Simulation of Shallow-Water Flows on General Terrain

Elena Bachini¹, Ilaria Fent², Mario Putti¹

¹ Department of Mathematics “Tullio Levi-Civita”,
University of Padua, Italy
² Institute of Mechanics, Materials and Civil Engineering,
Université Catholique de Louvain, Belgium

Key words: shallow-water, arbitrary topography, Godunov-type scheme

Shallow Water Equations are classically used as models of environmental fluid dynamics when the horizontal (longitudinal and lateral) components of the flow field are predominant with respect to the vertical components. This is the so called Shallow Water (SW) hypothesis. Application of SW equations ranges from large-scale models in meteorology, oceanography [1], or tsunami modeling [2], but also to smaller scale models of river morphology, avalanches [3], debris flows, landslides and other phenomena. These applications must consider a general topography, commanded for example by mountain landscapes, introducing mathematical difficulties that have not yet a comprehensive solution. This is in contrast with the increasing necessity of reliable models, both in environmental and industrial applications.

The presence of a general terrain plays an important role, increasing the geometrical complexity of the fluid streamlines. It is then difficult to accurately identify the negligible velocity component under the SW hypothesis, as the average flow field drastically departs from a rectilinear behavior. To address this problem, Savage and Hutter [4, 5] developed a formulation of the SW model in local curvilinear coordinates based on depth integration along the normal to the topography. Their approach is valid only for small and essentially one-dimensional bottom curvatures and in practice assumes that the fluid surface is parallel to the bottom. This strategy was extended by Bouchut and Westdickenberg [6] to consider less restrictive bottom geometries. Starting from the NS equations, written in curvilinear coordinates, the flow velocity component perpendicular to the bottom is considered negligible and a hydrostatic pressure distribution is assumed along local normals. The resulting SW equations are derived by depth integration along the normal direction, under the further hypotheses of a linear velocity distribution (equivalent to assume constant depth averaged velocity) and of a fluid depth sufficiently small to guarantee the invertibility of the coordinate transformation. We propose a more accurate approach [7], performing depth integration following the so called “cross-flow” path, i.e., a path along which the tangential component of the fluid velocity is zero. Unfortunately this definition is implicit and thus impractical, as it requires the knowledge of the velocity field, an unknown of the problem. To solve this difficulty, Bresh and Noble [8], and Noble and Vila [9], propose a discretization of the cross-flow path by means of a discrete Fourier transformation. However, the resulting model approaches the complexity of a fully 3D simulation.

The objective here is to maintain the two-dimensionality of the SW models, thus preserving the computational efficiency of a reduced dimensional model, and at the same time to implement the idea of the cross-flow paths. We derive a covariant formulation of the SW equations intrinsically defined on a local reference frame that is anchored on the bottom surface. The derivation is obtained by integrating the Navier-Stokes (NS) equations along a direction locally normal to the bed topography. This direction is used as an approximation of the “cross-flow” integration path, which are defined by the condition of being at each point orthogonal to the NS velocities. Along these paths it is possible to precisely state the hydrostatic pressure condition, thus enabling the actual reduction of the three-dimensional NS equations to the two-dimensional SW model once the SW hypothesis of flow dynamics occurring prevalently along the bottom surface is invoked.

The resulting system of SW equations turns out to be closely related to the model developed by [6], and shares similar approximations and limitations in terms of geometry of the bed topography. We study the formal order of approximation of the model, proving a second order of approximation with respect to an appropriate “geometric” aspect ratio parameter that includes information on local curvatures. Moreover, we prove that the system admits a conservative energy equation and that it preserves the steady state of a lake at rest.

The previous considerations lead to a final hyperbolic system of SW equations on curvilinear coordinates, characterized by the emergence of geometric source terms and flux functions that explicitly vary in space. These features complicate the development of an efficient and accurate numerical scheme. This same problem arises in the simulations of shallow water flows on the surface of a sphere, but much more challenging is when the problem is defined on a generally curved surface. Rossmanith et al. [10] are among the first to study a numerical method based on a finite volume scheme defined on a quadrilateral grid for the solution of hyperbolic systems.
on a general manifold. However, the discretization of geometrical quantities based on the surface fundamental forms by quadrilateral meshes is not consistent [11]. For this reason, in this work we develop a triangular first order Godunov type finite volume scheme. To this aim, we first verify the rotation invariance of our equations and define a proper geometrically adapted one dimensional Riemann problem on the curvilinear triangle edges. The exact Riemann solver, proposed by Toro [12] for dam-break problems in classical SW system, is extended to take into consideration the geometry of the bottom surface intrinsic to our model. The same study is applied to adapt the HLL scheme in order to address more general wave patterns.

A number of test cases performed over irregular bottom topography are used to show the effectiveness of the numerical approach and to verify the importance of considering the geometric features of the bed topography in the equations. The numerical results obtained show that it is important to take into full consideration the geometrical features of the terrain even for relatively mild and slowly varying curvatures.

References