Upscaling of coupled geomechanics, flow, and heat in a poro-elastic medium in the quasi-static situation

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Abstract

Motivated by geothermal energy storage in the subsurface, we undertake a formal derivation of a linear poro-thermo-elastic system within the quasi-static framework. This derivation is based upon the well known derivation of the quasi-static poroelastic equations (also known as the Biot consolidation model) from the micro structure, except that we now include energy conservation equations in the micro-scale model. These are coupled to the fluid/structure model by using linear thermo-elasticity for the solid structure instead of the usual linear elasticity. The resulting upscaled system is similar to the linear poro-elastic equations, but with an added conservation of energy equation, fully coupled to the momentum and mass conservation equations. We start at the pore scale, and apply the technique of homogenization to derive the upscaled model in the case of periodically distributed pores. Assuming the homogenization ansatz holds true, we obtain a fully coupled system of equations on the macro-scale accounting for the effects of geomechanics, heat transfer, and fluid flow within a fully saturated porous material.

1 Introduction

The theory of consolidation of soils goes back to the work of M. A. Biot (see e.g. [4; 5; 6; 7]), and since then a range of different authors have contributed to the field, extending the models to different situations and providing more rigorous results for the equations. Without giving a complete reference list, we refer to Clopeau et al. [11], Gilbert and Mikelić [14], Burridge and Keller [10], Nguetseng [25], Lévy [19], Sánchez-Palencia [28], and the references therein. Today, this field is better known as 'poroelasticity', and is of great importance in a range of different engineering disciplines, such as reservoir engineering and biomechanics. In the following we shall discuss briefly the linear theory of poro-elasticity in order to set the stage for the rest of this paper, which is concerned with the derivation of a (semi-) linear thermo-poroelasticity model.

The motivation for the present work is the growing interest in geothermal energy storage in the subsurface, specifically in geological permeable layers such as saline aquifers and depleted oil and gas reservoirs. To date, there has been some commercial applications of geothermal energy storage, but there is need to better understand the how thermal stresses in the grains are influenced by the forces exerted on the pore walls by the fluid. The present work can be considered a contribution to the modeling aspect of such an endeavor, as our focus lies on including only the most dominant physical processes, and a fluid which is fully saturating the pore space. Let us mention also some previous work which has been done is this direction; in Lee and Mei [18] the homogenization of a similar model was undertaken, but with a different scaling. In Bringedal et al. [9] a formal upscaling of non-isothermal reactive flow in porous media was undertaken, but the solid matrix was assumed rigid. In Eden and Muntean [12] homogenization of a fully coupled thermoelasticity problem was undertaken, but not in the context of fluid/structure interaction.

Our micro-scale model consists of a fluid/structure interaction model, together with energy conservation equations for both phases (the solid and fluid), where we scale the fluid/structure equations corresponding to the diphasic macroscopic behavior of the system (i.e. small fluid viscosity). A rigorous study of this situation in the isothermal case can be found in [11]. Different scalings are of course possible, and for different values of the reference quantities, the homogenization process may result in
vastly different macro-scale models. A discussion around the characterization of the behavior of porous media according to the values of such reference quantities can be found in Auriault [3]. Regarding the energy conservation, we consider two different scaling regimes; one corresponding to a Péclét number of order one, giving a (nonlinear) convective term in the upscaled energy conservation equation, and one corresponding to a small Péclét number, resulting in no such convective term, giving a fully linear model on the macro-scale. Depending on the physics involved, both may be relevant.

The upscaling procedure is done via the two-scale asymptotic expansion method of homogenization. This is a well known method for qualitatively assessing the structure of the upscaled equations. For an accurate physical model, the values of the homogenized coefficients should be confirmed by experiments, as the the asymptotic expansion method only provides formulas for these in the case of a severely simplified micro-scale geometry. Our justification for the upscaled model comes from the similarity with the isothermal poro-elastic equations, and the analogy to the thermo-elasticity equations in mechanics.

The article is organized as follows: first we shall discuss the linear poro-elastic equations, such that this model can easily be compared to the one we derive here. Then, we briefly present the steps involved in the two-scale asymptotic expansion method of homogenization, and establish the assumptions governing our modeling. A presentation of the micro-scale governing equations will follow, together with a dimensional analysis, giving the micro-scale model in dimensionless form. Next, we undertake the formal upscaling, and provide formulas for all the homogenized coefficients. These are determined by the cell problems posed in the rescaled unit pore cell, and while some of these are the same as in the isothermal case, some are also new. We provide proofs of positive definiteness and symmetry for these. In the concluding section we discuss the upscaled model and give a comparison with the isothermal linear poroelastic equations.

2 Preliminaries

2.1 The linear poro-elastic equations

In this section we briefly present the linear poro-elastic equations, following Showalter [29]. We also note that in [29] the well-posedness of this model was first established.

The linear poro-elastic equations are the analog of the linear elasticity equations in mechanics, only extended to account for the effect of the saturating fluid. This is seen as an additional fluid pressure term in the stress tensor, since the total stress in a fully saturated porous medium must account for both the usual mechanical stress and for the 'fluid stress', which in this case is the fluid pressure. This is because viscous forces are usually very small within in the saturating fluid. The deformation rate of the solid material is also very small compared to the flow rate of the fluid, and thus a quasi-static assumption is made, meaning that the acceleration term is ignored. Given a body force, \( \mathbf{b} \), this gives the momentum balance equation as:

\[
-\nabla \cdot \mathbf{\sigma}_{tot} = \mathbf{b},
\]

where the total stress is given by: \( \mathbf{\sigma}_{tot} = \mathbf{\sigma} - \alpha \nabla p \), where \( p \) is the fluid pressure, and \( \mathbf{\sigma} = \mathbf{\sigma}(\mathbf{w}) \) is the standard Cauchy stress tensor from elasticity (linearly dependent on the displacement, \( \mathbf{w} \)). The coefficient \( \alpha \) is the fluid/structure coupling coefficient, and is referred to as the Biot-Willis constant. It is in general a full tensor, but reduces to a scalar in the isotropic case (as presented here).

Since the primary variables in the poro-elastic momentum balance equation are the fluid pressure and the displacement field, an additional equation is needed. This is the conservation of mass, which states that the rate of change of the water content of some fixed control volume within the medium should be equal to the net flux of fluid going into the same control volume. Denoting by \( \mathbf{q} \) the fluid flux and by \( \eta \) the water content, this can be written as:

\[
\frac{\partial \eta}{\partial t} = -\nabla \cdot \mathbf{q}
\]

It now remains to give constitutive equations relating the secondary variables, \( \mathbf{\sigma}, \mathbf{q} \) and \( \eta \) to the primary variables \( p \) and \( \mathbf{w} \). In the isotropic case the Cauchy stress can be written using the Lamé
coefficients, $\mu$ and $\lambda$ as: $\sigma(w) = \mu \mathbf{e}(w) + \lambda \nabla \cdot w \mathbf{I}$, where $\mathbf{e}(\cdot)$ is the usual symmetric gradient. The fluid flux is assumed to obey the Darcy law, given by $q = -\frac{1}{\mu_f} K (\nabla p - \mathbf{b})$, where $\mu_f$ is the fluid viscosity, and $K$ is the permeability. For the fluid content the constitutive relation is: $\eta = \eta(w, p) = c_0 p + \alpha \nabla \cdot w$, where the term $c_0 p$ measures the amount of fluid that can be injected into a fixed control volume, and $\alpha \nabla \cdot w$ measures the amount of fluid that can be squeezed out. It is important to note that the coefficient $\alpha$ here is indeed the same Biot-Willis coefficient from the momentum equation, $c_0$ is usually referred to as the ‘constrained specific storage coefficient’, and it satisfies $c_0 > 0$.

The constitutive relations given here were first justified through experimentation, the most famous example being the Darcy law, but they are also possible to obtain by upscaling the fluid/structure interaction using some homogenization technique. In particular, upscaling the equations of linear elasticity for the solid skeleton and the linearized Navier-Stokes equations for the fluid, coupled at the mutual interface by requiring continuity of contact forces and zero flow (in this case that means the fluid velocity should equal the displacement velocity), will produce the equations discussed above, if an appropriate scaling is used. If the homogenization technique of two-scale asymptotic expansion is applied (which we will do in the following), this will also provide explicit formulas for the coefficients, and therefore giving the possibility to deduce their precise meaning as well as to prove some properties they should posses, such as being symmetric and positive definite (calculating their value is also possible, though in general not very useful).

The derivation of the linear poro-elastic equations is a well known example of the application of homogenization theory, as was mentioned in the introduction. In the following, we shall do a similar derivation, except including also thermal effects. The resulting upscaled system is similar to the one described above, but due to the introduction of a third primary variable, the temperature, there will also be a conservation of energy equation. We propose this system as the analog of the linear thermo-elasticity equations in mechanics, only extended to account for the effect of the saturating fluid.

### 2.2 Homogenization

For the upscaling procedure, we use the technique of homogenization, in particular the two-scale asymptotic expansion method. For a detailed review of this method and its applications we refer to the book Hornung [15]. This section is devoted to briefly presenting the steps involved:

First, the two scales characterizing the system is identified, and their ratio is denoted by $\varepsilon$. In the context of porous media these will be the typical pore size, $l$ and the size of the entire domain under consideration, $L$, i.e. $\varepsilon = l/L$. This ratio will typically be around $10^{-6}$, and for reservoir applications even smaller.

The unknowns are then written as functions of the ‘slow’ spatial scale, $x = \tilde{x}/L$, the ‘fast’ spatial scale $y = \tilde{y}/l$ (where $\tilde{x}$ is the physical coordinate), and of $\varepsilon$, and the existence of perturbation series of the form:

$$u^\varepsilon = u^0(x, y, t) + \varepsilon u^1(x, y, t) + \varepsilon^2 u^2(x, y, t) + \cdots$$  \hspace{1cm} (1)

is postulated, and inserted back into the rescaled equations, while also expressing any spatial differential operators in both slow and fast spatial scales. This is the key assumption, namely that the two scales can be separated. By equating terms of equal $\varepsilon$-power, an infinite sequence of boundary value problems is produced, but typically everything of $O(\varepsilon)$, or higher, is neglected. This sequence of boundary value problems can be solved successively, and in order to eliminate the dependence of the fast spatial scale, the ‘last’ boundary value problem (the highest order one which is still deemed non-negligible) is averaged with respect to the fast spatial scale, giving the homogenized equation. The coefficients of this problem are averages of so-called auxiliary functions, which can be found by solving a set of boundary value problems called auxiliary problems, having only the fast spatial scale as the independent variable, while the slow spatial scale takes the role of a parameter.

The goal of the two-scale asymptotic expansion method is to find the boundary value problem satisfied by the unknowns in the limit $\varepsilon \to 0$, and it is a successful one at that, except for the fact that the limit problem is not obtained rigorously using some notion of convergence in the appropriate
function spaces. Without going into any detail we mention that one such notion is the 'two-scale convergence', developed by G. Allaire and G. Nguetseng (see e.g. Allaire [2] and Nguetseng [24]), but this is outside the scope of this article. The existence of a perturbation series like above for the unknowns will remain a postulate, and we shall refer to this as the 'homogenization ansatz'.

3 The pore-scale model

3.1 Assumptions

In this paper we model the interaction between an elastic matrix and an incompressible fluid which is saturating the pore space, together with energy conservation for both the solid and fluid phase. Our physical domain is \( \Omega \subset [0, L]^3 \), which consists of a solid skeleton, \( \Omega_s \), and a fluid filled void space, \( \Omega_f \), where the internal boundary between the solid and void parts is denoted by \( \Gamma \), i.e. in the reference configuration we have: \( \Omega = \Omega_s \cup \Omega_f \cup \Gamma \) where \( \Omega_s \cap \Omega_f = \emptyset \), and \( \Gamma = \partial \Omega_s \cap \partial \Omega_f \). We let our time domain be the interval \( J = (0, T_{\text{end}}] \), where \( T_{\text{end}} > 0 \) is the final time. We denote by \( x = (x_1, x_2, x_3) \) the coordinates of the reference configuration, and by \( t \) the time coordinate.

The modeling of fluid and elastic solid structure interaction is in general very complicated, since for an elastic solid Lagrangian coordinates are the preferred reference frame, while for the fluid it is the Eulerian coordinates. Thus, when coupling the two processes at the mutual interface, one needs to take into account the movement of the interface itself. This leads to a nonlinear problem, even if the two respective processes are described by linear equations. For more details on this, see Iliev et al. [16]. In order to avoid nonlinear coupling conditions, we make the assumption that the displacements of the material points in the solid are very small compared to the movement of the fluid, such that the same coordinates can be used when describing both processes. The coupling conditions at the mutual interface can then be stated with respect to the undeformed configuration, i.e. \( \Gamma \).

We let \( l \) be a typical pore size, and let \( L \) be the size of the macro-scale domain, and set as usual \( \varepsilon = l/L \). We let \( w(t) \) denote the displacement field of the solid, defined on \( \Omega_s \), and assume its magnitude is no larger than \( l \). Thus, we can neglect any second order term in \( w \), and in particular terms of \( \nabla w \cdot (\nabla w)^T \), such that linear strain is justified. We let \( v(t) \) be the fluid velocity, defined on \( \Omega_f \), and assume we have a slow moving fluid at low Reynold’s number, i.e. the fluid should take a long time to traverse the distance \( L \). We let \( T_s(t) \) and \( T_f(t) \) be the temperature distributions of the solid and fluid, defined on \( \Omega_s \) and \( \Omega_f \), respectively, and assume the temperature variations from the reference temperature, which we denote by \( \theta_0 \), are small, and that the two phases are in local thermodynamic equilibrium. Then, we can use the reference densities for both phases, and in particular, we can regard the fluid as being incompressible, i.e. \( \nabla \cdot v = 0 \) in \( \Omega_f \). We can also neglect terms of order \( w \cdot \nabla v \), \( w \cdot \nabla T_s \), and \( w \cdot \nabla T_f \), and higher, but note we must keep terms of \( v \cdot \nabla T_f \).

3.2 Presentation of the equations

In this section we present the governing equations to be used throughout the rest of this article. This will include a brief discussion of the linear equations of thermo-elasticity for an anisotropic solid, relevant for the present work. For more information on the linear theory of thermoelasticity, we refer to Pabst [26] and the references therein.

The linear momentum balance for an elastic solid is given by:

\[
\rho_s \frac{\partial^2 w}{\partial t^2} - \nabla \cdot \sigma = b \quad \text{in} \quad \Omega_s \times J,
\]

where \( \rho_s \) is the solid density, and the constitutive equation for the stress is:

\[
\sigma = \sigma(F, T_s),
\]

in the non-isothermal case. Since we are within the completely (i.e. geometrically and physically) linearized framework, the deformation gradient, \( F \), can be considered to be approximately identity,
and the first Piola-Kirchhoff stress and the Cauchy stress tensors coincide (which is why we denote it as $\sigma$), and the above constitutive equation above takes the form:

$$\sigma(w, T_s) = C e(w) - M(T_s - \theta_0),$$

which is a generalized Hooke’s law, extended to include thermal effects. $C = (C_{ijkl})^{3}_{i,j,k,l=1}$ is the stiffness tensor of the material (or more precisely, the referential tensor of isothermal elasticities), which satisfies: $C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}$. $M = (M_{ijkl})^{3}_{i,j,k,l=1}$ is the thermal stress tensor, satisfying $M_{ij} = M_{ji}$, and $\theta_0$ is the reference temperature, considered here to be uniform (i.e. constant). In order to have symmetric positive definite coefficients in the upscaled problem, the same must be true for $C$ and $M$, i.e.:

$$e^T C e > 0, \forall e \in \mathbb{R}^{3\times 3}, \text{ and } x^T M x > 0, \forall x \in \mathbb{R}^3.$$

Denoting by $c_s$ the specific internal energy of the solid, the conservation of energy equation is given by:

$$\rho_s \frac{\partial c_s}{\partial t} = \sigma : e(\partial_t w) - \nabla \cdot h_s,$$

where the heat flux, $h_s$, obey Fourier’s law:

$$h_s = -K_s \nabla T_s.$$

$K_s = (K^s_{ijkl})^{3}_{i,j,k,l=1}$ is the thermal conductivity of the material, and is assumed to satisfy the usual symmetry and positive definiteness: $K^s_{ijkl} = K^s_{jikl}$, and $x^T K_s x > 0, \forall x \in \mathbb{R}^3$.

In the dissipative term, we neglect the contribution from the mechanical stress (which is second order in the gradient of $w$), and assume the heat generation from the thermal stress can be approximated by the reference temperature of the medium, i.e.: $\sigma : e(\partial_t w) \approx -\theta_0 M : e(\partial_t w)$. Using the specific heat capacity of the solid, $c_s$, we can now write the conservation of energy equation (5) as:

$$\rho_s c_s \frac{\partial T_s}{\partial t} + \theta_0 M : e(\partial_t w) - \nabla \cdot (K_s \nabla T_s) = 0, \quad \text{in } \Omega_s \times J.$$

In the pore space the flow is governed by the Navier-Stokes equations:

$$\rho_f \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) - \nabla p + \mu \Delta v = b, \quad \text{in } \Omega_f \times J,$$

and the incompressible mass conservation:

$$\nabla \cdot v = 0, \quad \text{in } \Omega_f \times J,$$

where $\rho_f$ is the fluid density, $p$ is the fluid pressure, and $\mu$ is the dynamic fluid viscosity.

Since there are no heat generation from dissipative effects in the fluid, we use a simple convection-conduction equation for the energy conservation:

$$\rho_f c_f \left( \frac{\partial T_f}{\partial t} + v \cdot \nabla T_f \right) - \nabla \cdot (K_f \nabla T_f) = 0, \quad \text{in } \Omega_f \times J,$$

where $c_f$ is the specific heat capacity of the fluid, and $K_f = (K_{ijkl}^f)^{3}_{i,j,k,l=1}$ is the thermal conductivity, satisfying: $K_{ij}^f = K_{jik}^f$, and $x^T K_f x > 0, \forall x \in \mathbb{R}^3$.

We now turn to the fluid/structure coupling conditions at the internal interface, which, due to our previous assumptions, have been linearized, and are stated with respect to the reference configuration. We denote by $\nu$ the outward unit normal field of $\Omega_f$ (i.e. pointing into the solid).

By Newton’s third law we must have a balance of normal forces coming from both sides:

$$(p I + 2\mu e(w)) \nu = (C e(w) - M(T_s - \theta_0)) \nu, \quad \text{on } \Gamma \times J,$$

where $I$ denotes the $3 \times 3$ identity tensor.
The no-flow condition at the internal interface now takes the form:

\[ v = \partial_t w, \quad \text{on} \quad \Gamma \times J. \quad (12) \]

Finally, continuity of heat flux and continuity of temperature at the internal interface gives (due to our assumption of the two phases being in local thermodynamic equilibrium):

\[ K_f \nabla T_f \cdot \nu = K_s \nabla T_s \cdot \nu, \quad \text{on} \quad \Gamma \times J, \quad (13) \]

and

\[ T_f = T_s, \quad \text{on} \quad \Gamma \times J. \quad (14) \]

We note that the upscaling of equations (2) (without the thermal stress term), (8), (9), (11), and (12), will produce the classical poro-elastic equations discussed in the introduction.

3.3 Dimensional analysis

In this section we introduce dimensionless variables, and scale the system according to the quasi-static diphasic macro-scale behaviour (see [3] for more details). In short, this means the fluid pressure should be of the same order as the normal stress coming from the elastic matrix.

With a slight abuse of notation we write now the dimensional quantities with a tilde, and the new dimensionless quantities in the same way as before:

\[ \tilde{x} = Lx, \quad \tilde{t} = \tau t, \quad \tilde{C} = C_{\text{ref}} C, \quad \tilde{M} = M_{\text{ref}} M, \quad \tilde{K}_f = K_{\text{ref}}^f K_f, \quad \tilde{K}_s = K_{\text{ref}}^s K_s, \]

where we choose the time scale, \( \tau = \frac{L}{V_{\text{ref}}}, \) as the characteristic transport time. As mentioned before, \( \tau \) should be quite large when considering a slow moving fluid within a porous medium. We note also that the elasticity and thermal stress coefficients of the solid, i.e. \( C_{\text{ref}} \) and \( M_{\text{ref}} \), are very large. The exact values of the above reference quantities are of secondary importance, as we are only interested in identifying terms that differ in size by an order of \( \varepsilon \) or more.

We denote by \( \Omega^\varepsilon = \frac{1}{L} \Omega \) our new dimensionless domain, which now is \( \Omega^\varepsilon \subset [0,1]^3 \), such that \( \Omega_s^\varepsilon \) and \( \Omega_f^\varepsilon \) are the corresponding dimensionless solid and void parts, respectively, and \( \Gamma^\varepsilon = \frac{1}{L} \Gamma \) is the corresponding dimensionless internal interface. We continue with the notation \( J \) for the time interval, keeping in mind time is now also dimensionless.

The balance of contact forces at the interface, equation (11), in dimensionless variables is:

\[ (-P_{\text{ref}} p I + \frac{V_{\text{ref}} M_{\text{ref}}}{L} 2 \mu \varepsilon \dot{e}(v)) \nu = (C_{\text{ref}} \varepsilon C \dot{e}(w) - M_{\text{ref}} \theta_0 T_s) \nu, \quad \text{on} \quad \Gamma^\varepsilon \times J. \]

Due to the diphasic scaling regime, we have \( P_{\text{ref}} \sim C_{\text{ref}} \varepsilon \sim M_{\text{ref}} \theta_0 \), and set the reference viscosity as: \( \mu_{\text{ref}} = \varepsilon^2 \frac{P_{\text{ref}} L}{V_{\text{ref}}} \). Thus, we can simplify the above equation and get the dimensionless form of equation (11) as:

\[ (p I + 2 \mu \varepsilon^2 \dot{e}(v)) \nu = (C \dot{e}(w) - M T_s) \nu, \quad \text{on} \quad \Gamma^\varepsilon \times J. \quad (15) \]

The momentum equation for the solid, equation (2), in dimensionless form is:

\[ \rho_s \frac{l}{\tau^2} \frac{\partial^2 w}{\partial t^2} - \mathcal{O}(\frac{C_{\text{ref}} \varepsilon}{L}) \nabla \cdot (C \dot{e}(w) - M T_s) = b, \quad \text{on} \quad \Gamma^\varepsilon \times J. \]

We see that the factor \( \rho_s \frac{l}{\tau^2} \) multiplying the acceleration term is very small, while the \( \mathcal{O}(\frac{C_{\text{ref}} \varepsilon}{L}) \)-factor, on the other hand is close to unity since \( C_{\text{ref}} \) is very large. We therefore divide by it, absorb it into the right hand side, yielding a new dimensionless body force (still denoted \( b \), and giving a
dimensionless constant in front of the acceleration term which we set as: \( \rho_s \frac{L}{\tau^2 C_{ref}} = O(\varepsilon^2) \). Thus, we get the dimensionless form of equation (2) as:

\[
\varepsilon^2 \frac{\partial^2 w}{\partial t^2} - \nabla \cdot (C e(w) - M T_s) = b, \quad \text{on} \quad \Gamma^e \times J.
\]

(16)

The dimensionless form of the momentum equation for the fluid, (8), is:

\[
\rho_f \frac{L}{\tau^2} \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) - \frac{P_{ref}}{L} \nabla p + \frac{P_{ref}}{L} \mu^2 \Delta v = b, \quad \text{in} \quad \Omega_f^e \times J,
\]

where we used the expression \( \mu_{ref} = \frac{\varepsilon^2}{V_{ref}} \). Again, we have \( \frac{P_{ref}}{L} \sim \frac{C_{ref}}{L} \) which is close to unity, so we divide by this factor, giving again a dimensionless body force, and a small dimensionless constant in front of the material derivative, i.e. we set \( \rho_f \frac{L}{\tau^2 C_{ref}} = O(\varepsilon^2) \). Thus, the dimensionless form of equation (8) is:

\[
\varepsilon^2 \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) - \nabla p + \varepsilon^2 \mu \Delta v = b, \quad \text{in} \quad \Omega_f^e \times J.
\]

(17)

The mass conservation equation for the fluid, equation (9) and the no-flow condition at the boundary, equation (12), in dimensionless variables are:

\[
\nabla \cdot v = 0, \quad \text{in} \quad \Omega_f^e \times J,
\]

(18)

and

\[
v = \partial_t w, \quad \text{on} \quad \Gamma^e \times J,
\]

(19)

respectively.

We now turn to the energy conservation equations. That for the fluid, equation (10), in dimensionless variables is:

\[
\rho_f c_f \left( \frac{1}{\tau} \frac{\partial T_f}{\partial t} + \frac{V}{L} v \cdot \nabla T_f \right) - \frac{K_{ref}^f}{L^2} \nabla \cdot (K_f \nabla T_f) = 0, \quad \text{in} \quad \Omega_f^e \times J.
\]

Multiplying by \( \frac{L^2}{K_{ref}^f} \), we get the Péclet number in front of the convective term, which in this case is given by \( Pe = \frac{\rho_f c_f V L}{K_{ref}^f} \), and we now make the choice \( Pe \sim O(1) \), meaning heat is transported within the fluid by convection and conduction at an equal rate. This is because we wish to keep a convection term in the upscaled equation. The other natural choice would be \( Pe << 1 \), in which conduction is the dominant process. However, when we undertake the upscaling procedure, it will become clear that this choice can be seen just as a special case of the more general \( Pe \sim O(1) \). In the concluding section we will however present the homogenized model corresponding to both choices of scaling.

In front of the time derivative term, we get the dimensionless constant \( \rho_f c_f \frac{L^2}{\tau K_{ref}^f} \), and due to the characteristic transport time, \( \tau \), being large, we set: \( \rho_f c_f \frac{L^2}{\tau K_{ref}^f} \sim O(\varepsilon) \).

Thus, the dimensionless form of equation (10) is:

\[
\varepsilon \frac{\partial T_f}{\partial t} + v \cdot \nabla T_f - \nabla \cdot (K_f \nabla T_f) = 0, \quad \text{in} \quad \Omega_f^e \times J.
\]

(20)

The energy conservation equation for the solid, (7), is:

\[
\rho_s c_s \left( \frac{1}{\tau} \frac{\partial T_s}{\partial t} + \frac{M_{ref}^l}{L} \mathbf{M} : \mathbf{e} \left( \partial_t w \right) - \frac{K_{ref}^s}{L^2} \nabla \cdot (K_s \nabla T_s) = 0, \quad \text{in} \quad \Omega_s^e \times J.
\]
Similarly as with the fluid, we multiply by \( \frac{L^2}{K_{\text{ref}}} \), and get the dimensionless constants, \( \frac{\rho_s c_s L^2}{\tau K_{\text{ref}}^s} \sim O(\varepsilon) \) and \( \frac{M_{\text{ref}} L}{\tau K_{\text{ref}}^s} \sim O(1) \), and we therefore write equation (7) in dimensionless form as:

\[
\varepsilon \frac{\partial T_s}{\partial t} + \mathbf{M} : \mathbf{e}(\partial_t w) - \nabla \cdot (\mathbf{K}_s \nabla T_s) = 0, \quad \text{in } \Omega_s^\varepsilon \times J, \quad (21)
\]

The reference values of the thermal conductivities of the two phases can be regarded as approximately the same order (i.e. \( K_{\text{ref}}^s \sim K_{\text{ref}}^v \)), and we therefore write the dimensionless form of equations (22) and (23) as:

\[
\mathbf{K}_f \nabla T_f \cdot \nu = \mathbf{K}_s \nabla T_s \cdot \nu, \quad \text{on } \Gamma^\varepsilon \times J, \quad (22)
\]

\[
T_f = T_s, \quad \text{on } \Gamma^\varepsilon \times J, \quad (23)
\]

respectively.

### 3.4 The complete dimensionless pore-scale model

For convenience, we summarize the dimensionless equations at the micro-scale below:

\[
\varepsilon^2 \frac{\partial^2 w^\varepsilon}{\partial t^2} - \nabla \cdot (\mathbf{C} \mathbf{e}(w^\varepsilon) - \mathbf{M} T_s^\varepsilon) = \mathbf{b}, \quad \text{in } \Omega_s^\varepsilon \times J, \quad (24a)
\]

\[
\varepsilon^2 \left( \frac{\partial v^\varepsilon}{\partial t} + v^\varepsilon \cdot \nabla v^\varepsilon \right) - \nabla \cdot (p^\varepsilon \mathbf{I} - 2\mu\varepsilon^2 \mathbf{e}(v^\varepsilon)) = \mathbf{b}, \quad \text{in } \Omega_f^\varepsilon \times J, \quad (24b)
\]

\[
\nabla \cdot v^\varepsilon = 0, \quad \text{in } \Omega_f^\varepsilon \times J, \quad (24c)
\]

\[
(-p^\varepsilon \mathbf{I} + 2\mu\varepsilon^2 \mathbf{e}(v^\varepsilon))\nu = (\mathbf{C} \mathbf{e}(w^\varepsilon) - \mathbf{M} T_s^\varepsilon)\nu, \quad \text{on } \Gamma^\varepsilon \times J, \quad (24d)
\]

\[
\partial_t w^\varepsilon = v^\varepsilon, \quad \text{on } \Gamma^\varepsilon \times J, \quad (24e)
\]

\[
\varepsilon \frac{\partial T_s^\varepsilon}{\partial t} + \mathbf{M} : \mathbf{e}(\partial_t w^\varepsilon) - \nabla \cdot (\mathbf{K}_s \nabla T_s^\varepsilon) = 0, \quad \text{in } \Omega_s^\varepsilon \times J, \quad (24f)
\]

\[
\varepsilon \frac{\partial T_f^\varepsilon}{\partial t} + v^\varepsilon \cdot \nabla T_f^\varepsilon - \nabla \cdot (\mathbf{K}_f \nabla T_f^\varepsilon) = 0, \quad \text{in } \Omega_f^\varepsilon \times J, \quad (24g)
\]

\[
\mathbf{K}_f \nabla T_f^\varepsilon \cdot \nu = \mathbf{K}_s \nabla T_s^\varepsilon \cdot \nu, \quad \text{on } \Gamma^\varepsilon \times J, \quad (24h)
\]

\[
T_s^\varepsilon = T_f^\varepsilon, \quad \text{on } \Gamma^\varepsilon \times J. \quad (24i)
\]

We omit initial and boundary conditions since they are not important for the homogenization procedure. Note also that we have included an epsilon-superscript on the dependent variables to emphasize the implicit dependence on both the slow and fast scales.

### 3.5 The domain

Before undertaking the separation of scales, we impose further assumptions on the domain, specifically that it is made up of a periodic repetition of a single pore cell, such that the geometry of the whole solid skeleton is determined by the geometry inside a single microscopic cell. We shall follow Allaire [1] in this presentation.

Let \( Y = (0, 1)^3 \) be the rescaled unit cube in \( \mathbb{R}^3 \), consisting of a solid part, \( Y_s \subset \bar{Y} \), which is a closed subset of strictly positive measure, and a void space, \( Y_f = Y \setminus Y_s \), which is an open and connected subset of strictly positive measure with a smooth boundary. We make a periodic repetition of \( Y_s \) over \( \mathbb{R}^3 \), and set \( Y_s^{\varepsilon,k} = \varepsilon (Y_s + k) \), where \( k \in \mathbb{Z}^3 \). Let \( K = \{ k \in \mathbb{Z}^3 : Y_s^{\varepsilon,k} \subset \Omega^\varepsilon \} \), such that \( \Omega_s^\varepsilon = \bigcup_{k \in K} Y_s^{\varepsilon,k} \) yields the solid skeleton (grain space), and \( \Omega_f^\varepsilon = \Omega^\varepsilon \setminus \Omega_s^\varepsilon \) is the fluid filled void space. The fluid/solid
internal interface is now given by \( \Gamma^\varepsilon = \partial \Omega^\varepsilon_s \setminus \partial \Omega^\varepsilon_f \). By construction, both \( \Omega^\varepsilon_s \) and \( \Omega^\varepsilon_f \) are now connected sets of strictly positive measure, and \( \Gamma^\varepsilon \) is a smooth surface.

Note again that the epsilon-superscript on \( \Omega^\varepsilon \) implies the implicit dependence of the domain on both length scales, but later when we impose the homogenization ansatz, we separate these two scales, and let the size of the domain become arbitrarily large (that is, we let \( \varepsilon \to 0 \)). Then, behind each infinitesimal point \( x \) (seen from the macro scale) there is a pore cell with its own geometry which can only be seen by the fast variable \( y \).

When this scale separation is done, we shall denote the macro scale domain simply by \( \Omega \), which is no longer possible to separate into solid and void parts because the porous structure is now seen as a single (fictitious) uniform material.

An example of pore cell geometry which satisfies the above assumptions are:

![Image](image_url)

Figure 1: Geometry inside unit cell (Mikelić and Wheeler [23])

4 Two-scale asymptotic expansions

4.1 Homogenization ansatz

We now undertake the separation of scales, and introduce the homogenization ansatz for the unknowns:

\[
\begin{align*}
v_{\varepsilon}(x,t) &= v^0(x,y,t) + \varepsilon v^1(x,y,t) + \varepsilon^2 v^2(x,y,t) + \cdots, \\
w_{\varepsilon}(x,t) &= w^0(x,y,t) + \varepsilon w^1(x,y,t) + \varepsilon^2 w^2(x,y,t) + \cdots, \\
T^f_{\varepsilon}(x,t) &= T^0_f(x,y,t) + \varepsilon T^1_f(x,y,t) + \varepsilon^2 T^2_f(x,y,t) + \cdots, \\
T^s_{\varepsilon}(x,t) &= T^0_s(x,y,t) + \varepsilon T^1_s(x,y,t) + \varepsilon^2 T^2_s(x,y,t) + \cdots, \\
p_{\varepsilon}(x,t) &= p^0(x,y,t) + \varepsilon p^1(x,y,t) + \varepsilon^2 p^2(x,y,t) + \cdots.
\end{align*}
\]

Note that we now have an added dependence on the spatial variable \( y \in Y \), in which all unknowns are \( Y \)-periodic due to the scaling and the periodic arrangement of the porous structure.

We must also reformulate the differential operators according to the chain rule: \( \nabla = \nabla_x + \varepsilon^{-1} \nabla_y \), and \( e(\cdot) = e_x(\cdot) + \varepsilon^{-1} e_y(\cdot) \).
We may now insert the asymptotic expansions into the governing equations, where we discard all terms of $O(\varepsilon)$ or higher. We start with equation (24a) for the elastic solid structure:

\[
\mathbf{b} = -\varepsilon^{-2} \nabla_y \cdot (\mathbf{C} \mathbf{e}_y (w^0)) \\
- \varepsilon^{-1} \left[ \nabla_y \cdot (\mathbf{C} (\mathbf{e}_x (w^0) + \mathbf{e}_y (w^1)) - \mathbf{M} T_s^0) + \nabla_x \cdot (\mathbf{C} \mathbf{e}_y (w^0)) \right] \\
- \varepsilon^0 \left[ \nabla_y \cdot (\mathbf{C} (\mathbf{e}_x (w^1) + \mathbf{e}_y (w^2)) - \mathbf{M} T_s^1) + \nabla_x \cdot (\mathbf{C} (\mathbf{e}_x (w^0) + \mathbf{e}_y (w^1)) - \mathbf{M} T_s^0) \right] \\
+ O(\varepsilon), \quad \text{in} \quad \Omega \times Y_s \times J,
\]  

(25)

The conservation of momentum and mass for the fluid, equations (24b) and (24c), yields:

\[
\mathbf{b} = \varepsilon^{-1} \nabla_y p^0 + \varepsilon^0 \left[ \nabla_x p^0 + \nabla_y p^1 - \mu \Delta_y v^0 \right] + O(\varepsilon), \quad \text{in} \quad \Omega \times Y_f \times J,
\]  

(26)

and

\[
0 = \varepsilon^{-1} \nabla_y \cdot v^0 + \varepsilon^0 \left[ \nabla_x \cdot v^0 + \nabla_y \cdot v^1 \right] + O(\varepsilon), \quad \text{in} \quad \Omega \times Y_f \times J.
\]  

(27)

At the internal interface, continuity of contact forces and continuity of displacement velocity, equations (24d) and (24e), gives:

\[
0 = \varepsilon^{-1} \mathbf{C} \mathbf{e}_y (w^0) \nu \\
+ \varepsilon^0 \left[ \mathbf{C} (\mathbf{e}_x (w^0) + \mathbf{e}_y (w^1)) - \mathbf{M} T_s^0 + p^0 \mathbf{I} \right] \nu \\
+ \varepsilon^1 \left[ \mathbf{C} (\mathbf{e}_x (w^1) + \mathbf{e}_y (w^2)) - \mathbf{M} T_s^1 + p^1 \mathbf{I} - 2 \mathbf{e}_y (v^0) \right] \nu \\
+ O(\varepsilon^2), \quad \text{on} \quad \Omega \times \Gamma \times J,
\]  

(28)

and

\[
0 = (v^0 - \partial_t w^0) + \varepsilon (v^1 - \partial_t w^1) + O(\varepsilon^2) \quad \text{on} \quad \Omega \times \Gamma \times J.
\]  

(29)

The energy conservation equations for the solid and fluid, equations (24f) and (24g), yields:

\[
0 = -\varepsilon^{-2} \nabla_y \cdot (\mathbf{K}_s \nabla_y T_s^0) \\
+ \varepsilon^{-1} \left[ \mathbf{M} : \mathbf{e}_y (\partial_t w^0) - \nabla_y \cdot (\mathbf{K}_s (\nabla_x T_s^0 + \nabla_y T_s^1)) - \nabla_x \cdot (\mathbf{K}_s \nabla_y T_s^0) \right] \\
+ \varepsilon^0 \left[ \mathbf{M} : (\mathbf{e}_x (\partial_t w^0) + \mathbf{e}_y (\partial_t w^1)) - \nabla_x \cdot (\mathbf{K}_s (\nabla_x T_s^0 + \nabla_y T_s^1)) - \nabla_y \cdot (\mathbf{K}_s (\nabla_x T_s^1 + \nabla_y T_s^2)) \right] \\
+ O(\varepsilon), \quad \text{in} \quad \Omega \times Y_s \times J,
\]  

(30)
and
\[ 0 = -\varepsilon^{-2} \nabla_y \cdot (K_f \nabla_y T^0_f) + \varepsilon^{-1} \left[ \nabla_y \cdot (K_f \nabla_y T^0_f) - \nabla_y \cdot (K_f \nabla_y T^j_f) \right] + \varepsilon \left[ \partial_t u^0 \cdot \nabla_x T^0_f - \nabla_x \cdot (K_f (\nabla_x T^0_f + \nabla_y T^1_f)) + v^0 \cdot \nabla_y T^1_f + v^1 \cdot \nabla_y T^0_f - \nabla_y \cdot (K_f (\nabla_x T^1_f + \nabla_y T^2_f)) \right] + \mathcal{O}(\varepsilon), \quad \text{in} \quad \Omega \times Y_s \times J. \] (31)

At the internal interface, continuity of energy and temperature, equations (24h) and (24i), gives:
\[ 0 = \varepsilon^{-1} \left[ K_f \nabla_y T^0_f - K_s \nabla_y T^0_s \right] \cdot \nu + \varepsilon^0 \left[ K_f (\nabla_x T^0_f + \nabla_y T^1_f) - K_s (\nabla_x T^0_s + \nabla_y T^1_s) \right] \cdot \nu + \varepsilon^1 \left[ K_f (\nabla_x T^1_f + \nabla_y T^2_f) - K_s (\nabla_x T^1_s + \nabla_y T^2_s) \right] \cdot \nu + \mathcal{O}(\varepsilon^2), \quad \text{on} \quad \Omega \times \Gamma \times J, \] (32)

and
\[ (T^0_s - T^0_f) + \varepsilon (T^1_s - T^j_f) + \mathcal{O}(\varepsilon^2) = 0 \quad \text{on} \quad \Omega \times \Gamma \times J. \] (33)

It is evident from the above equations that at the lowest order, the displacement, pressure, and temperature has no \( y \)-dependence, so we write:
\[ p^0(x, y, t) = p^0(x, t), \quad \text{in} \quad \Omega \times J, \] (34a)
\[ w^0(x, y, t) = w^0(x, t), \quad \text{in} \quad \Omega \times J, \] (34b)
\[ T^0_s(x, y) = T^0_f(x, y) = T^0(x, t), \quad \text{in} \quad \Omega \times J. \] (34c)

However, as seen from equations (26) and (27), at the lowest order, there is still a \( y \)-dependence in the fluid velocity.

### 4.2 The flow

In this section we derive an upscaled equation for the flow, namely the well known Darcy law, giving the flux of fluid through porous media.

We now consider the equation (26) at order \( \mathcal{O}(\varepsilon^0) \), equation (27) at order \( \mathcal{O}(\varepsilon^{-1}) \), and the boundary condition, equation (29) at order \( \mathcal{O}(\varepsilon^0) \), which gives the problem:
\[ \nabla_x p^0 - \mu \Delta_y v^0 = b - \nabla_y p^1, \quad \text{in} \quad Y_f, \]
\[ \nabla_y \cdot v^0 = 0, \quad \text{in} \quad Y_f, \]
\[ v^0 = \partial_t w^0, \quad \text{on} \quad \Gamma, \]
\[ v^0(x, \cdot, t) \text{ and } p^1(x, \cdot, t) \text{ are } Y\text{-periodic,} \quad \forall(x, t) \in \Omega \times J. \]
Note that since $\nabla_y v^0 = 0$ for $y \in Y_f$, we have: $\nabla_y \cdot 2\mathbf{e}_y(v^0) = \Delta_y v^0$. By defining $q = v^0 - \partial_t w^0$, we can rewrite the above problem as:

$$\begin{align*}
\nabla_x p^0 - \mu \Delta_y q &= b - \nabla_y p^1, & \text{in } Y_f, \\
\nabla_y \cdot q &= 0, & \text{in } Y_f, \\
q &= 0, & \text{on } \Gamma,
\end{align*}$$

which is the well-known cell problem in the homogenization of the filtration through rigid porous media. By using the identities $b = \sum_{j=1}^3 b_j e_j$, and $\nabla_x p^0 = \sum_{j=1}^3 \frac{\partial p^0}{\partial x_j} e_j$, where $b_j$ is the $j$'th component of the body force, and $\{e_j\}_{j=1,2,3}$ the canonical basis of $\mathbb{R}^3$, we can solve for $q$ and $p^1$ as follows:

$$\begin{align*}
q(x, y, t) &= \frac{1}{\mu} \sum_{j=1}^3 \Lambda^j(y) \left( b_j(x, t) - \frac{\partial p^0}{\partial x_j}(x, t) \right), \\
p^1(x, y, t) &= \sum_{j=1}^3 \Pi^j(y) \left( b_j(x, t) - \frac{\partial p^0}{\partial x_j}(x, t) \right),
\end{align*}$$

where $\Lambda^j$ and $\Pi^j$ ($\Lambda^j(y) \in \mathbb{R}^3, \Pi^j(y) \in \mathbb{R}$), are determined by the following cell problems (for $j = 1, 2, 3$):

$$\begin{align*}
- \Delta_y \Lambda^j + \nabla_y \Pi^j &= e_j, & \text{in } Y_f, \\
\nabla_y \cdot \Lambda^j &= 0, & \text{in } Y_f, \\
\Lambda^j &= 0, & \text{on } \Gamma,
\end{align*}$$

where $\Lambda^j$ and $\Pi^j$ are $Y$-periodic.

We integrate over $Y_f$ and obtain the Darcy flux:

$$q(x, t) = \int_{Y_f} q(x, y, t) dy = -\frac{1}{\mu} K^H (\nabla_x p^0(x, t) - b(x, t)),$$

where the effective coefficient, $K^H$, (known as the permeability tensor) is given by:

$$(K^H)_{ij} = \int_{Y_f} (\Lambda^j(y))_{i} dy, \quad i, j = 1, 2, 3.$$  

We get a similar expression for the average of $v^0$:

$$\int_{Y_f} v^0(x, y, t) dy = \partial_t v^0(x, t) |_{Y_f} - \frac{1}{\mu} K^H (\nabla_x p^0(x, t) - b(x, t)).$$

It can be shown that the tensor $K^H$ is symmetric and positive definite. We refer to [Mikelic [20]] for a proof.

By using the expressions for $q^0$ and $p^1$, we also obtain the following which will be useful later:

$$\left(2\mu \mathbf{e}_y(v^0) - p^1 \mathbf{I}\right) \cdot \nu = \sum_{j=1}^3 (2\mathbf{e}_y(\Lambda^j) - \Pi^j \mathbf{I}) \left( b_j - \frac{\partial p^0}{\partial x_j} \right) \cdot \nu \quad \text{on } \Omega \times \Gamma \times J.$$  

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4.3 Momentum conservation

In this section we homogenize the equations for the solid in order to obtain the upscaled conservation of momentum balance equation.

From equations (25) and (28) at order $O(\varepsilon^{-1})$ we obtain:

$$\nabla_y \cdot (\mathbf{C}(\mathbf{e}_x(w^0) + \mathbf{e}_y(w^1)) - \mathbf{M} T^0) = 0,$$

in $\Omega \times Y_s \times J$  \hspace{1cm} (43a)

$$(\mathbf{C}(\mathbf{e}_x(w^0) + \mathbf{e}_y(w^1)) - \mathbf{M} T^0)\nu = -p^0 I\nu,$$

on $\Omega \times \Gamma \times J$,  \hspace{1cm} (43b)

$w^1(x, \cdot, t)$ is $Y$-periodic,  \hspace{1cm} $\forall (x, t) \in \Omega \times J$.  \hspace{1cm} (43c)

Using the tensor outer product (denoted '$\otimes$'), we now make use of the following identity:

$$\mathbf{e}_x(w^0) = \sum_{i,j=1}^{3} \frac{1}{2} \partial w^0_i \partial x_j (\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i),$$

such that we can use $\partial w^0_i \partial x_j$ as scalars in the expression for $w^1$:

$$w^1(x,y,t) = \sum_{i,j=1}^{3} \frac{1}{2} \partial w^0_i \partial x_j (x,t) U_{ij} + T^0(x,t) V(y) + p^0(x,t) W(y),$$

(44)

where the functions $U_{ij}$, $V$, and $W$, $(U_{ij}(y), V(y), W(y) \in \mathbb{R}^3)$, are determined by the following cell-problems (for $i,j = 1, 2, 3$):

$$\nabla_y \cdot (\mathbf{C} \mathbf{e}_y(U_{ij})) = 0,$$

in $Y_s$,  \hspace{1cm} (45a)

$$\mathbf{C} \left( \mathbf{e}_y(U_{ij}) + \frac{\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i}{2} \right) \nu = 0,$$

on $\Gamma$,  \hspace{1cm} (45b)

$U_{ij}$ is $Y$-periodic,  \hspace{1cm} (45c)

and

$$\nabla_y \cdot (\mathbf{C} \mathbf{e}_y(V)) = 0,$$

in $Y_s$,  \hspace{1cm} (46a)

$$\mathbf{C} \mathbf{e}_y(V) \nu = \mathbf{M} \nu,$$

on $\Gamma$,  \hspace{1cm} (46b)

$V$ is $Y$-periodic,  \hspace{1cm} (46c)

and

$$\nabla_y \cdot (\mathbf{C} \mathbf{e}_y(W)) = 0,$$

in $Y_s$,  \hspace{1cm} (47a)

$$\mathbf{C} \mathbf{e}_y(W) \nu = -I\nu,$$

on $\Gamma$,  \hspace{1cm} (47b)

$W$ is $Y$-periodic.  \hspace{1cm} (47c)

We now continue with the solid at order $O(\varepsilon^0)$, where we make use of the expression (42) from the last section. We thus obtain the following problem:

$$\nabla_x \cdot \left( (\mathbf{C}(\mathbf{e}_x(w^0) + \mathbf{e}_y(w^1)) - \mathbf{M} T^0) \right) + \mathbf{b} = -\nabla_y \cdot \left( \mathbf{C}(\mathbf{e}_x(w^1) + \mathbf{e}_y(w^2)) - \mathbf{M} T^1_s \right),$$

in $\Omega \times Y_s \times J$,  \hspace{1cm} (48a)

$$(\mathbf{C}(\mathbf{e}_x(w^1) + \mathbf{e}_y(w^2)) - \mathbf{M} T^1_s)\nu = \sum_{j=1}^{3} \left( 2 \mathbf{e}_y(A^j) - \Pi^j \right) \left( b_j - \frac{\partial p^0}{\partial x_j} \right) \nu,$$

on $\Omega \times \Gamma \times J$,  \hspace{1cm} (48b)

$w^2(x, \cdot, t)$ is $Y$-periodic,  \hspace{1cm} $\forall (x, t) \in \Omega \times J$.  \hspace{1cm} (48c)
Integrating the right side over $Y_s$, using also equation (38a), yields:

$$-\int_{Y_s} \nabla_y \cdot \left( C(e_x(w)) + e_y(w^2) \right) - MT_s dy = \int_{\Gamma} (C(e_x(w)) + e_y(w^2)) - MT_s dy ds_y$$

$$= \sum_{j=1}^{3} \int_{Y_f} \nabla_y \Pi^j - \Delta_y \lambda^j \left( \frac{\partial P^0}{\partial x_j} - b_j \right) = \sum_{j=1}^{3} \int_{Y_f} e_j dy \left( \frac{\partial P^0}{\partial x_j} - b_j \right) = (\nabla_x P^0 - b)|Y_f|.$$  

Using the expression for $w^1$, equation (44), we get for the left hand side:

$$b |Y_s| + \nabla_x \cdot \left( \int_{Y_s} (C(e_x(w)) + e_y(w^2)) - MT^0 dy \right)$$

$$= b |Y_s| + \nabla_x \cdot \left( \int_{Y_s} C e_y(W) dy \right) + \nabla_x \cdot \left( T^0 \int_{Y_s} C e_y(V) - M dy \right)$$

$$+ \nabla_x \cdot \left( \sum_{ij=1}^{3} \int_{Y_s} C \left( e_y(U_{ij}(y)) + \frac{e_i \otimes e_j + e_j \otimes e_i}{2} \right) dy \right).$$

Putting the two sides together gives the upscaled momentum equation:

$$-\nabla_x \cdot \left( A^H e_x(w^0) - (|Y_f| I - B^H)P^0 - (|Y_s| M - U^H)T^0 \right) = b, \quad \text{in } \Omega \times J, \quad (48)$$

where (for $i, j, k, l, = 1, 2, 3$):

$$\left( A^H \right)_{ijkl} = \int_{Y_s} C \left( e_y(U_{ij}(y)) + \frac{e_i \otimes e_j + e_j \otimes e_i}{2} \right)_{kl} dy, \quad (49)$$

$$\left( B^H \right)_{ij} = \int_{Y_s} C \left( e_y(W(y)) \right)_{ij} dy, \quad (50)$$

$$\left( U^H \right)_{ij} = \int_{Y_s} C \left( e_y(V(y)) \right)_{ij} dy. \quad (51)$$

It can be shown that the effective tensors $A^H$ and $B^H$ are symmetric and positive definite. We refer to (Sanchez-Palencia [27]) for a proof. That $U^H$ is symmetric and positive definite is shown the same way as for $B^H$, except that it now relies on the same properties for $M$.

### 4.4 Mass conservation

In order to derive the upscaled mass conservation equation, we take take terms of $O(\varepsilon^0)$ from equation (27), together with $O(\varepsilon^1)$ terms from the boundary condition (29), and obtain the following problem:

$$\nabla_y \cdot v^1 = -\nabla_x \cdot v^0, \quad \text{in } \Omega \times Y_f \times J, \quad (52a)$$

$$v^1 = \partial_t w^1, \quad \text{on } \Omega \times \Gamma \times J, \quad (52b)$$

$$v^1(x, \cdot, t) \text{ is } Y\text{-periodic}, \quad \forall (x, t) \in \Omega \times J. \quad (52c)$$

Integrating the left hand side of the first equation over $Y_f$, and using the expression for $w^1$, equation
(44), yields:
\[
\begin{align*}
\int_{Y_f} \nabla_y \cdot v^1 dy &= - \int_{Y_s} \nabla_y \cdot \partial_t w^1 dy \\
&= - \partial_t \left( \sum_{i,j=1}^3 \frac{\partial w^0}{\partial x_j} \int_{Y_s} \nabla_y \cdot U^{ij} dy + T^0 \int_{Y_s} \nabla_y \cdot V dy + p^0 \int_{Y_s} \nabla_y \cdot W dy \right) \\
&= - D^H : e_x (\partial_t w^0) - \partial_t T^0 E^H + \partial_t p^0 G^H,
\end{align*}
\]
where:
\[
\begin{align*}
D^H_{ij} &= \int_{Y_s} \nabla_y \cdot U^{ij} dy, \\
E^H &= \int_{Y_s} \nabla_y \cdot V dy, \\
G^H &= - \int_{Y_s} \nabla_y \cdot W dy.
\end{align*}
\]

Integrating the right hand side over \(Y_f\), and using the expression for the average of \(v^0\), equation (41), yields:
\[
- \nabla_x \cdot \left( \int_{Y_f} v^0 dy \right) = - \nabla_x \cdot \left( \partial_t w^0 |_{Y_f} + q \right).
\]

Putting the two sides together, we obtain the upscaled mass conservation equation:
\[
D^H : e_x (\partial_t w^0) + \partial_t T^0 E^H - \partial_t p^0 G^H = \nabla_x \cdot \left( \partial_t w^0 |_{Y_f} + q \right). \tag{53}
\]

**Lemma 1.** \(G^H > 0\)

**Proof.** Test with \(W\) in the the cell problems (46):
\[
G^H = - \int_{Y_s} \nabla_y \cdot W dy = \int_{Y_s} C e_y(W) : e_y(W) dy > 0 \tag{54}
\]

**Lemma 2.** \(D^H = B^H\)

**Proof.** Test with \(U^{ij}\) in the cell problem (46):
\[
D^H_{ij} = \int_{Y_s} \nabla_y \cdot U^{ij} dy = - \int_{Y_s} C e_y(W) : e_y(U^{ij}) dy, \tag{55}
\]

Test with \(W\) in cell problem (45):
\[
B^H_{ij} = \int_{Y_s} (C e_y(W))_{ij} dy = - \int_{Y_s} C e_y(U^{ij}) : e_y(W) dy. \tag{56}
\]

We can now rewrite equation (53) as:
\[
\partial_t (p^0 G^H - T^0 E^H) + \nabla_x \cdot \left( |Y_f| (I - B^H) \partial_t w^0 + q \right) = 0 \tag{57}
\]
4.5 Energy conservation

In this section we homogenize the equations for the heat transfer in the solid and fluid, in order to obtain the upscaled energy conservation equation.

We take terms of order $O(\varepsilon^{-1})$ from equations (30), (31) and (32), and obtain the following problem:

\[
\nabla_y \cdot (K_f (\nabla_x T^0 + \nabla_y T^1_f)) = 0, \quad \text{in} \quad \Omega \times Y_f \times J, \quad (58a)
\]

\[
\nabla_y \cdot (K_s (\nabla_x T^0 + \nabla_y T^1_s)) = 0, \quad \text{in} \quad \Omega \times Y_s \times J, \quad (58b)
\]

\[
K_f (\nabla_x T^0 + \nabla_y T^1_f) \cdot \nu = K_s (\nabla_x T^0 + \nabla_y T^1_s) \cdot \nu, \quad \text{on} \quad \Omega \times \Gamma \times J, \quad (58c)
\]

\[
T^1_f = T^1_s \quad \text{on} \quad \Omega \times \Gamma \times J, \quad (58d)
\]

$T^1_f(x, \cdot, t)$ and $T^1_s(x, \cdot, t)$ are $Y$-periodic, \quad $\forall (x, t) \in \Omega \times J. \quad (58e)$

Using the identity $\nabla_x T^0 = \sum_{j=1}^{3} \frac{\partial T^0}{\partial x_j} \nu_j$, we can solve for $T^1_f$ and $T^1_s$ as:

\[
T^1_f(x, y, t) = \sum_{j=1}^{3} \frac{\partial T^0(x, t)}{\partial x_j} \nu_j(y) \quad \text{and} \quad T^1_s(x, y, t) = \sum_{j=1}^{3} \frac{\partial T^0(x, t)}{\partial x_j} \nu_j(y), \quad (59)
\]

where $\nu_j$ and $\nu_j(y)$ ($\nu_j(y) \in \mathbb{R}$) are determined by (for $j = 1, 2, 3$):

\[
\nabla_y \cdot (K_f \nabla_y \nu_j) = 0, \quad \text{in} \quad Y_f, \quad (58a)
\]

\[
\nabla_y \cdot (K_s \nabla_y \nu_j) = 0, \quad \text{in} \quad Y_s, \quad (58b)
\]

\[
K_f (\nabla_y \nu_j) \cdot \nu = K_s (\nabla_y \nu_j) \cdot \nu, \quad \text{on} \quad \Gamma, \quad (58c)
\]

\[
\nu_j = \nu_j, \quad \text{on} \quad \Gamma, \quad (58d)
\]

$\nu_j$ and $\nu_j(y)$ are $Y$-periodic.

By defining:

\[
\nu_j(y) = \begin{cases} 
\nu_j(y), & \text{if } y \in Y_f, \\
\nu_j(y), & \text{if } y \in Y_s,
\end{cases} \quad (60)
\]

due to the boundary condition, and using the properties of $K_s$ and $K_f$, we can write the more convenient problem:

\[
\Delta_y \nu_j = 0, \quad \text{in} \quad Y_s \cup Y_f, \quad (61a)
\]

\[
(\nabla_y \nu_j) \cdot \nu = 0, \quad \text{on} \quad \Gamma, \quad (61b)
\]

$\nu_j$ is $Y$-periodic. \quad (61c)

Continuing to the next order, $O(\varepsilon^0)$, we obtain the problem:

\[
v^0 \cdot \nabla_x T^0 + v^0 \cdot \nabla_y T^1_f - \nabla_x \cdot (K_f (\nabla_x T^0 + \nabla_y T^1_f)) = \nabla_y \cdot (K_f (\nabla_x T^0 + \nabla_y T^1_f)), \quad \text{in} \quad \Omega \times Y_f \times J, \quad (61a)
\]

\[
M : (e_x (\partial_t w^1) + e_y (\partial_t w^1)) - \nabla_x \cdot (K_s (\nabla_x T^0 + \nabla_y T^1_s)) = \nabla_y \cdot (K_s (\nabla_x T^0 + \nabla_y T^1_s)), \quad \text{in} \quad \Omega \times Y_s \times J, \quad (61b)
\]

\[
K_f (\nabla_x T^1_f + \nabla_y T^2_f) \cdot \nu = K_s (\nabla_x T^1_s + \nabla_y T^2_s) \cdot \nu, \quad \text{on} \quad \Omega \times \Gamma \times J, \quad (61c)
\]

\[
T^2_f = T^2_s \quad \text{on} \quad \Omega \times \Gamma \times J, \quad (61d)
\]

$T^2_f(x, \cdot, t)$ and $T^2_s(x, \cdot, t)$ are $Y$-periodic, \quad $\forall (x, t) \in \Omega \times J.$
Integrating the first equation over $Y_f$, and using the expressions for $T^1_f$ and the average of $v^0$, equations (59) and (41), together with the boundary conditions (58d) and (35a) yields:

\[
\int_{\Gamma} K_f(\nabla_x T_f^1 + \nabla_y T_f^2) \cdot \nu ds \\
= (\{Y_f|\partial_t w^0 + q \cdot \nabla_x T^0 + \int_{Y_f} v^0 \cdot \nabla_y T_f^1 dy - \nabla_x \cdot (\int_{Y_f} K_f(\nabla_x T^0 + \nabla_y T_f^1) dy)) \\
= q \cdot \nabla_x T^0 + \partial_t w^0 \cdot \sum_{j=1}^{3} \frac{\partial T^0}{\partial x_j} \int_{Y_f} e_j + \nabla_y \theta_f^j dy - \nabla_x \cdot \left(\sum_{j=1}^{3} \frac{\partial T^0}{\partial x_j} \int_{Y_f} K_f(e_j + \nabla_y \theta_f^j) dy\right)
\]

Integrating the second equation over $Y_s$, using also the expressions for $T^1_s$ and $w^1$, equations (59) and (44), yields:

\[
\int_{\Gamma} K_s(\nabla_x T_f^1 + \nabla_y T_f^2) \cdot \nu ds \\
= \nabla_x \cdot \left(\int_{Y_s} K_s(\nabla_x T^1_s + \nabla_y T_s^2) dy\right) - |Y_s| M : e_x(\partial_t w^0) - \int_{Y_s} M : e_y(\partial_t w^1) dy \\
= \nabla_x \cdot \left(\sum_{j=1}^{3} \frac{\partial T^0}{\partial x_j} \int_{Y_s} K_s(e_j + \nabla_y \theta_s^j) dy\right) - |Y_s| M : e_x(\partial_t w^0) \\
- e_x(\partial_t w^0) \cdot \sum_{i,j=1}^{3} \int_{Y_s} M : e_y(U^{ij}) dy - \partial_t T^0 \int_{Y_s} M : e_y(V) dy - \partial_t \nu^0 \int_{Y_s} M : e_y(W) dy
\]

Since the left hand sides of the two above equations are equal, we put them together and obtain the upscaled energy conservation equation:

\[
\partial_t T^0 M^H + \partial_t \nu^0 N^H + \partial_t w^0 \cdot \Xi^H \nabla_x T^0 + q \cdot \nabla_x T^0 + \nabla_x \cdot ((R^H + |Y_s| M)\partial_t w^0 - \Theta^H \nabla_x T^0) = 0
\]

where (for $i, j = 1, 2, 3$):

\[
(\Theta^H)_{ij} = \int_{Y_f} (K_f(e_j + \nabla_y \theta_f^j))_i dy + \int_{Y_s} (K_s(e_j + \nabla_y \theta_s^j))_i dy,
\]

\[
(\Xi^H)_{ij} = \int_{Y_f} (e_j + \nabla_y \theta_f^j)_i dy,
\]

\[
(R^H)_{ij} = \int_{Y_s} M : e_y(U^{ij}) dy,
\]

\[
M^H = \int_{Y_s} M : e_y(V) dy,
\]

\[
N^H = \int_{Y_s} M : e_y(W) dy.
\]

Lemma 3. $R^H = -U^H$

Proof. Test with $U^{ij}$ in the cell problem (46) to obtain:

\[
(R^H)_{ij} = \int_{Y_s} M : e_y(U^{ij}) dy = \int_{Y_s} C e_y(V) : e_y(U^{ij}) dy.
\]

Test now the cell problem (45) with $V$:

\[
(U^H)_{ij} = \int_{Y_s} (C e_y(V))_{ij} dy = -\int_{Y_s} C e_y(U^{ij}) : e_y(V) dy
\]
Lemma 4. \( M^H > 0 \)

Proof. Test with \( V \) in the cell problems (46) to obtain:
\[
M^H = \int_{Y_s} M : \mathbf{e}_y(V)\,dy = \int_{Y_s} C \mathbf{e}_y(V) : \mathbf{e}_y(V)\,dy > 0
\]

\[\square\]

Lemma 5. \( N^H = -E^H \)

Proof. Test with \( V \) in the cell problem (47):
\[
E^H = \int_{Y_s} \nabla_y \cdot V\,dy = -\int_{Y_s} C \mathbf{e}_y(W) : \mathbf{e}_y(V)\,dy,
\]
test also with \( W \) in the cell problem (47):
\[
N^H = \int_{Y_s} M : \mathbf{e}_y(W)\,dy = \int_{Y_s} C \mathbf{e}_y(V) : \mathbf{e}_y(W)\,dy.
\]

\[\square\]

Lemma 6. \( \Theta^H \) and \( \Xi^H \) are symmetric and positive definite.

Proof. Test with \( \theta^i \) in the \( j \)'th cell problem (61), and by \( \theta^j \) in the \( i \)'th problem to obtain:
\[
\int_{Y_f} K_f(e_j + \nabla_y \theta^j) \cdot \nabla_y \theta^i \,dy = \int_{Y_f} K_f(e_i + \nabla_y \theta^i) \cdot \nabla_y \theta^j \,dy = 0,
\]
and
\[
\int_{Y_s} K_s(e_j + \nabla_y \theta^j) \cdot \nabla_y \theta^i \,dy = \int_{Y_s} K_s(e_i + \nabla_y \theta^i) \cdot \nabla_y \theta^j \,dy = 0.
\]
Thus, we can write \( \Theta^H \) as:
\[
(\Theta^H)_{ij} = \int_{Y_f} K_f(e_j + \nabla_y \theta^j) \cdot (e_i + \nabla_y \theta^i) \,dy + \int_{Y_s} K_s(e_j + \nabla_y \theta^j) \cdot (e_i + \nabla_y \theta^i) \,dy,
\]
and it follows that \( \Theta^H \) is symmetric. For the positive definiteness, observe that for non-negative \( \alpha_{1,2,3} \in \mathbb{R} \) not all equal to zero we have:
\[
\sum_{i,j=1}^{3} (\Theta^H)_{ij} \alpha_i \alpha_j = \sum_{i,j=1}^{3} \int_{Y_f} K_f \nabla_y (\alpha_j(y_j + \theta^j)) \cdot \nabla_y (\alpha_i(y_i + \theta^i)) \,dy
\]
\[
+ \sum_{i,j=1}^{3} \int_{Y_s} K_s \nabla_y (\alpha_j(y_j + \theta^j)) \cdot \nabla_y (\alpha_i(y_i + \theta^i)) \,dy > 0.
\]
That \( \Xi^H \) is symmetric and positive definite is shown in the same way.

\[\square\]

We can now rewrite the upscaled energy conservation equation as:
\[
\partial_t (T^0 M^H - p^0 E^H) + (\Xi^H \partial_t w^0 + q) \cdot \nabla_x T^0 + \nabla_x \cdot \left( (|Y_s| M - U^H) \partial_t w^0 - \Theta^H \nabla_x T^0 \right) = 0. \quad (62)
\]
5 Conclusion

5.1 The upscaled quasi-static poro-thermo-elastic system

In this section we summarize the homogenized equations derived in the previous sections. We omit now all superscripts in the variables, subscripts in the differential operators (with the understanding they are now all taken with respect to the slow variable $x$), and introduce a more familiar notation for the coefficients, similar to what is normally used in the quasi-static poro-elastic equations:

\[
\begin{align*}
\alpha &:= (|Y_f| I - B^H), & \beta &:= (|Y_f| M - U^H), & A^H &:= A \\
K &:= K^H, & \Theta &:= \Theta^H, & \Xi &:= \Xi^H, \\
c_0 &:= G^H, & a_0 &:= M^H, & b_0 &:= E^H,
\end{align*}
\]

where $\alpha$ is the Biot-Willis constant, and $c_0$ is the specific storage coefficient, discussed in the introduction, and $A$ is the fourth order Gassman tensor, containing the elastic coefficients of the porous medium.

Thus, we write the upscaled system as:

\[
\begin{align*}
q &= -\frac{1}{\mu} K(\nabla p - b), & \text{in } \Omega \times J, & (63a) \\
- \nabla \cdot (A \epsilon(w) - \alpha p - \beta T) &= b, & \text{in } \Omega \times J, & (63b) \\
\partial_t(c_0p - b_0T + \nabla \cdot \alpha w) + \nabla \cdot q &= 0, & \text{in } \Omega \times J, & (63c) \\
\partial_t(a_0T - b_0p + \nabla \cdot \beta w) + (\Xi \partial_t w + q) \cdot \nabla T - \nabla \cdot (\Theta \nabla T) &= 0, & \text{in } \Omega \times J. & (63d)
\end{align*}
\]

We can now return to the discussion in section 2.1, where we presented the linear poro-elastic system, and compare our equations with these. We see that the stress in the momentum equation (63b) now has an additional linear dependency on the temperature of the medium, i.e. $\sigma = \sigma(w, p, T) = A \epsilon(w) - \alpha p - \beta T$, just as in the linear thermo-elastic equations in mechanics. The homogenized tensor $\beta$ can be interpreted as an upscaled thermal stress coefficient, giving the induced thermal stress coming from a unit temperature gradient. In the mass conservation equation (63c) we see that the water content is also linearly dependent on the temperature, i.e. $\eta = \eta(w, p, T) = c_0p - b_0T + \nabla \cdot \alpha w$. In other words, the amount of fluid that can be injected into an arbitrary fixed control volume is now given by: $c_0p - b_0T$, where the homogenized coefficient $b_0$ can be interpreted as a thermal expansion coefficient. The linear temperature dependency of the stress and of the water content is quite reasonable, and is what could be expected when including thermal effects in the linear poroelastic equations.

It remains to discuss the energy conservation equation (63d). If we had used the different scaling corresponding to a small Péclet number, i.e. $Pe \sim O(\varepsilon)$, the dimensionless energy conservation equation for the fluid at the micro scale, equation (24g), would take the form:

\[
\varepsilon \left( \frac{\partial T_f^\varepsilon}{\partial t} + \partial_t w^\varepsilon \cdot \nabla T_f^\varepsilon \right) = 0, & \text{in } \Omega_f^\varepsilon \times J. & (64)
\]

Then, after separating the scales, the terms: $\varepsilon \left( \frac{\partial T_f^\varepsilon}{\partial t} + \partial_t w^\varepsilon \cdot \nabla T_f^\varepsilon \right)$ give no contribution to the $O(\varepsilon^{-1})$-problem, and for the $O(\varepsilon^0)$-problem, we only retain the term: $\partial_t \nu^0 \cdot \nabla \cdot \nu^0 T^0$, which is evidently equal to zero. Thus, the upscaled energy conservation equation corresponding to a small Péclet number is:

\[
\partial_t(a_0T - b_0p + \nabla \cdot \beta w) - \nabla \cdot (\Theta \nabla T) = 0, & \text{in } \Omega \times J, & (65)
\]

and we have a fully linear upscaled system.

Denoting by: $\xi = \xi(w, p, T) = a_0T - b_0p + \nabla \cdot \beta w$, the energy present in some arbitrary control volume, we see from equation (63d) that the rate of change of energy present, $\partial_t \xi$, is balanced by the net energy flux into the same control volume, either by conduction: $-\nabla \cdot (\Theta \nabla T)$, or convection:
\((\Xi \partial_t w + q) \cdot \nabla T\). We see also from equation (64) that in the case of a small Péclet number, the rate of change in energy present is balanced only by the conduction. The homogenized tensors \(\Theta\) and \(\Xi\) can be interpreted as a kind of upscaled thermal conductivities, while \(a_0\) gives the energy present by a unit temperature rate of change.

References


