# Implicit schemes for 2D non-linear problems: Performance and efficiency assessment of two solvers

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## Introduction

A wide variety of problems in applied mathematics and physics require the solution of large linear systems of equations having a penta-diagonal matrix structure. In particular, the numerical resolution in a rectangular mesh of differential equations or systems of differential equations in 2D using techniques of implicit temporal discretization is characterized by this structure of five diagonals within a sparse matrix. Implicit schemes are chosen in order to relax the restriction imposed by explicit schemes over the computational time step. the particular solver chosen to deal with the resulting penta-diagonal matrix can be determinant in the final efficiency of the computational model. this is specially crucial when solving 2D systems of non-linear equations.

In this work, two widely used ways to find the solution of the system of equations are explored and compared in terms of performance and efficiency: Penta-Diagonal Matrix Algorithm (PDMA) [1] and BiConjugate Gradient Stabilized (BiCGStab) [2]. As the nature of the chosen equations has a non-linear character, a linearization procedure is required as a previous step to the linear solver.

Three 2D non-linear problems are studied in this work, commonly used in the framework of fluid mechanics and, more specifically, within the surface flow characterization: 1) Burgers' inviscid equation, 2) Zero-Inertia equation and 3) Shallow Water system of equations. All of them are solved by means of a first-order finite volume scheme using an implicit temporal discretization. This particular technique generates a matrix containing a number of differential equations equal to the number of computational cells multiplied by the number of variables per cell. This usually implies that huge matrices appears even for small discretized domains. Hence, an efficient linear solver is mandatory in order to keep the efficiency of the implicit numerical method.

## Brief description of the solvers

**PDMA:** Penta-Diagonal Matrix Algorithm splits the problem in $x$ and $y$ directions and solves each one independently, essentially being a double Tri-Diagonal Matrix Algorithm (TDMA), often termed as Thomas algorithm [1], which is a simplified form of Gaussian elimination.

**BiCGStab:** The BiConjugate Gradient Stabilized method is an iterative method developed by [2] for the numerical solution of non-symmetric linear systems. It belongs to the Krylov subspace family of methods and has faster and smoother convergence than the original BiConjugate Gradient as well as many other iterative methods.

## Application to physical models

Three 2D non-linear problems are considered in this work to test the efficiency of the solvers:

1) Inviscid Burgers' equation:

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{f} = 0, \qquad \mathbf{f} = \left( \frac{u^2}{2}, \frac{u^2}{2} \right) \tag{1}$$

with $u$ representing a generic variable and $\mathbf{f}$ the flux vector.

2) Zero-inertia model:

$$\frac{\partial h}{\partial t} + \nabla \cdot \mathbf{q} = 0, \qquad \mathbf{q} = \left( \frac{h^{5/3}}{n\sqrt{|\mathbf{S}|}} S_x, \frac{h^{5/3}}{n\sqrt{|\mathbf{S}|}} S_y \right) \tag{2}$$

being $h$ the water depth, $\mathbf{q}$ the unitary discharge vector along the ($x$,$y$) coordinates, $n$ the Manning roughness coefficient and $\mathbf{S}$ the water surface slope vector:

$$\mathbf{S} = -\nabla(h + z) \tag{3}$$

with $z$=topographic bed elevation.

3) Shallow Water equations:

$$\mathbf{U} = (h, q_x, q_y)^T, \qquad q_x = hv_x, \qquad q_y = hv_y \tag{4}$$

$$\mathbf{F} = \left(q_x, \frac{q_x^2}{h} + \frac{1}{2}gh^2, \frac{q_x q_y}{h}\right)^T, \qquad \mathbf{G} = \left(q_y, \frac{q_x q_y}{h}, \frac{q_y^2}{h} + \frac{1}{2}gh^2\right)^T \tag{5}$$

$$\mathbf{S} = (0, gh\left(S_{0x} - S_{fx}\right), gh\left(S_{0y} - S_{fy}\right))^T \tag{6}$$

with ($v_x$,$v_y$) the depth-averaged components of the velocity vector along the ($x$,$y$) coordinates.

The slope terms are defined as $S_{0x} = -\frac{\partial z}{\partial x}$, $S_{0y} = -\frac{\partial z}{\partial y}$ and the friction-associated loses are calculated as $S_{fx} = \frac{n^2 v_x \sqrt{v_x^2+v_y^2}}{h^{4/3}}$, $S_{fy} = \frac{n^2 v_y \sqrt{v_x^2+v_y^2}}{h^{4/3}}$.

## Numerical example: Flood over complex terrain

This test presents the simulation of a flloding wave over a complex terrain with several obstacles. The $50m \times 30m$ domain is discretized by means of a $\sim 1500$ cells mesh (see Figure 1). A constant Manning's roughness value of $0.03sm^{-1/3}$ is imposed and $h = 0m$ is set as initial condition for the entire domain. All the boundaries are closed except the left one in which a constant water depth of $h = 0.3m$ is assumed. This particular case is simulates by means of the implicit Shallow Water model. Figure 2 shows the evolution of the water depth $h$ at $t = 3s$, $t = 9s$ and $t = 600s$.

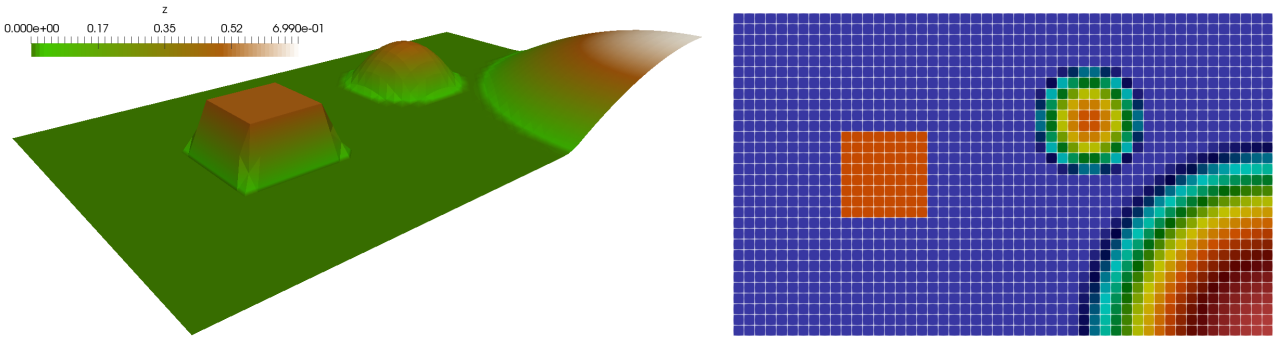


*Figure 1:* 3D representation of the domain (left) and rectangular computational mesh (right).

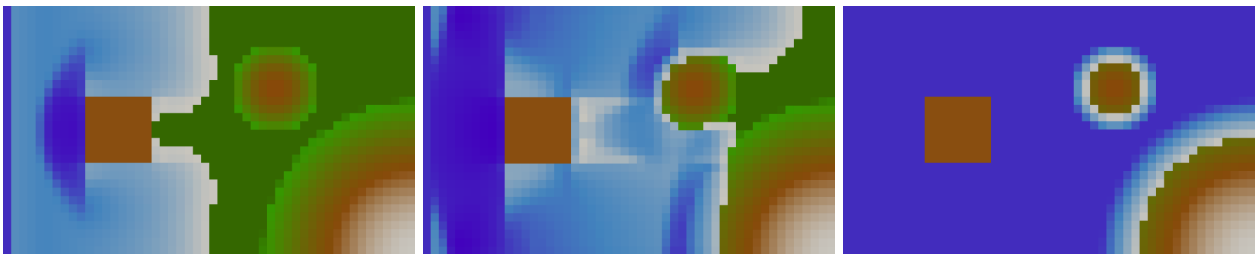

*Figure 2:* Water depth ($h$) at $t = 3s, 9s$ and $600s$

## References

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