Space-time discretization for non-linear Biot's model

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Introduction

Coupled flow and geomechanics models have several applications including groundwater hydrology, CO_2 sequestration, geothermal energy, and subsidence phenomena. Biot's model is often used for describing this coupled behaviour. The development of robust, efficient and accurate numerical methods to solve Biot's model has recently attracted a lot of attention [1, 2, 3]. In particular, space-time discretization, have been proposed and analysed [4, 5]. So far, the studies of the space-time discretization has been obtained only for the linear problems (both flow and mechanics).

In this work, we discretize the model the non-linear Biot model using continuous Galerkin FE in time. The non-linear variational problem is linearized by L-scheme [6, 7] where the well knows splitting schemes such as fixed stress and fixed strain becomes particular case of our scheme. For the non-linear model, we consider the case when the volumetric strain and the fluid compressibility are non-linear functions with certain conditions. The stability of this approach is illustrated by numerical experiments.

Non linear Biot's model

We use Biots consolidation model considering a non linear elastic, homogeneous, isotropic, porous medium $\Omega \in \mathbb{R}^d$, saturated with a slightly compressible fluid. On the space-time domain $\Omega \times [0,T]$ the governing equations reads as follows:

$$\begin{aligned} -\nabla \cdot [2\mu\varepsilon(\mathbf{u}) + \mathfrak{h}(\nabla \cdot \mathbf{u}) - \alpha(pI)] &= \rho_b \mathbf{g}, && (1)\\ \partial_t \left(\mathfrak{b}(p) + \alpha \nabla \cdot \mathbf{u}\right) + \nabla \cdot \mathbf{q} &= f, \quad \mathbf{q} = -\mathbf{K}\left(\nabla p - \rho_f \mathbf{g}\right). && (2) \end{aligned}$$

The variable $\mathbf{u}$ is the displacement, p is the fluid pressure, q is the Darcy flux. We denote by $\mathfrak{h}(\cdot)$ the non linear volumetric stress in the mechanic equation Eq.1. Further $\varepsilon(\mathbf{u}) = \frac{\nabla \mathbf{u} + (\nabla \mathbf{u})}{2}$ is the linearised strain tensor, μ is the shear modulus, α is the Biot coefficient, ρ_b is the bulk density and g the gravity vector. The non linear compressibility is denote by $\mathfrak{b}(p)$, f is a volume source term , K is the permeability tensor divided by fluid viscosity and ρ_f is the fluid density. For simplicity, we assume homogeneous boundary $u = 0$, $p = 0$ on $\partial\Omega \times [0,T]$ and initial conditions $u = u_0$, $q = q_0$, $p = p_0$ in $\Omega \times 0$.

Linearization with L-Scheme

In order to solve the non linear Biot model Eq. (1)-(2) we introduce the monolithic L-scheme which iterate as follows:

Problem: Given $\mathbf{u}^0 = \mathbf{u}_0$, $\mathbf{q}^0 = \mathbf{q}_0$ and $p^0 = p_0$. For $s > 0$, find (until convergence) $\mathbf{u}^s \in H^1\left(I; \mathbf{H}^1(\Omega)\right) \cap L^2\left(I; \mathbf{H}_0^1(\Omega)\right)$, $\mathbf{q}^s \in L^2\left(I; \mathbf{H}(\mathrm{div}; \Omega)\right)$ and $p^s \in H^1\left(I; L^2(\Omega)\right)$ such that:

$$\begin{aligned} 2\mu \int_I \langle \varepsilon(\mathbf{u}^s) : \varepsilon(\mathbf{v})\rangle d\tau + \int_I \langle L_2(\nabla \cdot \delta \mathbf{u}^s) - \alpha p^s, \nabla \cdot \mathbf{v}\rangle d\tau &= \int_I \langle \rho_b \mathbf{g} - \mathfrak{h}(\nabla \cdot \mathbf{u}^{s-1}), \nabla \cdot \mathbf{v}\rangle d\tau,\\ \int_I \langle \mathbf{K}^{-1}\mathbf{q}^s, \mathbf{z}\rangle d\tau - \int_I \langle p^s, \nabla \cdot \mathbf{z}\rangle d\tau &= \int_I \langle \rho_f \mathbf{g}, \mathbf{z}\rangle d\tau,\\ \int_I \langle \partial_t \left(L_1 \delta p^s + \alpha \nabla \cdot \mathbf{u}^s\right), w\rangle d\tau + \int_I \langle \nabla \cdot \mathbf{q}^s, w\rangle d\tau &= \int_I \langle f - \partial_t \mathfrak{b}(p^{s-1}), w\rangle d\tau. && (3) \end{aligned}$$

For all $\mathbf{v} \in L^2\left(I; H_0^1(\Omega)\right)$, $\mathbf{z} \in L^2\left(I; \mathbf{H}(div; \Omega)\right)$ and $w \in L^2\left(I; L^2(\Omega)\right)$, where $\delta(\cdot)^s = (\cdot)^s - (\cdot)^{s-1}$.

This monolithic L-scheme can be slightly modified by replacing $\nabla \cdot \mathbf{u}^s \approx \nabla \cdot \mathbf{u}^{s-1}$ in Eq. 3. This allows to split the non linear problem in two sub problems as it shows in [7].

	Newton scheme		L-scheme	
#DOF	iterations	Cpu [s]	iterations	Cpu time [s]
256	8	22	13	12
1024	8	92	13	48
4096	8	359	13	205

Table 1: Comparison between Newton scheme and L-scheme,

Space-time discretization

The continuous Galerkin FE method in time for the linear cas of Biot's model have been study and analyse by Bause et al. [4, 5]. The time interval $(0, T]$ is decompose by N subintervals $I_n = (t_{n-1}, t_n]$, where $n \in 1, ..., N$, $0 = t_0 < t_1 < ... < t_{n-1} < t_n = T$ and $\tau = t_n - t_{n-1}$. We introduce the function spaces of piecewise polynomials of order r in time:

$$\mathcal{X}^r(X) = \left\{ \phi_\tau \in C(\bar{I}; X) \;\middle|\; \phi_{\tau|\bar{I}_n} = \phi_n \in \mathbb{P}_r(\bar{I}_n; X); \forall n \in \{1, ..., N\} \right\},$$
$$\mathcal{Y}^r(Y) = \left\{ \psi_\tau \in L^2(I; Y) \;\middle|\; \psi_{\tau|I_n} = \psi_n \in \mathbb{P}_r(I_n; Y); \forall n \in \{1, ..., N\} \right\}.$$

For the space discretization, let $\Omega = \cup_{K \in \mathcal{K}_h} K$ be a regular decomposition of Ω into d-simplices. We denote by h the mesh size. The discrete spaces are given by $\mathbf{V}_h = \{\mathbf{v}_h \in H^1(\Omega)^d \,;\, \mathbf{z}_{h|K} \in \mathbb{P}_1^d \,,\, \forall K \in \mathcal{K}_h\}$, $W_h = \{w_h \in L^2(\Omega) \,;\, w_{h|K} \in \mathbb{P}_0 \,,\, \forall K \in \mathcal{K}_h\}$ and $\mathbf{Z}_h = \{\vec{z}_h \in H(\mathrm{div}; \Omega) \,;\, \vec{z}_{h|K}(\vec{x}) = \vec{a} + b\vec{x},\, \vec{a} \in \mathbb{R}^d,\, b \in \mathbb{R},\, \forall K \in \mathcal{K}_h\}$.

The test functions $\psi_{\tau|I_n} = x_h \psi_n(t)$ for $i = 1, ..., r$, are define at every I_n and vanish at $I \setminus In$. This allows us to define a problem at each time interval I_n as follows:

Problem: For each n=1,...,N, find $\mathbf{u}_{n,h}^{s,j}, \in \mathbf{Z}_h$, $\mathbf{q}_{n,h}^{s,j} \in \mathbf{V}_h$ and $p_{n,h}^{s,j} \in W_h$ for every $j = 1, ..., r,$, such that:

$$\begin{aligned}
2\mu\langle \varepsilon(\mathbf{u}_{n,h}^{s,i}), \varepsilon(z_h)\rangle + \langle L_2(\nabla \cdot \delta \mathbf{u}_{n,h}^{s,i} - \alpha p_{n,h}^{s,i}, \nabla \cdot \mathbf{v}_h\rangle &= \langle \rho_b \mathbf{g} - \mathfrak{h}(\nabla \cdot \mathbf{u}_{n,h}^{s-1}), \mathbf{v}_h\rangle, \\
\langle \mathbf{K}^{-1} \mathbf{q}_{n,h}^{s,i}, z_h\rangle - \langle p_{n,h}^{s,i}, \nabla \cdot z_h\rangle &= \langle \rho_f \mathbf{g}, \mathbf{z}_h\rangle, \\
\sum_{j=0}^{r} \left\{ \alpha_{ij} \langle L_1 \delta p_{n,h}^{s,i} + \alpha \nabla \cdot \mathbf{u}_{n,h}^{s,i}, w_h\rangle \right\} + \tau_n \beta_{ii} \langle \nabla \cdot \mathbf{q}_{n,h}^{s,i}, w_h\rangle &= \tau_n \beta_{ii} \langle f(t_{n,i}), w_h\rangle - \sum_{j=0}^{r} \left\{ \alpha_{ij} \langle \partial_t \mathfrak{b}(p_{n,h}^{s-1}), w_h\rangle \right\}.
\end{aligned}$$

For every $i = 1, .., r$ and for all $\mathbf{v}_h \in \mathbf{V}_h$, $\mathbf{z}_h \in \mathbf{Z}_h$ and $w_h \in Q_h$. Where the coefficients $\alpha_{ij} = \int_{In} \frac{d(\phi_n^j)}{dt} \psi_n^i dt$ and $\beta_{ii} = \int_{In} \phi_n^i \psi_n^i dt$ following the notation of Bause et al. in [4, 5].

Numerical results

The linear convergence of the monolithic L-scheme cause it to have lower convergence rate compared with Newton methods. Nevertheless, previous studies [7] shows how to set L_1 and L_2 in other improve the rate of convergence and the performance of the L-scheme. This improvement in the L-scheme make it possible to compare it with Newton method, as you can see in the table 1, where the L-scheme shows better performance in CPU time.

Discussion

The high order discretization in time shows higher order accuracy compared with backward euler. This allows us to treat problem when, for isntance, the pressure have high fluctuations. The convergence rate of the monolithic and splitting L-scheme is not affected by space-time discretization. Nevertheless, the algebraic system obtained by using high order discretizaton in time became more challenge to solve. Thus, a preconditioner base on the splitting L-scheme would be a promising choice to solve the algebraic system eficiently.

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