Poromechanics based on Minimization – Models and Solvers

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Introduction

Classically models coupling mechanical deformation and fluid flow in porous media are stated as PDEs, naturally arising from conservation principles and additional constitutive laws [1]. Recently, for fully saturated porous media, the equivalence of Biot’s equations to a minimization formulation has been discussed [2], yielding an alternative to the standard approach.

In this work, on the one hand, we discuss the applicability of minimization principles as modeling tool for nonlinear poromechanics. And on the other hand, we exploit the minimization formulation and apply optimization methods in order to solve the coupled problem efficiently in a decoupled fashion. In the following, we demonstrate the idea for the linear Biot’s equations written as minimization problem and solved by a simple block coordinate descending method. For nonlinear poromechanics. And on the other hand, we exploit the minimization formulation and apply alternative to the standard approach.

Linear poromechanics based on minimization

We consider a linearly elastic porous medium $\Omega \subset \mathbb{R}^3$, fully saturated by a slightly compressible fluid. Let the mechanical deformation $u$ and the fluid flux $q$, at given time $t$, be defined as the minimizer of an energy potential $\Pi(u, q; t)$, i.e., they solve

$$\text{Find } (u^*, q^*) \in \mathcal{U}_D \times \mathcal{Q}_N \text{ satisfying } \Pi(u^*, q^*; t) = \inf_{(u, q) \in \mathcal{U}_D \times \mathcal{Q}_N} \Pi(u, q; t). \quad (1)$$

for suitably defined function spaces $\mathcal{U}_D$, $\mathcal{Q}_N$ and energy potential $\Pi$. The ideas of the derivation follow from [2].

Function spaces. Let $\Gamma_D \subset \partial \Omega$, $\Gamma_N \subset \partial \Omega$ have non-zero measure. Given prescribed data $u_D$ and $q_N$ for displacement and normal flux, respectively, we define the functional spaces

$$\mathcal{U}_D = \{ u \in H^1(\Omega; \mathbb{R}^d) \mid u = u_D(t) \text{ on } \partial \Gamma_D \},$$

$$\mathcal{Q}_N = \left\{ q \in L^2(\Omega \times (0, t)) \left| \int_0^t q(\tau) d\tau \in H(\text{div}; \Omega), \ q \cdot n = q_N \text{ on } \partial \Omega \times (0, t) \right. \right\},$$

where $L^2(\Omega; \mathbb{R}^d)$ contains all vector-valued, square-integrable functions, and $H^1(\Omega; \mathbb{R}^d)$ and $H(\text{div}; \Omega)$ contain $L^2(\Omega; \mathbb{R}^d)$ functions with weak derivative and weak divergence, respectively.

Energy. For fixed mechanical deformation $u$ and fluid flux $q$, at given time $t$, the energy potential $\Pi$ is identified with the sum of Helmholtz free energies of the skeleton and the fluid, the dissipation functional accounting for viscous dissipation of the fluid flowing through the porous medium, and the work performed by the skeleton and the fluid due to external body forces

$$\Pi(u, q; t) = \int_{\Omega} \psi_s(e(u); t) + \psi_f(e(u), q; t) \, dx + \int_{\Omega} D_f(q; t) \, dx + \int_{\Omega} \mathcal{P}_{\text{ext}}(u, q; t) \, dx, \quad t \in [0, T].$$

For simplicity, in the following, we neglect external forces and set $\mathcal{P}_{\text{ext}} \equiv 0$. Under the above assumptions, the energy densities are defined by quadratic functions

$$\psi_s(\varepsilon(u); t) = \frac{1}{2} \left( 2\mu \varepsilon(u) : \varepsilon(u) + \lambda (\nabla \cdot u)^2 \right),$$

$$\psi_f(\varepsilon(u), q; t) = \frac{M}{2} \left( \frac{m_{\text{eff}}}{\rho_f} - \int_{t \, t_{\text{ref}}} \nabla \cdot q(\tau) d\tau - \alpha \text{tr} \varepsilon(u)(t) \right)^2,$$

$$D_f(q; t) = \int_{0}^{t} \frac{1}{2K} q(\tau) \cdot q(\tau) d\tau.$$

Here, $\mu$ and $\lambda$ are the Lamé parameters, $\varepsilon(u)$ is the symmetric gradient of $u$, $M$ is the compressibility of the fluid combined with the Biot modulus, $m_{\text{eff}}$ is the fluid mass at some arbitrary but fixed reference time $t_{\text{ref}}$, $\rho_f$ is the fluid density and $K$ is the absolute permeability.

Find $(u^*, q^*) \in \mathcal{U}_D \times \mathcal{Q}_N$ satisfying $\Pi(u^*, q^*; t) = \inf_{(u, q) \in \mathcal{U}_D \times \mathcal{Q}_N} \Pi(u, q; t).$
Properties of the Minimization formulation. From convex analysis it follows:

Lemma 1. The energy potential \( \Pi \) is strictly convex, weakly lower semi-continuous and coercive. Hence, problem (1) is well-posed. Furthermore, the unique solution also solves the linear Biot’s model \([\Pi]\), with the fluid pressure \( p \) defined as the \( L^2(\Omega) \) projection of \( M \left( \frac{m_{\text{ref}}}{\rho_f} - \int_{t_{\text{ref}}}^t \nabla \cdot q(\tau) d\tau - \alpha \tau \varepsilon(u) \right) \).

Undrained Splitting scheme via Block Coordinate Descending Methods

In order to solve numerically Biot’s equations written in its classical form, iterative splitting methods are widely used, requiring only the successive solution of the mechanical and the fluid flow subproblems and hence allowing the use of separate simulators for each subproblem. Due to their unconditional stability, the Fixed Stress Splitting and the Undrained Splitting schemes are particularly popular splitting schemes \([3, 4, 5]\). In order to solve the minimization problem (1), we apply an iterative optimization method, which in the same spirit decouples the mechanical and the fluid flow subproblems.

Nonlinear Two-Block Gauss Seidel method. Block coordinate descending methods successively minimize an energy, by sequentially minimizing the energy only with respect to single components while keeping the remaining components fixed. Here, in particular, we apply the Nonlinear Two-Block Gauss Seidel method, where we assign the two independent blocks to be the mechanical deformation \( u \) and the fluid flux \( q \). By this, the application of the optimization method decouples the problem into a mechanical and a fluid flow subproblem. The method reads: Let \((u^0, q^0)\) be an initial guess for fixed time \( t \). Then we iterate until convergence is met, where in each iteration \( k \), given a current guess \((u^{k-1}, q^{k-1})\), we find \((u^k, q^k)\) in two steps. We solve successively

\[
\begin{align*}
\mathbf{u}^k &= \arg\min_{\mathbf{u} \in \mathcal{U}} \Pi(u, q^{k-1}), \\
q^k &= \arg\min_{q \in \mathcal{Q}} \Pi(u^k, q).
\end{align*}
\]

The resulting subproblems are solved exactly by solving two linear problems. In the first step, we solve for the mechanical displacement \( \mathbf{u}^k \), which satisfies for all suitable test functions \( \delta \mathbf{u} \)

\[
\int_{\Omega} 2\mu \varepsilon(\mathbf{u}^k) : \varepsilon(\delta \mathbf{u}) + \lambda \nabla \cdot \mathbf{u}^k \nabla \cdot \delta \mathbf{u} \, dx - \int_{\Omega} \alpha M \left( \frac{m_{\text{ref}}}{\rho_f} - \alpha \nabla \cdot \mathbf{u}^k - \int_{t_{\text{ref}}}^t \nabla \cdot q^{k-1} d\tau \right) \nabla \cdot \delta \mathbf{u} \, dx = 0.
\]

Afterwards, given \( \mathbf{u}^k \), we solve for the fluid flux \( q^k \), which satisfies for all suitable test functions \( \delta q \)

\[
\int_{\Omega} K^{-1} q \cdot \delta q \, dx - \int_{\Omega} M \left( \frac{m_{\text{ref}}}{\rho_f} - \alpha \nabla \cdot \mathbf{u}^k - \int_{t_{\text{ref}}}^t \nabla \cdot q^{k-1} d\tau \right) \nabla \cdot \delta q \, dx = 0.
\]

Lemma 2 (cf. [6]). The Nonlinear Two-Block Gauss Seidel method as described above converges unconditionally.

Lemma 3. The Nonlinear Two-Block Gauss Seidel method as described above is equivalent with the Undrained Splitting scheme, but here written in a non-standard two-field formulation with \((\mathbf{u}, q)\) as independent variables.

Discussion

Combining Lemma 2 and Lemma 3 yields that the Undrained Splitting scheme is unconditionally stable. This result is not new but it demonstrates that the above approach is very promising in the view of the application of minimization principles for general poromechanics. Nonlinear generalizations of the above model can be achieved by modifying the energy potential \( \Pi \). Then applying suitable optimization methods allows to construct efficient solvers for nonlinear poromechanics. In the frame of this work, we discuss the possibilities and limitations of the approach for modeling and solving nonlinear poromechanics.

References