A reduced-order model based on stochastic moment equations for the solution of groundwater flow problems

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Keywords: Moment Equations, Groundwater flow, Stochastic hydraulic transmissivity, Model order reduction

Abstract

The computational burden associated with a Monte Carlo-based solution of groundwater flow settings in random transmissivity fields is reduced by using a projection-based surrogate model. The basis functions for the projection are traditionally obtained through the snapshot technique. As an alternative, we propose to use the leading eigenvectors of the covariance computed by solving the equations describing the head moments. This approach enables us to construct a reduced order model circumventing the need to solve the full system model.

Introduction

Let us consider the steady-state groundwater flow problem in a confined aquifer:

$$\begin{cases} \nabla_x\left(T\left(\mathbf{x}\right)\nabla_x h\left(\mathbf{x}\right)\right)+f\left(\mathbf{x}\right)=0 & \mathbf{x}\in\Omega \\ h(\mathbf{x})=H(\mathbf{x}) & \mathbf{x}\in\Gamma_D \\ \left[T\left(\mathbf{x}\right)\nabla_x h\left(\mathbf{x}\right)\right]\cdot\mathbf{n}(\mathbf{x})=Q(\mathbf{x}) & \mathbf{x}\in\Gamma_N \end{cases} \tag{1}$$

where h represents hydraulic head [L]; f is a forcing term [T^{-1}]; T represents transmissivity [L^2T^{-1}]; H is a fixed hydraulic head at Dirichlet boundary Γ_D [L]; and Q represents the flux at the Neumann boundary Γ_N [L^3T^{-1}]. The solution of (1) through a finite element or finite difference method on n nodes can be expressed through a system $\mathbf{A}(Y)\mathbf{h}(Y)=\mathbf{b}$, where matrix $\mathbf{A}(Y)$ is symmetric positive definite and sparse, $h(Y)$ is the vector of the unknown heads, and vector $\mathbf{b}$ does not depend on Y under the assumption of $Q = 0$. We refer to (1) as the Full System Model (FSM).

A frequent assumption in groundwater flow models is to describe uncertainty associated with the spatial distribution of transmissivity by modeling $Y = \ln T$ as a second order stationary random field, typically characterized by a normal distribution. The associated probability distribution function (*pdf*) of hydraulic heads, $p(h)$, can be empirically estimated through a computationally expensive suite of numerical Monte Carlo (MC) simulations, where N FSM solutions, $\mathbf{h}^{(1)},\ldots,\mathbf{h}^{(N)}$, are numerically computed for N samples, $Y^{(1)},\ldots,Y^{(N)}$, of the transmissivity field. As an alternative to the solution of the FSM, Pasetto et al. [1] propose the use of a reduced-order model (ROM). The latter projects the head solution onto the space of nh basis vectors $\mathbf{p}_1,\ldots,\mathbf{p}_{nh}$,

$$\mathbf{h}^{(i)} \approx \langle\mathbf{h}\rangle + \mathbf{P}\boldsymbol{\alpha}^{(i)} \tag{2}$$

where $\langle\mathbf{h}\rangle$ is the (ensemble) mean vector of hydraulic head and $\mathbf{P}$ the matrix collecting the basis vectors of dimension $n\times nh$. The coefficient vector $\boldsymbol{\alpha}^{(i)}$ is obtained by solving the projected equations

$$\mathbf{P}^{\mathrm{T}}\mathbf{A}^{(i)}\mathbf{P}\,\boldsymbol{\alpha}^{(i)}=\mathbf{P}^{\mathrm{T}}\mathbf{b}-\mathbf{P}^{\mathrm{T}}\mathbf{A}^{(i)}\langle\mathbf{h}\rangle \tag{3}$$

Methods

Pasetto et al. [1] compute the basis vectors for the projection through the snapshot technique, i.e., as eigenvectors of the correlation matrix associated with a certain number, $Ns << N$, of FSM solutions. The ROM model obtained by using this snapshot technique is here termed SnapROM. Here, we propose to replace this empirical computation of the mean head and head covariance by the second-order solutions of the stochastic Moment Equations (MEs, [2,3]) associated with the groundwater flow problem described by (1). MEs are deterministic and yield a solution which is more accurate than the one which could be obtained relying solely on a few MC realizations. The reduced model obtained in this way is here termed as MEsROM. To further decrease the computational cost of the ROM models avoiding the projection of the full system matrix for each MC realization, we explore the possibility of approximating the transmissivity field by its truncated Karhunen-Loeve (KL) representation, using a finite number (nT) of terms. This approximation enables us to compute the ROM matrices offline, and significantly reduces the online costs. Otherwise, while the application of KL can remarkably improve the computation efficiency, it might result in accuracy loss. Models derived and solved through the above mentioned KL approximation of the transmissivity field are here indicated as FSM-KL, SnapROM-KL and MEsROM-KL.

Results and Conclusions

All of the models described above are compared for a simple two-dimensional numerical exemplary setting depicting a flow in a rectangular domain across which Y is characterized by an anisotropic exponential covariance function with zero mean, and variance equal to 0.1. Figure 1 shows that the first basis vectors respectively computed with the MEs solution and the snapshot technique display a good agreement only when a very large snapshot size is employed. Using only a few snapshots yields poor representations of the basis vectors, i.e., poor SnapROM accuracy. This is clearly shown in Figure 2, where the errors associated with MEsROM-KL are seen to be clearly lower than their counterparts associated with SnapROM-KL (Ns=500) in reproducing the head variance.

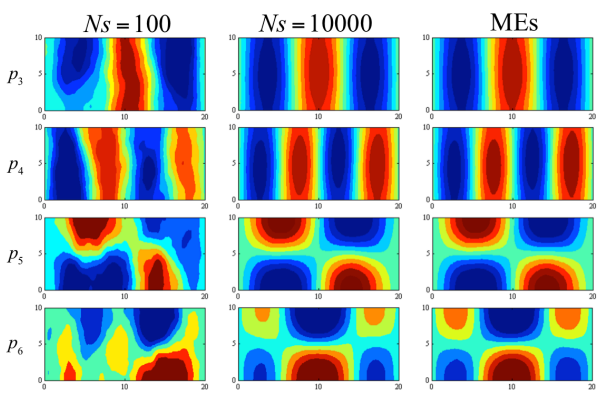

Figure 1. Basis vectors computed through 100, 10000 snapshots and the MEs.

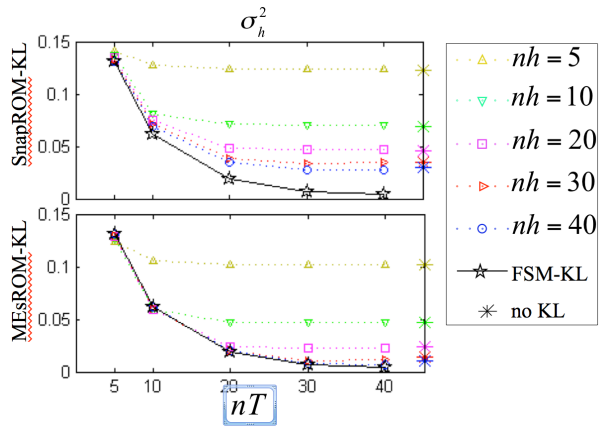

Figure 2. Convergence of the ROM error on the head variance (σ_h^2)

Our results suggest that the proposed reduced-order model obtained with the aid of the second-order mean head and head covariance computed by the MEs provides accurate estimations of the probability distribution of the heads, with online CPU times for N=10,000 ranging from 80 s (FSM) to 15 s (MEsROM-KL, nh=20, nT=20).

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