),$$

with $\text{cov}[k](x, z) := \mathbb{E}[k(., x)k(., z)] - \mathbb{E}[k(., x)]\mathbb{E}[k(., z)]$. The uncorrelated random variables $\{Y_n\}_{n=1}^{\infty}$ are given by $Y_n = \dfrac{1}{\sqrt{\lambda_n}} \displaystyle\int_D (k(., x) - \mathbb{E}[k(x)])\phi_n(x)dx$. We define the truncated (K-L) expansion k_N by

$$k_N(\omega, x) = \mathbb{E}[k](x) + \sum_{n=1}^{N} \sqrt{\lambda_n} \phi_n(x) Y_n(\omega). \tag{3}$$

This truncated (K-L) expansion satisfies Assumptions 1 and 2.

Example 2 (A nonlinear coefficient)

In some cases, to ensure some properties for data such as the positivity, the parameter k is given by a nonlinear transformation of a Gaussian field G, like $k = g(G)$, where $g > 0$ is non-negative and smooth. A widely used example is a log-normal coefficient $k(\omega, x) = e^{G_N}$, where G_N is the (K-L) truncation of the Gaussian field G.

2.2. Strong formulation

If the stochastic functions k and f are parameterized by the vector $\mathbf{Y}$ as in Assumption 2, then by Doob-Dynkin's lemma, the solution u of $(\mathcal{P}_s)$ can also be parameterized by $\mathbf{Y}$ so that $u(\omega, x) = u(\mathbf{Y}(\omega), x)$. Therefore, we introduce the strong deterministic problem obtained by projecting $(\mathcal{P}_s)$ in $H_0^1(D)$ for all

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y in Γ (a.e):

$$\begin{cases} \text{Find } u(y,.) \in H_0^1(D) \text{ such that} \\ \int_D k(y,.)\nabla u(y,x) \cdot \nabla v(x)dx = \int_D f(y,x)v(x)dx \ \ \forall v \in H_0^1(D). \end{cases} \quad (4)$$

For all y in Γ (a.e), by Lax-Milgram theorem, the problem (4) is well posed and $u(y,.)$ satisfies:

$$\|u(y,.)\|_{H_0^1(D)} \leq C_D k_{min}^{-1}(y)\|f(y,.)\|_{L^2(D)}, \quad (5)$$

where C_D is the Poincaré constant. Under Assumptions (**H1**, **H2**), the solution u belongs to $L_\varrho^p(\Gamma) \otimes H_0^1(D)$ for all $p \geq 1$, since:

$$\mathbb{E}[\|u\|_{H_0^1(D)}^p] \leq C_D^p \mathbb{E}[k_{min}^{-p}\|f\|_{L^2(D)}^p] \leq C_D^p\sqrt{\mathbb{E}[(k_{min}^{-1})^{2p}]\mathbb{E}[\|f\|_{L^2(D)}^{2p}]}.$$

3. Stochastic collocation method

90   The stochastic collocation method is based on polynomial interpolation. Let $\mathcal{P}_p(\Gamma) := \otimes_{n=1}^N \mathcal{P}_{p_n}(\Gamma_n)$ be the set of tensor product polynomials with degree at most $p = (p_1, \ldots, p_n, \ldots, p_N)$.

Collocation methods define a a set of polynomials $L_{\mathbf{j}}$ in $\mathcal{P}_p(\Gamma)$ and a grid $\mathcal{H}$ of collocation points $y_{\mathbf{j}} = (y_{1,j_1}, \ldots, y_{n,j_n}, \ldots, y_{N,j_N})$, where $y_{n,j_n} \in \Gamma_n$. Then the multi-dimensional polynomial interpolation $\mathcal{A}$ is defined in the following way, for any function v in $\mathcal{C}^0(\Gamma; H_0^1(D))$:

$$\mathcal{A}v(y,x) := \sum_{y_{\mathbf{j}} \in \mathcal{H}} v(y_{\mathbf{j}}, x)L_{\mathbf{j}}(y). \quad (6)$$

This interpolation requires computing the function $v(y_{\mathbf{j}}, x)$ at each collocation point $y_{\mathbf{j}}$. Thus the cost is proportional to the size of the grid $\mathcal{H}$. Although

95   it is embarrassingly parallel, the cost can become prohibitive for two reasons: the size of $\mathcal{H}$ can grow rapidly with the stochastic dimension N and the cost of computing $v(y,x)$ for a given y can be quite high.

Indeed, the solution $u(y,x)$ is approximated in practice by $u_h(y,x)$, where u_h is obtained for example by a finite element method. Therefore, for a given

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$y \in \Gamma$, computing $u_h(y, x)$ requires solving the deterministic problem

$$
\begin{cases}
\text{Find } u_h(y, .) \in V_h(D) \text{ such that} \\
\int_D k(y, .) \nabla u_h(y, x) \cdot \nabla v_h(x) dx = \int_D f(y, x) v_h(x) dx, \quad \forall v_h \in V_h(D),
\end{cases}
\tag{7}
$$

where $V_h(D)$ is a standard finite element space.

Now, we recall three different types of grids: a full tensor product of one-dimensional intervals or a sparse grid given by the Smolyak algorithm [26, 28], which can be isotropic or anisotropic.

3.1. Full tensor product interpolation

The full tensor product interpolation is given by a product of one-dimensional interpolations, where a high degree is used in each dimension. For $n = 1, ..., N$, let $i_n \in \mathbb{N}^+$ be a level of a one-dimensional interpolation of degree p_n, and $\mathcal{X}^{i_n} := \{y_{n,1}, ..., y_{n,m(i_n)}\}$ be a set of $m(i_n) = p_n + 1$ collocation points in Γ_n.

For $v \in \mathcal{C}^0(\Gamma_n; V_h(D))$, we introduce a sequence of one-dimensional Lagrange interpolation operators $\mathcal{U}^{i_n} : \mathcal{C}^0(\Gamma_n; V_h(D)) \longrightarrow \mathcal{P}_{p_n}(\Gamma_n) \otimes V_h(D)$ defined by:

$$
\mathcal{U}^{i_n}(v)(z, x) := \sum_{j=1}^{m(i_n)} v(y_{n,j}, x) \ell_{n,j}^{i_n}(z),
\tag{8}
$$

where $\ell_{n,j}^{i_n}(z) = \prod_{r=1, r \neq j}^{m(i_n)} \frac{z - y_{n,r}}{y_{n,j} - y_{n,r}}$ is the Lagrange polynomial of degree p_n.

We define the full tensor grid

$$
\mathcal{H}_{\mathbf{i}} = \mathcal{X}^{i_1} \times \ldots \times \mathcal{X}^{i_N},
$$

with collocation points $y_{\mathbf{j}} = (y_{1,j_1}, \ldots, y_{n,j_n}, \ldots, y_{N,j_N})$, and the multidimensional Lagrange polynomials $L_{\mathbf{j}}^{\mathbf{i}}(y) = \otimes_{n=1}^{N} \ell_{n,j_n}^{i_n}(y_n)$. The full tensor product interpolation is given by

$$
\mathcal{A}_{\mathbf{i}} v(y, x) = \sum_{\mathbf{j} \in \mathcal{H}_{\mathbf{i}}} v(y_{\mathbf{j}}, x) L_{\mathbf{j}}^{\mathbf{i}}(y) = \mathcal{U}^{i_1} \otimes \ldots \otimes \mathcal{U}^{i_N} v(y, x).
\tag{9}
$$

The size of the grid $\mathcal{H}_{\mathbf{i}}$ is $\prod_{n=1}^{N} m(i_n)$. Since it grows very rapidly with N, it is more interesting to use a sparse grid.

3.2. Isotropic sparse grid method

The isotropic or standard sparse grid method is based on the Smolyak algorithm [26, 28], which provides an effective way to approach multivariate functions by a polynomial interpolation. It is given by linear combinations of product formulas (9), where in each dimension a small level of interpolation is used. This provides a significant reduction of the grid size.

For each $n = 1, \ldots, N$, we define $\Delta_{i_n} = \mathcal{U}^{i_n} - \mathcal{U}^{i_n - 1}$ with $\mathcal{U}^0 = 0$ and $\mathcal{U}^{i_n}$ given by (8).

We introduce a global level of interpolation, denoted by $w \in \mathbb{N}$. We consider the sets of multi-levels defined by

$$X(w, N) := \left\{ \mathbf{i} \in \mathbb{N}^N, \mathbf{i} \geq \mathbf{1}, \, |\mathbf{i}| := \sum_{n=1}^{N} i_n \leq w + N \right\}$$

and

$$Y(w, N) := \left\{ \mathbf{i} \in \mathbb{N}^N, w < |\mathbf{i}| \leq w + N \right\}.$$

The Smolyak interpolation operator $\mathcal{A}(w, N)$ with level w is given by the formula:

$$\mathcal{A}(w, N)v := \sum_{\mathbf{i} \in X(w, N)} (\Delta_{i_1} \otimes \ldots \otimes \Delta_{i_N})v. \tag{10}$$

An equivalent formula (see [28]) which is more practical than (10) is given by:

$$\mathcal{A}(w, N)v = \sum_{\mathbf{i} \in Y(w, N)} (-1)^{w + N - |\mathbf{i}|} \binom{N - 1}{w + N - |\mathbf{i}|} \mathcal{U}^{i_1} \otimes \ldots \otimes \mathcal{U}^{i_N} v. \tag{11}$$

The grid of collocation points is then

$$\mathcal{H}(w, N) = \bigcup_{\mathbf{i} \in Y(w, N)} \{ \mathcal{X}^{i_1} \times \ldots \times \mathcal{X}^{i_N} \}.$$

In the case when the nodes are nested, the Smolyak interpolant becomes more interesting, since these nodes can be used several times in the grid and this results in a significant reduction of the computational cost. Examples of nested points are Newton-Cotes, Clenshaw-Curtis, Gauss-Patterson points [8, 9, 13, 20, 21].

3.3. The anisotropic sparse grid method

In the standard sparse grid method, all directions are isotropic, since any permutation of an admissible multi-level in $X(w, N)$ is also admissible. However, when the dimensions have different behaviors, the solution is highly anisotropic and the convergence rate can be improved thanks to an anisotropic sparse grid.

We follow [21] closely to define the anisotropic sparse grid formula. We consider a set of weights $\alpha_1, \ldots, \alpha_N$, we let $\underline{\alpha} = \min_{1 \leq, n \leq N} \alpha_n$ and $|\alpha| = \sum_{n=1}^{N} \alpha_n$.

We consider the sets of multi-levels defined by

$$X_\alpha(w, N) := \{\mathbf{i} \in \mathbb{N}^N, \mathbf{i} \geq \mathbf{1}, \sum_{n=1}^{N} (i_n - 1)\alpha_n \leq w\underline{\alpha}\}$$

and

$$Y_\alpha(w, N) = \{\mathbf{i} \geq \mathbf{1}, w\underline{\alpha} < \sum_{n=1}^{N} \alpha_n i_n \leq w\underline{\alpha} + |\alpha|\}.$$

The anisotropic interpolation is given by

$$\mathcal{A}_\alpha(w, N)v := \sum_{\mathbf{i} \in X_\alpha(w, N)} (\Delta_{i_1} \otimes \ldots \otimes \Delta_{i_N})v, \tag{12}$$

or equivalently [14]:

$$\mathcal{A}_\alpha(w, N)v = \sum_{\mathbf{i} \in Y_\alpha(w, N)} c_\alpha(\mathbf{i}) \mathcal{U}^{i_1} \otimes \ldots \otimes \mathcal{U}^{i_N} v, \tag{13}$$

with $c_\alpha(\mathbf{i}) = \sum_{\mathbf{j} \in \{0,1\}^N, \mathbf{i}+\mathbf{j} \in X_\alpha(w, N)} (-1)^{|\mathbf{j}|}$.

The anisotropic sparse grid is here

$$\mathcal{H}_\alpha(w, N) = \bigcup_{\mathbf{i} \in Y_\alpha(w, N)} \{\mathcal{X}^{i_1} \times \ldots \times \mathcal{X}^{i_N}\}. \tag{14}$$

The size of the anisotropic sparse grid, denoted by M, should be as small as possible, since the computational cost of the stochastic collocation method is proportional to M.

4. Error estimation and adaptivity in the stochastic space

The main ingredients of the collocation method with an anisotropic sparse grid are the vector of weights α and the level of interpolation w. The vector α

takes into account the relative importance of each dimension whereas the level w controls the size of the grid. Our objective is to find suitable weights α and a level w at a small cost.

An adaptive approach increases the level w until a prescribed accuracy, defined by a tolerance parameter and an error estimation [2, 3]. Quite often, this error estimation requires solving deterministic problems (7) at several stochastic points. In order to avoid this overhead cost, we propose to use another error estimation, much cheaper because it does not involve the problem (7).

4.1. Error indicator

In what follows, we introduce an indicator λ to define an adaptive estimation of the weights and the level of interpolation. The problem $(\mathcal{P}_s)$ is equivalent to the following system:

$$\begin{cases} -\nabla u = k^{-1}p, \\ div(p) = f. \end{cases}$$

The first equation shows that each stochastic dimension of u and $k^{-1}p$ have the same behavior. From the second equation we expect that p and f have also the same behavior with respect to the stochastic variable. Therefore, we propose to define the indicator $\lambda := k^{-1}f$, in order to identify the behavior of different directions of u at low cost.

In [10], for a one-dimensional elliptic problem, we computed the Karhunen-Loève expansion for k^{-1}, and used its random variables as a basis to build a projection of u which gives a good approximation of the expectation $\mathbb{E}[u]$. In this work, we propose to use a similar methodology to prove an error bound in the interpolation error of u, where u is solution of $(\mathcal{P}_s)$.

We first recall a lemma with a bound for the error $\|u - \mathcal{A}_{\mathbf{i}}u\|_{L^2(\Gamma;H_0^1(D))}$ in u when using the full tensor interpolation (9). This is the main result given in [3], showing that each direction contributes to the global error with a decay rate depending on the size of the region where u is analytic.

Lemma 4.1. *([3]) Let $v \in L_\varrho^2(\Gamma) \otimes H_0^1(D)$ and $\mathcal{A}_{\mathbf{i}}v$ its full interpolation defined in (9). We assume that, in each direction, v admits an analytic extension in the*

region $\Sigma_n(\Gamma_n; \tau_n(v)) := \{z \in \mathbb{C}, \quad dist(z, \Gamma_n) \leq \tau_n(v)\}$, with the size $\tau_n(v) > 0$. Then, there exists $C_F > 0$, independent of each p_n, such that:

$$\|v - \mathcal{A}_\mathbf{i}v\|_{L^2_\varrho(\Gamma)\otimes H^1_0(D)} \leq C_F \sum_{n=1}^{N} \beta_n(p_n)e^{-\nu_n p_n^{\theta_n}}, \tag{15}$$

where, if Γ_n is bounded, $\theta_n = \beta_n = 1$ and $\nu_n = log\left(\frac{2\tau_n(v)}{|\Gamma_n|}(1 + \sqrt{1 + \frac{|\Gamma_n|^2}{4\tau_n^2(v)}})\right)$; if Γ_n is unbounded, then $\theta_n = 1/2, \beta_n = O(\sqrt{p_n})$ and $\nu_n = \tau_n(v)/2$.

We also recall a lemma given in [3], showing that the solution u of $(\mathcal{P}_s)$ is analytic.

Lemma 4.2. *([3]) We assume that k and f are analytic. Furthermore, if k is linear in y we also assume that f is polynomial in y. If k is non linear, we assume that $k = g(b_0(x) + \sum_{n=1}^{N} b_n(x)y_n)$, where g is an analytic and strictly positive function. Then the solution u of $(\mathcal{P}_s)$ is analytic.*

We are now able to prove that the indicator $\|\lambda - \mathcal{A}_\mathbf{i}\lambda\|_{L^2_\varrho(\Gamma)\otimes L^2(D)}$ is an upper bound of the error $\|u - \mathcal{A}_\mathbf{i}u\|_{L^2_\varrho(\Gamma)\otimes H^1_0(D)}$.

Proposition 4.1. *We make assumptions of Lemma 4.2. Let u be the solution of $(\mathcal{P}_s)$, $\lambda = k^{-1}f$ and $\mathcal{A}_\mathbf{i}u$ and $\mathcal{A}_\mathbf{i}\lambda$ their approximations by the formula (9). Then we have the following estimate:*

$$\|u - \mathcal{A}_\mathbf{i}u\|_{L^2_\varrho(\Gamma;H^1_0(D))} \leq C\|\lambda - \mathcal{A}_\mathbf{i}\lambda\|_{L^2_\varrho(\Gamma,L^2(D))}, \tag{16}$$

where the positive constant C depends on u and λ.

Proof. Thanks to Lemma 4.2, both functions u and λ are analytic, so that we can apply Lemma 4.1.

1) First, we consider the case where k is linear according to y: $k = b_0 + \sum_{n=1}^{N} b_n y_n$. Since f is polynomial on y, it is enough to compare the analyticity region of u with that of k^{-1}. We prove that $\tau_n(u) \geq \tau_n(k^{-1})$ for each $n = 1, \ldots, N$.

To do this, we adopt the following notations: for each $n = 1, \ldots, N$, we set $\Gamma_n^* = \prod_{j\neq n}^{N} \Gamma_j$, $y = (y_n, y_n^*)$ where $y_n \in \Gamma_n$ and $y_n^* \in \Gamma_n^*$.

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Since we can write $k^{-1}(z, y_n^*, x) = \dfrac{k^{-1}(y_n, y_n^*, x)}{1 + \dfrac{(z - y_n)b_n(x)}{k(y_n, y_n^*, x)}}$, then $k^{-1}(., y_n^*, x)$ can

be analytically extended in the region $\{|z - y_n| < \tau_n(k^{-1})\}$ and

$$\frac{1}{\tau_n(k^{-1})} = \|\frac{b_n}{k}\|_\infty.$$

The function $u(., y_n^*, x)$ as a function of the variable y_n can be extended analytically in the ball $\{|z - y_n| < \tau_n(u)\}$ where the radius $\tau_n(u)$ is computed by d'Alembert's ratio test as follows:

$$\frac{1}{\tau_n(u)} = \limsup_{m \longrightarrow \infty} \frac{\frac{\|\partial_{y_n}^m u\|_{H_0^1(D)}}{m!}}{\frac{\|\partial_{y_n}^{m-1} u\|_{H_0^1(D)}}{(m-1)!}}. \tag{17}$$

By deriving m-times problem (4) with respect to y_n we obtain:

$$\partial_{y_n}^m \int_D k\nabla u \nabla v dx = \int_D \sum_{l=0}^m \binom{m}{l} \partial_{y_n}^l k \partial_{y_n}^{m-l} \nabla u \nabla v dx = \partial_{y_n}^m \int_D f v dx \quad \forall v \in H_0^1(D).$$

For $l \geq 2$ we have $\partial_{y_n}^l k = 0$ and $\partial_{y_n} k = b_n$, then

$$\int_D k \frac{\partial_{y_n}^m \nabla u}{m!} \nabla v dx = -\int_D b_n \frac{\partial_{y_n}^{m-1} \nabla u}{(m-1)!} \nabla v dx + \int_D \frac{\partial_{y_n}^m f}{m!} v dx. \tag{18}$$

Taking $v = \partial_{y_n}^m \nabla u$ in (18) and by Cauchy-Schwartz inequality we obtain:

$$\frac{\|\sqrt{k}\partial_{y_n}^m \nabla u\|_{L^2(D)}}{m!} \leq \|\frac{b_n}{k}\|_\infty \frac{\|\sqrt{k}\partial_{y_n}^{m-1} \nabla u\|_{L^2(D)}}{(m-1)!} + \frac{C_D}{\sqrt{k_{min}(y)}} \frac{\|\partial_{y_n}^m f\|_{L^2(D)}}{m!}.$$

Since f is polynomial on y, we have $\limsup_{m \longrightarrow \infty} \frac{\|\partial_{y_n}^m f\|_{L^2(D)}}{m!} = 0$, thus

$$\limsup_{m \longrightarrow \infty} \frac{\frac{\|\sqrt{k}\partial_{y_n}^m \nabla u\|_{L^2(D)}}{m!}}{\frac{\|\sqrt{k}\partial_{y_n}^{m-1} \nabla u\|_{L^2(D)}}{(m-1)!}} \leq \|\frac{b_n}{k}\|_\infty.$$

Taking the energy norm in $H_0^1(D)$, from equality (17), we get $\frac{1}{\tau_n(u)} \leq \|\frac{b_n}{k}\|_\infty$, which implies $\tau_n(u) \geq \tau_n(k^{-1})$. Thus, by using Lemma 4.1, there exists $C > 0$ such that:

$$\|u - \mathcal{A_i}u\|_{L_\varrho^2(\Gamma; H_0^1(D))} \leq C\|\lambda - \mathcal{A_i}\lambda\|_{L_\varrho^2(\Gamma; L^2(D))}.$$

2) We suppose now that k is non linear according to y as $k(y, x) = g(b_0(x) + \sum_{n=1}^N b_n(x)y_n)$, where g is an analytic and strictly positive function. Let $\{\mathcal{V}_l\}_l$

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a discretization of the set Γ, where for each element $\mathcal{V}_l$ with the size h_l, the following estimate holds:

$$k(y, x) \approx \sum_{n=1}^{N} a_n(y^l, x) y_n \ \ \forall y \in \mathcal{V}_l.$$

Where y^l is the center of $\mathcal{V}_l$. Let $\mathcal{A}_{\mathbf{i}}^l u$ be the full interpolation for the restriction of u in each element $\mathcal{V}_l$, then by the previous case the following estimate holds:

$$||u - \mathcal{A}_{\mathbf{i}}^l u||_{L_\varrho^2(\mathcal{V}_l; H_0^1(D))} \leq \tilde{c} ||\lambda - \mathcal{A}_{\mathbf{i}}^l \lambda||_{L_\varrho^2(\mathcal{V}_l; L^2(D))}. \tag{19}$$

Furthermore, for all y in $\mathcal{V}_l$, if we set $\psi_l(y, \cdot) = u(y, \cdot) - \mathcal{A}_{\mathbf{i}} u(y, \cdot)$, then since $\mathcal{A}_{\mathbf{i}} u$ is polynomial, we get:

$\mathcal{A}_{\mathbf{i}}^l \psi_l = \mathcal{A}_{\mathbf{i}}^l u - \mathcal{A}_{\mathbf{i}}^l(\mathcal{A}_{\mathbf{i}} u) = \mathcal{A}_{\mathbf{i}}^l u - \mathcal{A}_{\mathbf{i}} u$. According to the classical h_p convergence (see [2, 7]), with a small size h_l, the following estimation holds,

$$||\psi_l - \mathcal{A}_{\mathbf{i}}^l \psi_l||_{L_\varrho^2(\mathcal{V}_l; H_0^1(D))} \approx \hat{c} h_l^p ||\psi_l||_{L_\varrho^2(\mathcal{V}_l; H_0^1(D))} \tag{20}$$

where p is the smallest order used in the directions of $\mathcal{A}_{\mathbf{i}}^l$. Therefore, we deduce

$$||u - \mathcal{A}_{\mathbf{i}}^l u||_{L_\varrho^2(\mathcal{V}_l; H_0^1(D))} \approx \hat{c} h_l^p ||u - \mathcal{A}_{\mathbf{i}} u||_{L_\varrho^2(\mathcal{V}_l; H_0^1(D))}. \tag{21}$$

Also by the same argument, we have:

$$||\lambda - \mathcal{A}_{\mathbf{i}}^l \lambda||_{L_\varrho^2(\mathcal{V}_l; H_0^1(D))} \approx \hat{c} h_l^p ||\lambda - \mathcal{A}_{\mathbf{i}} \lambda||_{L_\varrho^2(\mathcal{V}_l; L^2(D))}. \tag{22}$$

Combining (21) and (22) with (19) we obtain in each $\mathcal{V}_l$:

$$||u - \mathcal{A}_{\mathbf{i}} u||_{L_\varrho^2(\mathcal{V}_l; H_0^1(D))} \leq \tilde{C} ||\lambda - \mathcal{A}_{\mathbf{i}} \lambda||_{L_\varrho^2(\mathcal{V}_l; L^2(D))}.$$

Then the estimation holds in Γ. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 4.1. *The result of the proposition is still true with Smolyak interpolation $\mathcal{A}_\alpha(N, w)$, since the accuracy of $\mathcal{A}_\alpha(N, w)v$ depends on the size of the analyticity region of the function v.*

4.2. Estimation of the constant in the error bound

In order to use the error bound of Proposition 4.1, it is necessary to estimate the constant C in the formula (16). The idea is to estimate C by the ratio $\frac{\|u - \mathcal{A}_\alpha(w,N)u\|}{\|\lambda - \mathcal{A}_\alpha(w,N)\lambda\|}$.

The error committed by the interpolation $\mathcal{A}_\alpha(w, N)v$ satisfies the following estimation [21, 22]:

$$\|v - \mathcal{A}_\alpha(w, N)v\| \approx C_1 \|v\| M^{-\nu(v)}, \tag{23}$$

where M is the grid size and the decay rate $\nu(v)$ depends on the size $\tau(v)$ of the analyticity region of v. More precisely, $\nu(v)$ can be approximated by $\min_n \nu_n(v)$. Therefore we propose to estimate C by $\frac{\|u\|}{\|\lambda\|} M^{\nu(\lambda) - \nu(u)}$. Since we cannot compute $\|u\|$ we replace it by $\|\mathcal{A}_\alpha(0, N)u\|$. Finally, we have the constant estimation:

$$C \approx \frac{\|\mathcal{A}_\alpha(0, N)u\|}{\|\mathcal{A}_\alpha(0, N)\lambda\|} M^{-\mu}, \tag{24}$$

where $\mu = \min_n \nu_n(u) - \min_n \nu_n(\lambda)$.

It might happen that $\mu < 0$, yielding a large constant C, but only when k is not analytically extended in a complex region with a bigger size than that of λ. Thus we discard this case and assume that $\mu \geq 0$.

If the random field k is nonlinear and smooth, the function u can be locally approximated by a function which has the same regularity as λ, therefore we assume that $\mu = 0$ in that case.

On the other hand, if the random field is linear with respect to y, in general $\mu > 0$ and should be computed. In Subsection 4.5, we discuss how to estimate the decay rates ν_n and in Subsection 5.2, we discuss how to estimate the exponent μ in this linear case.

4.3. Determination of the level w

Applying Proposition 4.1, we propose to estimate the error $\|u - \mathcal{A}_\mathbf{i} u\|_{L^2_\varrho(\Gamma, H^1_0(D))}$ by $C \|\lambda - \mathcal{A}_\mathbf{i} \lambda\|_{L^2_\varrho(\Gamma, H^1_0(D))}$, where C is given by (24). In practice, the function λ is approximated by a reference value with a large level of interpolation.

This error estimation can be very useful to select the level w which satisfies a given accuracy. Choosing the smallest possible value is important because the level w controls the size of the grid. By using the indicator λ, the overhead cost for selecting the level w is very small. Indeed, the interpolation $\mathcal{A}_\alpha(w, N)\lambda$ requires only an evaluation of λ at the collocation points of the grid $\mathcal{H}_\alpha(w, N)$, without solving expensive deterministic problems.

4.4. Error estimation in the unbounded case

The global error committed by the interpolation $\mathcal{A}_\mathbf{i}v$ can be split into N parts, $\|v - \mathcal{A}_\mathbf{i}v\|_{L^2_\varrho(\Gamma, H^1_0(D))} \leq \sum_{n=1}^{N} \varepsilon_n$. Each part ε_n is the error contribution in the direction n which is estimated by [3, 20]:

$$
\begin{aligned}
\varepsilon_n &\approx c_n e^{-\nu_n p_n}, && \text{when } \Gamma_n \text{ is bounded }, && (25) \\
\varepsilon_n &\approx O(\sqrt{p_n}) e^{-\nu_n \sqrt{p_n}}, && \text{when } \Gamma_n \text{ is unbounded}. && (26)
\end{aligned}
$$

When Γ is unbounded, we prove another error estimation, based on Hermite interpolation.

We suppose that the density ϱ decreases like a Gaussian kernel, and each density ρ_n of Y_n satisfies the following estimate:

$$
\rho_n(z) \leq C_n e^{-\delta z^2}, \quad C_n > 0 \text{ et } \delta > 0, \quad \forall z \in \mathbb{R}. \tag{27}
$$

We denote by $H_p(t) \in \mathcal{P}_p(\mathbb{R})$ the normalized Hermite polynomials,

$$
H_p(t) = \frac{1}{\sqrt{p!2^p\pi^{\frac{1}{2}}}}(-1)^p e^{t^2}\frac{d^p}{dt^p}\left(e^{-t^2}\right).
$$

We recall that they form a complete orthonormal basis of the space $L^2_\mu(\mathbb{R})$, where $\mu(t) = e^{-t^2}$. The associated Hermite functions satisfy $h_p(t) = \sqrt{\mu(t)}H_p(t)$. The following proposition estimates the error committed by $\mathcal{U}^i\kappa$ for an entire function κ satisfying (27). The proof is based on an estimate of the Fourier-Hermite coefficients (see [6]) for an entire function decreasing as a Gaussian kernel.

Proposition 4.2. *We suppose that ρ satisfies (27). We consider a function κ in $L^2_\rho(\mathbb{R}; L^2(D))$ and assume that, for each $t \in \mathbb{R}$, the function $\kappa(t,.)$ has an entire extension in the complex plane, and there exists $\nu > 0$ such that*

$$\|e^{-\nu|z|}\kappa(z)\|_{L^2(D)} \le C_\nu < \infty, \quad \forall z \in \mathbb{C}. \tag{28}$$

Then, there is a decay $g > 0$ and a constant K which does not depend on $m(i)$ such that

$$\|\kappa(t,.) - \mathcal{U}^i(\kappa)(t,.)\|_{L^2_\rho(\mathbb{R}; L^2(D))} \le K e^{-gm(i)}, \tag{29}$$

where $\mathcal{U}^i(\kappa)$ is the Lagrange interpolation defined by $m(i)$ Gauss-Hermite points.

Proof. We consider the function $\tilde{\kappa}(t,x) = \kappa(w(t), x)$ where $w(t) = \dfrac{\sqrt{2}t}{\delta}$ and the expansion of $\tilde{\kappa}$ in Hermite polynomials basis:

$$\tilde{\kappa}(t,x) = \sum_{p=0}^{\infty} \kappa_p(x) H_p(t) \quad \text{where} \quad \kappa_p(x) = \int_{\mathbb{R}} \tilde{\kappa}(t,x) H_p(t) e^{-t^2} dt. \tag{30}$$

We set $F(t,x) := \tilde{\kappa}(t,x) e^{-\frac{t^2}{2}}$, and its Fourier-Hermite expansion is given by:

$$F(t,x) = \sum_{p=0}^{\infty} F_p(x) h_p(t) \quad \text{where} \quad F_p(x) = \int_{\mathbb{R}} F(t,x) h_p(t) dt. \tag{31}$$

The Hermite coefficients F_p in (31) are the same as in (30). Indeed,

$$F_p(x) = \int_{\mathbb{R}} F(t,x) h_p(t) dt = \int_{\mathbb{R}} \tilde{\kappa}(t,x) e^{-t^2} H_p(t) dt = \kappa_p.$$

The function κ satisfies (28), then F decreases as a Gaussian kernel, since

$$\|\tilde{\kappa}(t,x)\| e^{-t^2/2} \le C_\nu e^{\sqrt{2}\frac{\nu}{\delta}t} e^{-t^2/2} \le C_\nu e^{-\frac{1}{2}(t-\sqrt{2}\frac{\nu}{\delta})^2} e^{\frac{\nu^2}{\delta^2}} \le d e^{-qt^2},$$

where d and q are positive, finite, and depending on ν and δ. Clearly F is an entire function and decreases as a Gaussian kernel; therefore, (see [6]) the Fourier-Hermite coefficients F_p decreases geometrically as:

$$F_p \le K e^{-gp} \quad \forall p \in \mathbb{N}.$$

Where the constant $K > 0$ does not depend on p and the decay $g > 0$. The following error can be bounded (see [3], Lemma 7) as:

$$\|\kappa(t,.) - \mathcal{U}^i(\kappa)(t,.)\|_{L^2_\rho(\mathbb{R}; L^2(D))} \le \min_{v \in \mathcal{P}_{m(i)} \otimes L^2(D)} \max_{t \in \mathbb{R}} \left| \|\kappa - v\|_{L^2(D)} e^{-\frac{(t\delta)^2}{4}} \right|. \tag{32}$$

Taking $v = \Pi_i \kappa$ in (32) the truncated Hermite expansion of κ up the order $m(i) - 1$, and let $\tilde{v} = \Pi_i \tilde{\kappa}$ that of $\tilde{\kappa}$. We deduce:

$$
\begin{aligned}
\|\kappa(t,.) - \mathcal{U}^i(\kappa)(t,.)\|_{L_\rho^2(\mathbb{R};L^2(D))} &\leq \max_{z \in \mathbb{R}} \left| \|\kappa - \Pi_i(\kappa)\|_{L^2(D)} e^{-\frac{(z\delta)^2}{4}} \right| \\
&\leq \max_{t \in \mathbb{R}} \left| \|\tilde{\kappa} - \Pi_i(\tilde{\kappa})\|_{L^2(D)} e^{-\frac{t^2}{2}} \right| \\
&\leq \sum_{p=m(i)}^\infty \|\kappa_p\|_{L^2(D)} \max_{t \in \mathbb{R}} \left| H_p(t) e^{-\frac{t^2}{2}} \right| \\
&\leq C_1 \sum_{p=m(i)}^\infty e^{-gp} \max_{t \in \mathbb{R}} |h_p(t)|.
\end{aligned}
$$

Knowing that $\max_{t \in \mathbb{R}} |h_p(t)| \leq 1$ (see e.g. [6]), we get:

$$
\|\kappa(t,.) - \mathcal{U}^i(\kappa)(t,.)\|_{L_\rho^2(\mathbb{R};L^2(D))} \leq C_1 \sum_{p=m(i)}^\infty e^{-gp} \leq K e^{-gm(i)}.
$$

$\square$

Thus, when the assumptions of Proposition 4.2 are satisfied, we can replace the error estimation (26) by the following estimation:

$$
\varepsilon_n \approx c_n e^{-g_n p_n}, \tag{33}
$$

where c_n is independant of p_n.

4.5. Determination of the anisotropic weights

A suitable choice of the weights can reduce the size of an anisotropic sparse grid, thus the cost of stochastic collocation. The main idea proposed in [21] consists of choosing each anisotropic weight α_n as the exponential rate of the convergence according to the direction n, i.e $\alpha_n = \nu_n$, where ν_n is the rate given in Lemma 4.1. We follow the same algorithm as in [21]: we compute the errors ε_n in each dimension n for various values of p_n and apply a log-linear regression in order to get the constant c_n and the decay rate ν_n.

In the error estimation (25), the constant c_n is independant of p_n for Gauss nodes. Also, our new bound (33) is valid for Gauss-Hermite nodes. Therefore

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we use Gauss nodes to obtain suitable weights. Moreover, in order to reduce the cost of interpolation, we choose the growth rule $m(i_n) = i_n$ in each dimension.

For each direction n, we fix a reference point $y_n^* = (y_j)$ where y_n is variable and $y_j = 0$ for $j \neq n$. For the random field discussed in Section 2.1, this reference point is adequate, since there is no interaction between directions. To compute the errors, we must also approximate a function v by a high level interpolation. We fix a reference level i_n^* whith degree $p_n^* = i_n^* - 1$.

Then, for each value of p_n and a given function v, we compute the error by

$$\varepsilon_n(p_n) = \|\mathcal{U}^{i_n^*} v(y_n^*) - \mathcal{U}^{i_n} v(y_n^*)\| \tag{34}$$

In [21], the function v is the solution $u_h(y_n^*)$ of the deterministic problem (7). We propose to use instead the indicator λ, at a much lower cost since we do not solve deterministic problems.

For a bounded problem, using (25), we solve the linear least-squares problem

$$\min_{\alpha_n, c_n} \sum_{p_n=1}^{p_n^*} (\log(\varepsilon_n(p_n)) - log(c_n) + p_n \alpha_n)^2, \tag{35}$$

where $\varepsilon_n(p_n)$ is computed by (34).

For an unbounded problem, we solve also the linear least-squares problem (35) if the assumptions of Proposition 4.2 are satisfied. Otherwise, based on (26), we solve the linear least-squares problem

$$\min_{\alpha_n, c_n} \sum_{p_n=1}^{p_n^*} (\log(\varepsilon_n(p_n)) - log(c_n) + \sqrt{p_n} \alpha_n)^2. \tag{36}$$

Once we selected the weights, we can apply the stochastic collocation method to compute $\mathcal{A}_\alpha(w, N) u_h$. We can choose a new growth order, in general $m(i_n) = 2^{i_n - 1}$ or $m(i_n) = 2i_n - 1$, and new collocation points, in particular nested points. The computations done to estimate the weights are then useless, but the overhead is very small since it involves only interpolations of λ.

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5. Numerical Examples

This section illustrates the efficiency of our adaptive approach for the stochastic linear elliptic problem, as described in Section 4. We consider the problem

$$\begin{cases} -\text{div}(k\nabla u) = f & \text{in } \Omega \times D, \\ u\mid_{\partial D} = 0, \end{cases} \tag{37}$$

with $D :=]0; 1[^2$.

We developed a Matlab code to implement the stochastic collocation method with a sparse anisotropic grid. The finite element space of the spatial discretization is the span of piecewise linear functions over a uniform triangulation of D with 2712 unknowns.

First, we determine the weight vector with the algorithm defined in subsection 4.5. The collocation points are the Gauss-Hermite points in this preliminary step. For a bounded set Γ, we compute the errors ε_n with (25). For an unbounded set Γ, we use (33). We denote by α and $\tilde{\alpha}$ the weight vectors obtained with respectively our indicator λ and the approximate solution u_h.

Then we choose new collocation points and use this vector α to compute $\mathbb{E}[\mathcal{A}_\alpha(w, N)u_h]$ and $\text{var}[\mathcal{A}_\alpha(w, N)u_h]$ for various values of the level w. We compare the error in the solution with our indicator based on λ and the constant C.

We run computations for two test cases, with a random input data k which is first nonlinear, then linear with respect to y.

5.1. Example 1: nonlinear case

In the first example, we consider k_N which is nonlinear with respect to the stochastic variables and with one-dimensional spatial dependence as defined in [20, 21]:

$$k_N(\omega, x) = 0.5 + \exp\left(Y_1(\omega)\left(\frac{\sqrt{\pi l_c}}{2}\right)^{1/2} + \sum_{n=2}^{N} \zeta_n \varphi_n(x) Y_n(\omega) \right), \tag{38}$$

where l_c is the correlation length and $\{Y_n\}_{n=1}^{N}$ are independent Gaussian random variables with mean zero and unit variance.

The reals $(\zeta_n)_{n=1}^N$ are strictly positive and given by:

$$\zeta_n := \left(\sqrt{\pi}l_c\right)^{1/2} \exp\left(\frac{-(\lfloor \frac{n}{2}\rfloor \pi l_c)^2}{8}\right), \quad \text{if } n > 1.$$

The functions $\{\varphi_n\}_{n=1}^N$ are defined by:

$$\varphi_n(x) := \begin{cases} \sin\left(\frac{\lfloor \frac{n}{2}\rfloor \pi x}{2}\right), & \text{if } n \text{ even}, \\ \cos\left(\frac{\lfloor \frac{n}{2}\rfloor \pi x}{2}\right), & \text{if } n \text{ odd}. \end{cases}$$

The load function is defined by $f(\omega, x) = \cos(x_1)\sin(x_2)$, $x = (x_1, x_2) \in D$.

The parameter k_N is a truncation of a log-normal field $k = e^G$, where G is a Gaussian field with a Gaussian covariance function [20, 21]. A large value of l_c corresponds to a small value of N, with a strong anisotropy, and vice-versa [2, 20, 21]. We consider two test cases: $l_c = \frac{1}{2}$ with $N = 11$ and $l_c = \frac{1}{5}$ with $N = 41$.

Here, $\Gamma = \mathbb{R}^d$ is unbounded thus, for sake of comparison, we also use the estimation (26) to estimate weights, denoted by $\hat{\alpha}$.

We use the roots of the Hermite polynomials as collocation points, both for estimating the weights and for the anisotropic sparse grid.

l_c	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}
1/2	2.57	2.26	2.27	4.31	4.31	8.18	8.18	12.50	12.51	20.58
1/5	3.13	2.55	2.55	2.88	2.88	3.4	3.4	4.34	4.35	5.92
l_c	$\tilde{\alpha}_1$	$\tilde{\alpha}_2$	$\tilde{\alpha}_3$	$\tilde{\alpha}_4$	$\tilde{\alpha}_5$	$\tilde{\alpha}_6$	$\tilde{\alpha}_7$	$\tilde{\alpha}_8$	$\tilde{\alpha}_9$	$\tilde{\alpha}_{10}$
1/2	2.57	2.36	2.09	3.58	3.51	8.31	7.04	11.73	10.8	21.48
1/5	3.13	2.88	2.50	2.33	2.16	2.56	2.62	3.55	3.27	4.22
l_c	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_4$	$\hat{\alpha}_5$	$\hat{\alpha}_6$	$\hat{\alpha}_7$	$\hat{\alpha}_8$	$\hat{\alpha}_9$	$\hat{\alpha}_{10}$
1/2	9.47	7.91	7.92	15.14	15.14	21.87	21.87	30.16	30.17	42.96
1/5	10.98	9.25	9.27	10.24	10.24	12.03	12.04	15.34	15.34	16.17

Table 1: Example 1: the first ten values of the weights obtained by three adaptive approaches.

Table 1 shows the first ten values of vectors α, $\tilde{\alpha}$ and $\hat{\alpha}$. We remark that the vector α for the indicator λ is close to the vector $\tilde{\alpha}$ for u_h. However, α is

rather different from $\hat{\alpha}$, because they are obtained with different one-dimensional

305 bounds.

In Figure 1, we plot the size of $\mathcal{H}_\alpha(w, N)$ and $\mathcal{H}_{\hat{\alpha}}(w, N)$ for $1 \le w \le 7$. We observe that $\#\mathcal{H}_\alpha(w, N) < \#\mathcal{H}_{\hat{\alpha}}(w, N)$, thus the weights obtained with (33) are more efficient than those obtained with (26).
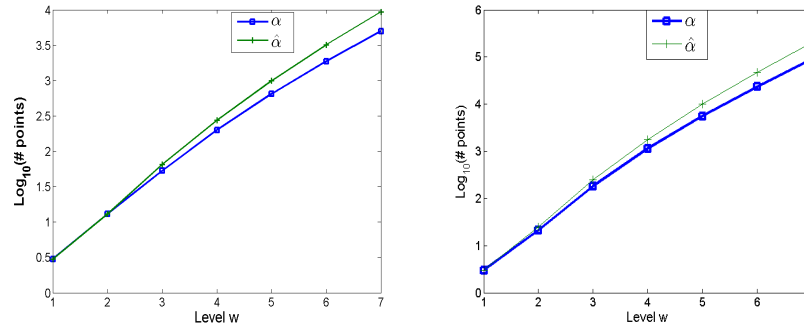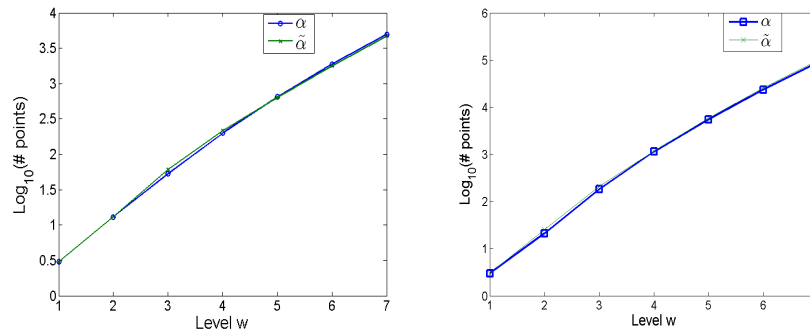

Figure 1: Example 1: $\#\mathcal{H}_{\hat{\alpha}}(w, N)$ versus $\#\mathcal{H}_\alpha(w, N)$, (left $l_c = 1/2$, right $l_c = 1/5$).

In Figure 2, we compare the grid size $\#\mathcal{H}_\alpha(w, N)$ versus $\#\mathcal{H}_{\tilde{\alpha}}(w, N)$ and observe that they are very close. It shows that our indicator λ is reliable.

Figure 2: Example 1: $\#\mathcal{H}_\alpha(w, N)$ versus $\#\mathcal{H}_{\tilde{\alpha}}(w, N)$, (left $l_c = 1/2$, right $l_c = 1/5$).

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In Figure 3, we plot, for various values of the level w, the error $\|u_h - \mathcal{A}_\alpha(w, N)u_h\|$ and our estimation $C\|\lambda - \mathcal{A}_\alpha(w, N)\lambda\|$, where C is estimated by (24) with $\mu = 0$. We have taken $w = 9$ when $l_c = \frac{1}{2}$ and $w = 7$ when $l_c = \frac{1}{5}$

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for the reference solutions u_{ref} and λ_{ref}. We observe that both errors have the

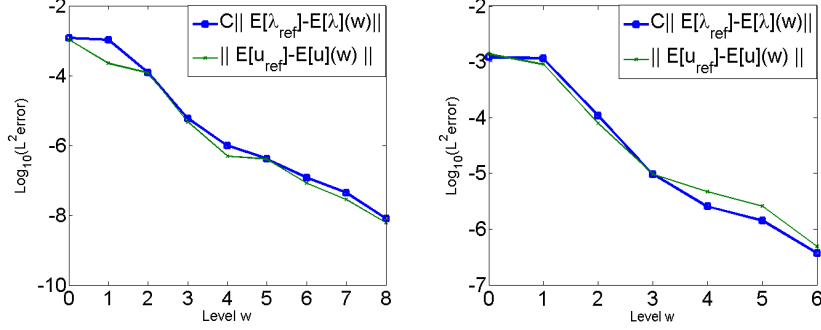315 same behavior, showing that our estimator is quite reliable.

Figure 3: Example 1: Error $C\|\mathbb{E}[\lambda_{ref} - \mathcal{A}_\alpha(w, N)\lambda]\|$ versus error in $\mathbb{E}[u_h]$ (left $l_c = 1/2$, right $l_c = 1/5$).

5.2. Example 2: linear case

In the second example, we consider a diffusion coefficient k_N which is linear with respect to the stochastic variable $y \in \Gamma$ where Γ is a bounded set in $\mathbb{R}^N$:

$$k_N(\omega, x) = \frac{\pi^2}{6} + \sum_{n=1}^{N} \frac{1}{1 + n^2} \cos(n\pi x_1) \sin(n\pi x_2) Y_n(\omega), \ x \in [0, 1]^2.$$

The variables $Y_1, \ldots, Y_N$ are independent and uniformly distributed in the box $(-1, 1)$. We consider two cases: $N = 51$ and $N = 351$, to see the effect of the stochastic dimension.

320 The load function is given by $f(\omega, x) = 1 + \sin(x_1)x_2 Y_1(\omega)$.

We use the roots of the Legendre polynomials to compute the vector of weights, whereas we use the Clenshaw-Curtis collocation points to exploit their nested property in the stochastic collocation method. Table 2 gives the first ten weights for $\lambda = k^{-1}f$ and u_h. We can see that the components of α are

325 slightly smaller than the components of $\tilde{\alpha}$. This is consistent with the theory, since when k is linear with respect to y, the analyticity region of u is larger than that of λ (proof of Proposition 4.1).

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	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}
α for λ	3.89	5.77	7.19	8.26	9.11	9.87	10.48	11.01	11.48	11.9
$\tilde{\alpha}$ for u	4.24	6.44	8.00	9.32	10.19	10.92	11.53	12.06	12.53	12.96

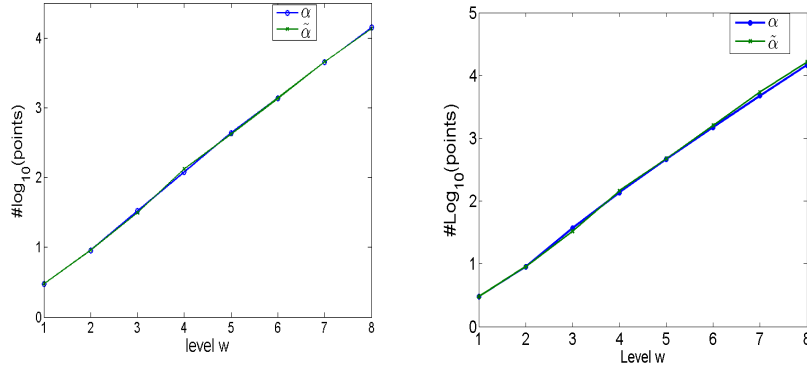Table 2: Example 2: the first ten values of the weights by two adaptive approaches.

Figure 4: Example 2: grid size $\#\mathcal{H}_\alpha(w, N)$ versus $\#\mathcal{H}_{\tilde{\alpha}}(w, N)$. Left: $N = 51$, Right: $N = 351$.

Figure 4 (right) shows that $\#\mathcal{H}_\alpha(w, N)$ is slightly smaller than $\#\mathcal{H}_{\tilde{\alpha}}(w, N)$. This could be explained by the fact that numerically we have $\dfrac{\alpha}{\min(\alpha)} \geq \dfrac{\tilde{\alpha}}{\min(\tilde{\alpha})}$.

We remark that $\#\mathcal{H}_\alpha(w, 51) \approx \#\mathcal{H}_\alpha(w, 351)$. This can be explained by the fact that the dimensions $n > 51$ are not effective in the construction of u and the indices corresponding to those dimensions are almost equal to one.

Now, we compare our error estimator with the error computed with u_h. The constant C is computed with (24) with the exponent $\mu = \min_n \nu_n(u) - \min_n \nu_n(\lambda)$. Since the weights are in fact the decay rates ν_n, we get directly an estimation of $\min_n \nu_n(\lambda)$. We assume that the minimum for u is achieved for the same value $\tilde{n}$ of n and estimate the associated decay rate for u_h. Then $\mu = \nu_{\tilde{n}}(u_h) - \nu_{\tilde{n}}(\lambda)$.

Figure 5 plots the error on $\mathbb{E}[u_h]$ and the estimator $C\|\mathbb{E}[\lambda_{ref} - \mathcal{A}_\alpha(w, N)\lambda]\|$, where the reference solutions are computed by taking a high level $w = 9$. Again, we observe that our error estimator is very reliable since the behaviors are the
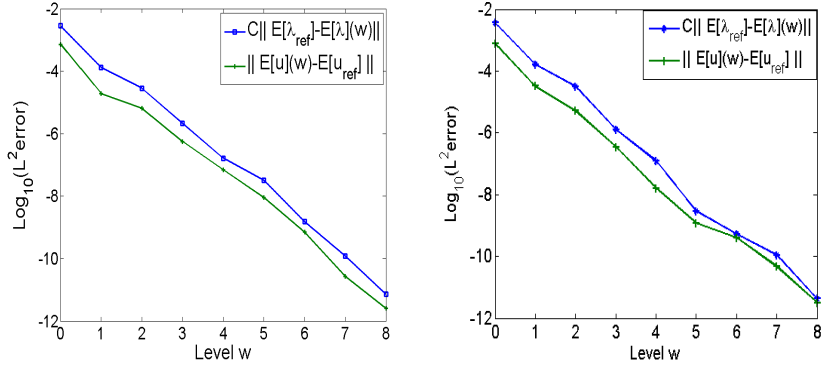
Figure 5: Example 2: Error $C\|\mathbb{E}[\lambda_{ref} - \mathcal{A}_\alpha(w, N)\lambda]\|_2$ versus error in $\mathbb{E}[u_h]$. Left: $N = 51$, Right: $N = 351$

same when we vary the level w.

6. Conclusion

In this work we propose a new strategy for an adaptive collocation method using an anisotropic sparse grid to approximate an elliptic problem with random input data. We have introduced an indicator based on the inverse of the diffusion coefficient, and we have shown that this indicator gives an estimation of the error in the solution.

This error estimation can be used to select the smallest possible level w of interpolation, in order to reduce the grid size, thus the computational cost of the stochastic collocation method. Moreover, this adaptive strategy requires only solving interpolation problems, with a very small cost overhead.

We also use our indicator to estimate suitable weights of anisotropy, again with a very small overhead.

We illustrate the efficiency and reliability of our indicator and adaptive strategy with two test cases, with bounded and unbounded random input data. The numerical results show that we obtain weights close to those computed by using the approximate solution, at a much lower cost. Our error estimation has the

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same behavior as the error computed with the approximate solution, thus it is very reliable and can be used to select an efficient level of interpolation.

A prospective next step is to study the impact of the spatial discretization on this error estimation.

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