Generation of a stationary Gaussian random field

Jocelyne Erhel
SAGE team, Inria, Rennes, France

co-authors
Mestapha Oumouni (SAGE team, Inria, Rennes)
Géraldine Pichot (SAGE team, Inria, Rennes)
Anthony Beaudoin (UMR Pprime, university of Poitiers)
Jean-Raynald de Dreuzy (UMR Geosciences, CNRS, Rennes)

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1 Introduction

Gaussian field

Random field $K(x)$
Domain Ω of $\mathbf{R}^d$, $x \in \Omega$

$$K(x) = \exp(\mu + \sigma Y(x))$$

where Y is a zero mean normal correlated field,
μ is the mean of K, σ is the variance of $K(x)$
Stationary field
$cov[Y](x, y) = cov[Y](x - y, 0)$
Covariance function:

$$cov[Y](x, y) = C(|x - y|)$$

Remark that $C(0) = 1$

Sampling a Gaussian field: 1d case

Covariance matrix

Domain $\Omega = [0, 1]$

Discrete grid of $N + 1$ equally spaced points $x_0 = 0, x_1, \ldots, x_N = 1$

Sampled covariance function $c_i = C(x_i)$ with $i = 0, 1, \ldots, N$

Covariance matrix

$$R_{ij} = C(|x_i - x_j|) = cov[Y](x_i, x_j)$$

Discrete sampled field

$$Y_i = Y(x_i)$$

random vector of normal variables with zero mean and covariance matrix R

$$K_i = \exp(\mu + \sigma Y_i)$$

Covariance matrix

Covariance matrix R

$$R_{ij} = C(|x_i - x_j|) = cov[Y](x_i, x_j)$$

R is a symmetric positive definite Toeplitz matrix of size $N+1$

$$R = \begin{pmatrix} c_0 & c_1 & . & . & . & c_{N-1} & c_N \\ c_1 & c_0 & c_1 & . & . & . & c_{N-1} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ c_{N-1} & . & . & . & c_1 & c_0 & c_1 \\ c_N & c_{N-1} & . & . & . & c_1 & c_0 \end{pmatrix}$$

Generating a sampled Gaussian field

Factorization

$$R = BB^T$$

Vector θ of random normal variables with zero mean and uncorrelated

$$Y = B\theta$$

is a random vector of normal variables with zero mean and covariance matrix R

Generation of $N + 1$ random variables

Factorizations of a covariance matrix

- Cholesky factorization $R = LL^T$
- Eigenvalue decomposition $R = V\Delta V^T = (V\sqrt{\Delta})(V\sqrt{\Delta})^T$
- Spectral decomposition

Cholesky and eigenvalue decompositions are quite expensive for large N
The matrix R can be ill-conditioned, implying numerical difficulties

Circulant embedding of a covariance matrix: 1d case

Circulant embedding of R

$a = (c_0, \dots, c_N, c_{N-1}, \dots, c_1) \in \mathbf{R}^{2N}$

Circulant matrix $A = \mathrm{circ}(a)$

The matrix A is real symmetric of size $2N$ and is spd if N is large enough

Example with $N = 4$

$$A = \begin{pmatrix} c_0 & c_1 & c_2 & c_3 & c_4 & c_3 & c_2 & c_1 \\ c_1 & c_0 & c_1 & c_2 & c_3 & c_4 & c_3 & c_2 \\ c_2 & c_1 & c_0 & c_1 & c_2 & c_3 & c_4 & c_3 \\ c_3 & c_2 & c_1 & c_0 & c_1 & c_2 & c_3 & c_4 \\ c_4 & c_3 & c_2 & c_1 & c_0 & c_1 & c_2 & c_3 \\ c_3 & c_4 & c_3 & c_2 & c_1 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_4 & c_3 & c_2 & c_1 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_4 & c_3 & c_2 & c_1 & c_0 \end{pmatrix}$$

Spectral decomposition of the circulant embedding

Eigenvalue decomposition of A

Discrete Fourier Transform: $F = (e^{-2i\pi kj/2N})_{0 \leq j,k \leq 2N-1}$

Inverse Discrete Fourier Transform: $F^{-1} = \frac{1}{2N} F^* = \frac{1}{2N} \overline{F}$

$$A = FDF^{-1} = \frac{1}{2N} FDF^*$$

where $D = diag(s)$ with $s = \overline{F}a = Fa$ (s is real)
$s_{2N-k} = s_k, \; k = 1, \ldots, N$

Generating two random fields with $4N$ random variables

Factorization of A

If A is spd then $A = (F\sqrt{D/2N})(F\sqrt{D/2N})^*$

$$A = (1/\sqrt{2N}F\mathbf{diag}(\sqrt{s}))(1/\sqrt{2N}F\mathbf{diag}(\sqrt{s}))^*$$

If A is not spd, a can be completed by a padding or N can be increased

Random complex vector of size $2N$

$$\theta = Re(\theta) + iIm(\theta)$$

$Re(\theta)$ and $Im(\theta)$ random normal zero mean uncorrelated real variables

$$Y_1 = \frac{1}{\sqrt{2N}}Re(F\mathbf{diag}(\sqrt{s})\theta)(0:N) \text{ and } Y_2 = 1/\sqrt{2N}Im(F\mathbf{diag}(\sqrt{s})\theta)(0:N)$$

Two vectors of random normal zero mean $N+1$ variables with correlation R

Generating one random field with $2(N+1)$ random variables

Random complex vector $u + iv$ **of size** $N+1$
u and v random Gaussian zero mean uncorrelated real variables with
$E[u_k^2] = E[v_k^2] = 1/2, k = 1, \ldots, N-1$
$v_0 = v_N = 0$
$E[u_0^2] = E[v_N^2] = 1$

Random complex vector of size $2N$

$$\theta = Re(\theta) + iIm(\theta)$$

with
$Re(\theta) = (u_0, \ldots, u_N, u_{N-1}, \ldots, u_1)$ and $Im(\theta) = (v_0, \ldots, v_N, -v_{N-1}, \ldots, -v_1)$

$$Y = \frac{1}{\sqrt{2N}}(Fdiag(\sqrt{s})\theta)(0:N)$$

Vector of real normal zero mean $N+1$ variables with correlation R

Using $(N+1)$ random variables with uniform law

Approximation of a Gaussian law by a uniform law
Real random vector ϕ of size $N+1$
with $\phi_k \in [0, 2\pi[$ random variable of uniform law
Random complex vector of size $2N$
$\theta_k = e^{i\phi_k}, k = 1, \ldots, N-1$ and $\theta_0 = \sqrt{2}\cos(\phi_0), \theta_N = \sqrt{2}\cos(\phi_N)$
and $\theta = (\theta_0, \ldots, \theta_N, \overline{\theta_{N-1}}, \ldots, \overline{\theta_1})$

$$Y = \frac{1}{\sqrt{2N}}(Fdiag(\sqrt{s})\theta)(0:N)$$

Vector of real zero mean $N+1$ variables with approximate Gaussian law and correlation R

Algorithm with uniform laws

- Sample the covariance function: vector c and circulant vector a
- Apply DFT to compute $s = Fa$
- Generate random phase vector ϕ and compute complex vector θ
- Apply DFT to compute $F(diag(\sqrt{s})\theta)$
- Compute random discrete field Y

Implementation using FFT and random number generator

Generation of a random field in 2d and 3d

Discrete regular grid in the domain Ω with $N+1$ points in each direction
The covariance matrix R is block-Toeplitz
The embedding uses symmetry in each direction to get a block-circulant matrix A
The random vectors also use symmetry in each direction

Parallel algorithm

Domain decomposition of the regular grid with one process per subdomain
Splitting of the circulant vector a in FFT with one process per block
Communications between processes to redistribute data

First numerical example: exponential covariance

Domain $\Omega = [0\ L_x] \times [0\ L_y]$
Covariance function

$$C(x) = \exp(-|x_1|/l_1 - |x_2|/l_2)$$

Ergodic field

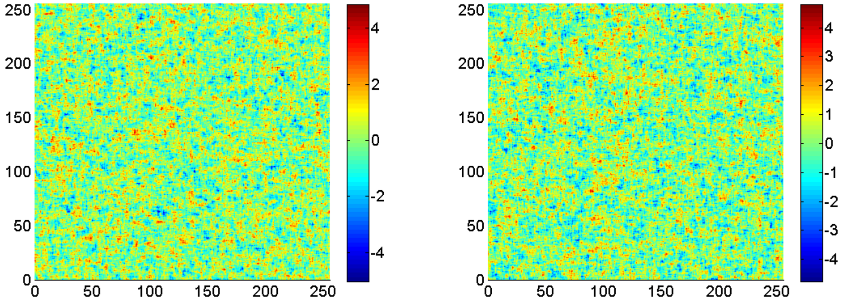
$$\lim_{|D|\to\infty} \frac{1}{|D|} \int_D Y(x)dx = E[Y] = 0$$

$$\lim_{|D|\to\infty} \frac{1}{|D|} \int_D Y(x)^2 dx = C(0) = 1$$

Numerical results

Domain size: $Lx = Ly = 2^8 = 256$ and correlation lengths: $l_1 = l_2 = 2$

Two discrete random fields

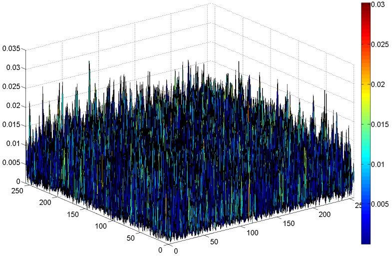


Numerical results

Monte-Carlo convergence: M **realizations of** $Y^{(m)}$

$$\tilde{C}_i = \frac{1}{M}\sum_{m=1}^{M} Y_i^{(m)} Y_0^{(m)}$$

Error $|C_i - \tilde{C}_i|$ with $M = 10^4$



Numerical results

Approximate mean and variance using ergodicity

$$\mu = \frac{1}{N+1} \sum_{i=0}^{N} Y_i$$

$$\sigma^2 = \frac{1}{N} \sum_{i=0}^{N} Y_i^2$$

N	$2^5 \times 2^5$	$2^8 \times 2^8$	$2^{10} \times 2^{10}$
μ	0.0281	-0.0102	0.004
σ^2	0.9579	0.9983	0.9988

Second numerical example: non separable exponential covariance

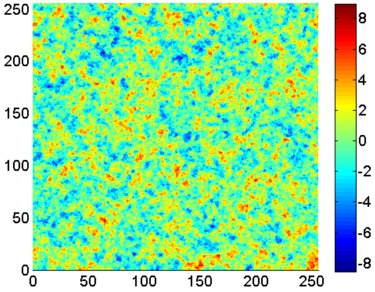
Domain $\Omega = [0\ L_x] \times [0\ L_y]$
Covariance function

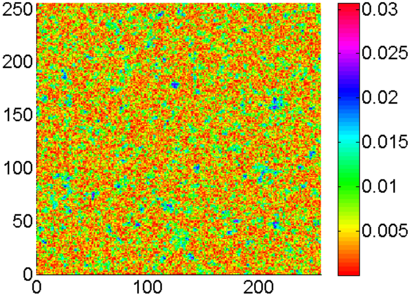
$$C(x) = \exp(-|x_1^2|/l_1^2 - |x_2|^2/l_2^2)$$

The circulant matrix A is spd only for large domains and large N

Numerical results

Domain size: $Lx = Ly = 2^8 = 256$ and correlation lengths: $l_1 = l_2 = 3$
One discrete random field σY with $\sigma = 2$
Monte-Carlo error on the covariance function with $M = 10^4$





Third numerical example: Gaussian covariance

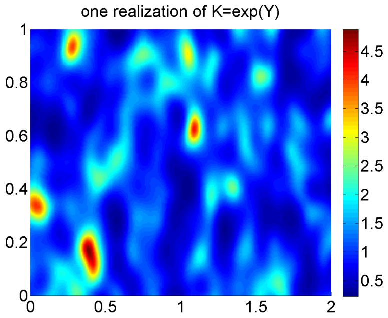
Domain $\Omega = [0\ L_x] \times [0\ L_y]$
Covariance function

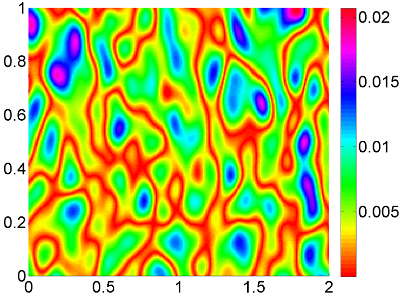
$$C(x) = \exp(-|x_1^2|/l_1^2 - |x_2|^2/l_2^2)$$

The circulant matrix A is spd only for large domains and large N

Numerical results

Domain size: $Lx = 2$ and $Ly = 1$ and correlation lengths: $l_1 = l_2 = 1/10$
One discrete random field $K = exp(\sigma Y)$ with $\sigma = 1$
Monte-Carlo error on the covariance function with $M = 10^4$





Conclusion

Summary

- Several variants for generating a correlated Gaussian field
- Parallel software GENFIELD (soon available)
- Use of random fields in hydrogeology

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Future work

- Comparison of K-L expansion and spectral decomposition
- Generation of K when the discrete density is given
- Use of random fields with multigrid methods