Zero-Inertia vs full shallow water equations: a comparison for rainfall-runoff modelling

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Introduction

Rainfall generated runoff is one of the key processes in Hydrology. Classically, and in many well-established hydrological models, the rainfall-runoff transformation is performed in a lumped, or at best, semi-distributed fashion, relying strongly on calibrated parameters. Arguably, physically-based, fully-distributed models can result in more robust, better-grounded, and more rigorous results [1]. Moreover, by actually representing and solving the actual physical processes at their necessary spatiotemporal scales, it becomes feasible to use such models as virtual laboratories that allow to test hypothesis and not only forecast behaviours, but to explain them [e.g. 1–3]. A wide knowledge base exists already on physically-based surface water modelling [4, 5], and indeed these models are now being used to replace rainfall-runoff transformations in hydrological models [1, 3]. Nonetheless, several approximations with different levels of physical and mathematical complexity, and therefore numerical and computational properties and costs, have been proposed. It is well-accepted that shallow flows are best represented by the shallow water equations (SWE), but approximations such as the zero-inertia (ZI) equation have received great attention and have been widely used [6–9], although mostly in the river-flooding context. In this work, the goal is to explore the applicability, accuracy and computational efficiency of the SWE and the ZI equation as models for rainfall-generated runoff on hillslopes and catchments, as to ascertain which model might be preferable in this context. To to this, both models were computationally implemented in equivalent formulations –as far as possible– for fair comparison, and were tested using the same discretisation and parametrisations on a set of growing-complexity benchmark tests with reference solutions.

Mathematical and numerical models

We solve the shallow water equations (SWE) in conservation form,

$$\begin{bmatrix} h \\ hu \\ hv \end{bmatrix} + \nabla \begin{bmatrix} h\mathbf{u} \\ hu^2 + huv + \dfrac{gh^2}{2} \\ hv^2 + huv + \dfrac{gh^2}{2} \end{bmatrix} = \begin{bmatrix} r + i \\ -gh\dfrac{\partial z}{\partial x} - gh\sigma_x \\ -gh\dfrac{\partial z}{\partial y} - gh\sigma_y \end{bmatrix} \qquad \begin{aligned} \sigma_x &= \frac{n^2 u\sqrt{u^2+v^2}}{h^{4/3}} \\ \sigma_y &= \frac{n^2 v\sqrt{u^2+v^2}}{h^{4/3}} \end{aligned} \tag{1}$$

with water depth $h\,[L]$, u and v, the cartesian components of the velocity vector $\mathbf{u}$ $[L/T]$, g is the acceleration of gravity $[L/T^2]$, z is bed elevation $[L]$. Friction slopes σ are typically approximated using Manning's law parametrized by Manning's roughness coefficient n $[T/L^{1/3}]$. i is the infiltration (when negative, or exfiltration rate when positive) $[L/T]$.

We also solve the zero-inertia (ZI) approximation –also often termed diffusive-wave approximation– to the SWE, obtained from neglecting acceleration terms in (1), thus obtaining a steady momentum equation. Considering Manning's friction law, the ZI equation is

$$\frac{\partial h}{\partial t} + \nabla\left(\frac{h^{5/3}}{n\sqrt{||\mathbf{Z}||}}\mathbf{Z}\right) = i \qquad \mathbf{Z} = \nabla(h+z) \tag{2}$$

The simplification inherent to the ZI approximation has implications, firstly in the applicability of this equation to generic shallow flows. It is clear that the ZI assumption is valid for potential-driven flow rather than kinetically-driven flows. It is also reasonable that the simplification has an impact on the accuracy of the computations, as compared to SWE results. In this work we explore this issues, specifically in the context of rainfall-runoff problems.

In terms of the numerical scheme, both the SWE (1) and the ZI (2) equation are discretised in space by means of a 2D first-order finite volume scheme. The scheme is implemented for general meshes but herein it is only used for structured cartesian meshes (regular square grids) since ZI solvers are usually implemented for such cases [e.g. 6, 7, 9]. The SWE solver is based on method proposed in [10] which harnesses the hyperbolic nature of the system, approximating edge-wise Riemann problems with the so-called Augmented Roe solver which accounts for the effects of source terms. The FV-SWE scheme is discretised with a first-order explicit approximation and is stable for CFL ≤ 1. The time discretisation scheme for the FV-ZI is equivalent, a first-order explicit scheme. The stability of this scheme is not clear [8]. Theoretically this type of non-linear diffusion equation exhibits stability properties which state that time step size must satisfy $\Delta t \leq \alpha(h)\delta x^{-2}$, but in practice a CFL condition –despite the equation not being hyperbolic– is often used [7, 9].

Performance comparison

We compared the FV-SWE solver against the FV-ZI solver for a series of rainfall-runoff test cases against benchmark solutions, ranging from 1D analytical solutions, to 1D and 2D laboratory experiments and field experiments over real topography. We systematically tested the response of both models under the same parametrisations and spatial discretisation (meshes) and we performed several simulations with various spatial discretisations to assess if mesh resolution could affect the relative response of the models.

The results suggest that on the question of applicability to rainfall-runoff simulations, both models are equally applicable under the studied conditions (in high-slope terrains this might not be the case [11]). In terms of accuracy, most simulations resulted in acceptable accuracy, even for rather coarse meshes. Interestingly, although the SWE exhibits the expected behaviour of accuracy loss proportional to mesh coarsening, the ZI solution does not clearly exhibit this behaviour, but rather seems to have a similar level of global accuracy despite (a certain degree of) mesh coarsening. Further analysis of this behaviour hints that this be related to the diffusive nature of the ZI equation. Finally, in terms of computational efficiency, the results strongly state that an efficient explicit SWE solver outperforms an efficient explicit ZI solver across all mesh sizes. The reason for this is the more-restrictive stability conditions of the ZI equation, despite the fact that per cell, and per time step, the ZI solver requires much less operations than the SWE solver. Only at very coarse meshes are the computational runtimes comparable. This result favours the use of the SWE model, which is already a-priori theoretically favoured due to its theoretical robustness, generality and wider applicability. The main disadvantage of the SWE solver seems to be its mathematical, numerical and computational complexity, i.e., it requires much more effort to understand and implement in comparison to a ZI solver. Nevertheless, the fact that coarse meshes results in ZI solutions that are comparatively better than coarse SWE solutions, together with a comparable runtime for coarse meshes suggests a practical conclusion: ZI solvers may have a niche in rainfall-runoff simulations which can do with coarse meshes, either because of scale, or data resolution. The results in this work suggest that ZI may outperform SWE solvers under such conditions in the context of rainfall-runoff simulation.

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