Uniformly accurate numerical schemes for a class of dissipative systems

Philippe Chartier, Mohammed Lemou, Léopold Trémant

Abstract

We consider a class of relaxation problems mixing slow and fast variations which can describe population dynamics models or hyperbolic systems, with varying stiffness (from non-stiff to strongly dissipative), and develop a multi-scale method by decomposing this problem into a micro-macro system where the original stiffness is broken. We show that this new problem can therefore be simulated with a *uniform* order of accuracy using standard explicit numerical schemes. In other words, it is possible to solve the micro-macro problem with a cost independent of the stiffness (a.k.a. uniform cost), such that the error is also uniform. This method is successfully applied to two hyperbolic systems with and without non-linearities, and is shown to circumvent the phenomenon of order reduction.

Keywords: dissipative problem, multi-scale, micro-macro decomposition, uniform accuracy, asymptotic approximation

1 Introduction

We are interested in problems of the form, for $x^\varepsilon(t) \in \mathbb{R}^{d_x}$ and $z^\varepsilon(t) \in \mathbb{R}^{d_z}$,

$$\begin{cases} \dot{x}^\varepsilon = a(x^\varepsilon, z^\varepsilon), & x^\varepsilon(0) = x_0, \\ \dot{z}^\varepsilon = -\dfrac{1}{\varepsilon} A z^\varepsilon + b(x^\varepsilon, z^\varepsilon), & z^\varepsilon(0) = z_0, \end{cases} \tag{1.1}$$

with $\varepsilon \in (0, 1]$ a small parameter, A a diagonal positive matrix with integer coefficients, and a, b are respectively the x-component and the z-component of an analytic map f which smoothly depends on ε. In this paper we shall more often write this problem as

$$\dot{u}^\varepsilon = -\frac{1}{\varepsilon}\Lambda u^\varepsilon + f(u^\varepsilon), \quad u^\varepsilon(0) = u_0, \tag{1.2}$$

where $u = \begin{pmatrix} x \\ z \end{pmatrix}$, $\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$ and $f(u) = \begin{pmatrix} a(x,z) \\ b(x,z) \end{pmatrix}$. We set $d = d_x + d_z$ the dimension of u such that $u \in \mathbb{R}^d$. In particular, the dimension of x^ε can be zero without impacting our results. The map $u \mapsto f(u)$ is assumed to be smooth. Our results do not consider the case where f involves a differential operator in space (i.e. the case of partial differential equations). Nonetheless, two of our examples are discretized hyperbolic partial differential equations (PDEs) for which the method is successfully applied, even though a special treatment is required.

Systems of this kind appear in population dynamics (see [GHM94; AP96; SAAP00; CCS15]), where A accounts for migration (in space and/or age) and a, b for both the demographic and inter-population dynamics. The migration dynamics is quantifiably faster than

the other dynamics involved, which explains the rescaling by ε in the model. When solving this kind of system numerically, problems arise due to the large range of values that ε can take.

Considering a numerical scheme of order $q > 1$, it is obvious by definition that for all ε, there exists a constant $C(\varepsilon)$ and a time-step $\overline{\Delta t}(\varepsilon)$ such that for all $\Delta t < \overline{\Delta t}(\varepsilon)$, the error $E_\varepsilon(\Delta t)$ when solving (1.2) is bounded by

$$E_\varepsilon(\Delta t) \leq C(\varepsilon)\Delta t^q.$$

Assume now that there exists Δt^* such that this scheme is stable for all $\varepsilon \in (0, 1]$ and $\Delta t < \Delta t^*$.[1] Order reduction means that there exist $s < q$ and $C > 0$, both independent of ε such that the uniform error $E(\Delta t) := \sup_\varepsilon E_\varepsilon(\Delta t)$ satisfies

$$\sup_{\varepsilon\in(0,1]} E_\varepsilon(\Delta t) \leq C\Delta t^s,$$

however generally such a constant C can only be found with s much smaller than q. This is a phenomenon known as *order reduction*, observed for instance in [HW96, Section IV.15] or in [HR07]. In order to obtain the same error for all ε, one must either accept this order reduction (if $s > 0$), as is done for asymptotic-preserving (AP) schemes (see for instance [Jin99]), or use an ε-dependent time-step $\Delta t = \mathcal{O}(\varepsilon^\alpha)$ for some $\alpha > 0$. In practice, both approaches cause the computational cost of the simulation to increase greatly, often prohibitively so.

Another handy way to tackle this is to invoke the *center manifold theorem* (see [Vas63; Car82; Sak90]) which dictates the long-time behaviour of the system and presents useful characteristics for numerical simulations: the dimension is reduced and the dynamics on the manifold is non-stiff. However, this approach does not capture the *transient solution* of the problem, i.e. the solution in short time before it reaches the stable manifold. This is troublesome when one wishes to describe the system after a sudden perturbation of environment (e.g. a change of A, a, b or a sudden drop in population). Furthermore, even if the solution is close to the manifold, these approximations are accurate up to a certain order $\mathcal{O}(\varepsilon^n)$, rendering them useless if ε is of the order of 1.

We first provide a systematic way to compute asymptotic models at any order in ε that approach the solution uniformly *even in short time*. Then we use the defect of this approximation to compute the solution with usual explicit numerical schemes and *uniform* accuracy (i.e. the cost and error of the scheme must be independent of ε). This approach automatically overcomes the challenges posed by both extremes $\varepsilon \ll 1$ and $\varepsilon \sim 1$.

In order to achieve this goal, for any non-negative integer k we construct a change of variable for the dissipative problem (1.2), $(\tau, u) \in \mathbb{R}_+ \times \mathbb{R}^d \mapsto \Omega_\tau^{[n]}(u) \in \mathbb{R}^d$, and a *non-stiff* vector field $u \in \mathbb{R}^d \mapsto F^{[n]}(u) \in \mathbb{R}^d$, such that

$$u^\varepsilon(t) = \Omega_{t/\varepsilon}^{[n]}\left(v^{[n]}(t)\right) + w^{[n]}(t) \tag{1.3}$$

where $v^{[n]}$ is the *macro* component dictated by the equation $\partial_t v^{[n]} = F^{[n]}(v^{[n]})$, and $w^{[n]}$ is the *micro* component of size $\mathcal{O}(\varepsilon^{n+1})$. The main result we prove is that from this decomposition, it is possible to compute u^ε with *uniform accuracy* up to order $n+1$ when using explicit

[1] In particular, the scheme cannot be any usual explicit scheme since it would require a stability condition of the form $\Delta t/\varepsilon < C$ with C independent of ε.

exponential Runge-Kutta schemes of the same order $n+1$ (see for instance [HO05]). Simply speaking, if $(t_i)_{0\leq i\leq N}$ is a discretisation of time-step Δt, and (v_i) and (w_i) are computed numerically using such a scheme, then there exists C *independent of* ε such that

$$\max_{0\leq i\leq N}\left|u^\varepsilon(t_i)-\Omega^{[n]}_{t_i/\varepsilon}(v_i)-w_i\right|\leq C\Delta t^{n+1}$$

where $|\cdot|$ is the usual Euclidian norm on $\mathbb{R}^d$. Furthermore, using a scheme of order n generates an error proportional to ε on the z-component of the solution. This is interesting as z^ε is of size ε after a time $\mathcal{O}(\varepsilon\log(1/\varepsilon))$. IMEX methods such as CNLF and SBDF (see [ARW95; ACM99]) are not the focus of the article, but their use is discussed.

Recently in [CCS16], asymptotic expansions of the solution of (1.1) were constructed using B-series in the case $A=I_{d_z}$. These expansions allowed an approximation of the solution of (1.1) with an error of size $\mathcal{O}(\varepsilon^{n+1})$. Implementing these expansions could be considered to compute $\Omega^{[n]}_{t/\varepsilon}\left(v^{[n]}(t)\right)$. However, this generates a truncation error which is incompatible with uniform accuracy. Encouraged by recent results on highly-oscillatory problems, namely [CLMV19] which do not involve truncation errors, our approach unfolds as follows. We start by considering the following problem

$$\dot{y}^\varepsilon=-ie^{-i\frac{t}{\varepsilon}\Lambda}f\left(e^{i\frac{t}{\varepsilon}\Lambda}y^\varepsilon\right),\qquad y^\varepsilon(0)=y_0:=u_0 \tag{1.4}$$

on which we apply averaging methods detailed in [CCMM15] that are in the vein of those initiated by [Per68] in order to approach the solution with the composition of a near-identity periodic map $(\theta,u)\mapsto\Phi^{[n]}_\theta(u)$ and a flow $(t,u)\mapsto\Psi^{[n]}_t(u)$ following a vector field $G^{[n]}$: $y^\varepsilon(t)=\Phi^{[n]}_{t/\varepsilon}\circ\Psi^{[n]}_t\circ\left(\Phi^{[n]}_0\right)^{-1}(y_0)+\widetilde{y}^{[n]}(t)$ for all $n\geq 0$, where $\widetilde{y}^{[n]}$ is of size $\mathcal{O}(\varepsilon^{n+1})$ and can be computed numerically with a uniform error. The change of variable $\Omega^{[n]}$ and the vector field $F^{[n]}$ are then deduced from $\Phi^{[n]}$ and $G^{[n]}$ using Fourier series. From this, the micro-macro problem defining $v^{[n]}$ and $w^{[n]}$ in (1.3) for the dissipative problem (1.2) is deduced.

The rest of the paper is organized as follows. In Section 2, we construct the change of variable and smooth vector field used to obtain the macro part in (1.3) for Problem (1.2). These maps are constructed using averaging methods on (1.4) and properties similar to those of averaging are proven, ensuring the well-posedness of the micro-macro equations on $(v^{[n]},w^{[n]})$ as defined in (1.3). In Section 3, we study the micro-macro problems associated with this new decomposition (1.3), and prove that the micro part $w^{[n]}$ is indeed of size ε^{n+1}, and that the problem is not stiff. We then state the result of uniform accuracy when using exponential RK schemes. In Section 4, we present some techniques to adapt our method to discretized hyperbolic PDEs. Namely, we study a relaxed conservation law and the telegraph equation, which can be respectively found for instance in [JX95] and [LM08]. In Section 5, we verify our theoretical result of uniform accuracy by successfully obtaining uniform convergence when numerically solving micro-macro problems obtained from a toy ODE and from the two aforementioned PDEs.

2 Derivation of asymptotic models with error estimates

Before describing the content of this section, let us start by stating the assumptions we make on problem (1.2) in order to conduct our study.

Assumption 2.1. *Let us set $d = d_x + d_z$ the dimension of Problem (1.2). There exists a compact set $X_1 \subset \mathbb{R}^{d_x}$ and a radius $\check{\rho} > 0$ such that for every x in X_1, the map $u \in \mathbb{R}^d \mapsto f(u) \in \mathbb{R}^d$ can be developed as a Taylor series around $\begin{pmatrix} x \\ 0 \end{pmatrix}$, and the series converges with a radius greater than or equal to $\check{\rho}$.*

It is therefore possible to naturally extend f to closed subsets of $\mathbb{C}^d$ defined by

$$\mathcal{U}_\rho := \left\{ u \in \mathbb{C}^d \,;\, \exists x \in X_1,\, \left| u - \begin{pmatrix} x \\ 0_{d_z} \end{pmatrix} \right| \leq \rho \right\},$$

for all $0 \leq \rho < \check{\rho}$ as it is represented by a Taylor series in $u \in \mathbb{C}^d$ on these sets. Here $|\cdot|$ is the natural extension of the Euclidian norm on $\mathbb{R}^d$ to $\mathbb{C}^d$. Writing $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, we define a map $(\theta, u) \in \mathbb{T} \times \mathcal{U}_{\check{\rho}} \mapsto g_\theta(u) \in \mathbb{C}^d$ by

$$g_\theta(u) := -ie^{-i\theta\Lambda} f\left(e^{i\theta\Lambda} u\right).$$

Thanks to assumption 2.1, g is well-defined and it is analytic w.r.t. both θ and u. In this section, we shall consider the highly-oscillatory problem

$$\dot{y}^\varepsilon = g_{t/\varepsilon}(y^\varepsilon), \qquad y^\varepsilon(0) = y_0 := u_0, \tag{2.1}$$

of which we construct a micro-macro decomposition using averaging techniques. After a short reminder of some results associated with these techniques, and a description of how the decomposition is computed, we deduce a decomposition for the dissipative problem (1.2). We finish the section by stating some properties of this new decomposition. We also make the following assumption.

Assumption 2.2. *We assume that there exist two radii $0 < \rho_0 < \rho_1 < \check{\rho}$ and a closed subset $X_0 \subset X_1 \subset \mathbb{R}^{d_x}$ such that the initial condition $u_0 \in \mathbb{C}^d$ satisfies*

$$\min_{x \in X_0} \left| u_0 - \begin{pmatrix} x \\ 0_{d_z} \end{pmatrix} \right| \leq \rho_0,$$

and for all $\varepsilon \in (0, 1]$, Problem (1.2) is well-posed on $[0, 1]$ with its solution u^ε in $\mathcal{U}_{\rho_1}$.

Note that this assumption is always met up to replacing $u \mapsto f(u)$ by $u \mapsto \alpha f(u)$ with α small enough (this can be seen as a time rescaling). Finally before proceeding, for $\rho \in [0, \check{\rho} - \rho_1)$, we introduce the set

$$\mathcal{K}_\rho := \mathcal{U}_{\rho_1 + \rho} = \left\{ u \in \mathbb{C}^d \,;\, \exists x \in X_1, \left| u - \begin{pmatrix} x \\ 0 \end{pmatrix} \right| \leq \rho_1 + \rho \right\}. \tag{2.2}$$

By definition, the solution of (1.2) is in $\mathcal{K}_0$ at all time.

2.1 Constructing an approximation of the periodic problem

The following construction and results are taken from [CLMV19], where they are described in (much) more detail. We first apply the averaging techniques of [CCMM15], which start by writing the solution of (2.1) as a composition

$$y^\varepsilon(t) = \Phi^{[n]}_{t/\varepsilon} \circ \Psi^{[n]}_t \circ \left(\Phi^{[n]}_0\right)^{-1}(y_0) + \mathcal{O}(\varepsilon^{n+1}) \tag{2.3}$$

where $\Phi^{[n]}$ is a change of variable $(\theta, u) \in \mathbb{T} \times \mathcal{U}_{\tilde{\rho}} \to \Phi^{\varepsilon}_{\theta}(u) \in \mathbb{C}^d$ and $\Psi^{[n]}$ is the flow map of an autonomous differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi^{[n]}_t(u) = G^{[n]}\Big(\Psi^{[n]}_t(u)\Big), \qquad \Psi^{[n]}_0 = \mathrm{id}$$

where $G^{[n]}$ is a smooth map which must be determined.

The idea behind this composition is that $\Psi^{[n]}$ captures the slow drift while $\Phi^{[n]}$ captures rapid oscillations. In this work, we will focus on standard averaging, meaning the change of variable is of identity average, i.e. $\langle \Phi^{[n]} \rangle = \mathrm{id}$. The average is simply defined by

$$\langle \varphi \rangle(u) := \frac{1}{2\pi} \int_0^{2\pi} \varphi_\theta(u) \mathrm{d}\theta. \tag{2.4}$$

The change of variable $\Phi^{[n]}$ is computed iteratively using the relation

$$\Phi^{[n+1]}_\theta = \mathrm{id} + \varepsilon \int_0^\theta T(\Phi^{[n]})_\sigma \mathrm{d}\sigma - \varepsilon \left\langle \int_0^\bullet T(\Phi^{[n]})_\sigma \mathrm{d}\sigma \right\rangle \tag{2.5}$$

with initial condition $\Phi^{[0]} = \mathrm{id}$. The operator T is defined for maps $(\theta, u) \mapsto \varphi_\theta(u)$ with identity average as

$$T(\varphi)_\sigma = g_\sigma \circ \varphi_\sigma - \partial_u \varphi_\sigma \cdot \langle g \circ \varphi \rangle. \tag{2.6}$$

From these changes of variable $\Phi^{[n]}$ we define vector fields $G^{[n]}$ and defects $\delta^{[n]}$ by

$$G^{[n]} := \langle g \circ \Phi^{[n]} \rangle, \qquad \delta^{[n]} := \frac{1}{\varepsilon} \partial_\theta \Phi^{[n]} + \partial_u \Phi^{[n]} G^{[n]} - g \circ \Phi^{[n]}. \tag{2.7}$$

Note that by definition, $\langle \delta^{[n]} \rangle = 0$.

In order to study and bound these maps, let us introduce the norms we will be using. Given a radius $\rho \geq 0$ and a map $(\theta, u) \in \mathbb{T} \times \mathcal{K}_\rho \mapsto \varphi_\theta(u)$ analytic in u and ν-times continuously differentiable in θ, we define the norms

$$\|\varphi\|_{\mathbb{T},\rho} := \sup_{(\theta,u)\in\mathbb{T}\times\mathcal{K}_\rho} |\varphi_\theta(u)|, \qquad \|\varphi\|_{\mathbb{T},\rho,\nu} := \sup_{0\leq\beta\leq\nu} \|\partial^\beta_\theta \varphi\|_{\mathbb{T},\rho}. \tag{2.8}$$

Property 2.3. *Assumptions 2.1 and 2.2 ensure the following properties:*

(i) There exists a final time $T > 0$ such that for all $\varepsilon \in (0, 1]$, Problem (2.1) is well-posed on $[0, T]$ with its solution y^ε in $\mathcal{K}_0$.

(ii) There exists a radius $R > 0$ such that for all $\theta \in \mathbb{T}$, the function $u \mapsto g_\theta(u)$ is analytic from $\mathcal{K}_{2R}$ to $\mathbb{C}^d$.

(iii) As the function $(\theta, u) \mapsto g_\theta(u)$ is analytic w.r.t. θ, we fix an arbitrary rank $p > 0$ and set $M > 0$ a constant such that for all $\sigma \in [0, 3]$,

$$\forall\ 0 \leq \nu \leq p + 2, \quad \frac{\sigma^\nu}{\nu!} \|\partial^\nu_\theta g\|_{\mathbb{T},2R} \leq M, \tag{2.9}$$

This allows us to get averaging results which can be summed up in the following theorem:

Theorem 2.4 (from [CLMV19] and [CCMM15]). *For $n \in \mathbb{N}^*$, let us denote $r_n = R/n$ and $\varepsilon_n := r_n/16M$ with R and M defined in Property 2.3. For all $\varepsilon > 0$ such that $\varepsilon \leq \varepsilon_n$, the maps $\Phi^{[n]}$ and $G^{[n]}$ are well-defined by (2.5) and (2.7). The change of variable $\Phi^{[n]}$ and the defect $\delta^{[n]}$ are both $(p+2)$-times continuously differentiable w.r.t. θ, and $\Phi_0^{[n]}$ is invertible with analytic inverse on $\mathcal{K}_{R/4}$. Moreover, the following bounds are satisfied for $0 \leq \nu \leq p+1$,*

$$(i) \quad \|\Phi^{[n]} - \mathrm{id}\|_{\mathbb{T},R} \leq 4\varepsilon M \leq \frac{r_n}{4}, \qquad (ii) \quad \|\partial_\theta^\nu \Phi^{[n]}\|_{\mathbb{T},R} \leq 8\varepsilon M \nu!$$

$$(iii) \quad \|G^{[n]}\|_{\mathbb{T},R} \leq 2M \qquad (iv) \quad \|\delta^{[n]}\|_{\mathbb{T},R,p+1} \leq 2M \left(2\mathcal{Q}_p \frac{\varepsilon}{\varepsilon_n}\right)^n$$

where $\mathcal{Q}_p$ is a p-dependent constant.

These properties ensure that the micro-macro problem is well-posed in [CLMV19]. We will now use these in order to define a decomposition for the dissipative problem (1.2), and show that similar properties are satisfied. The micro-macro problem associated with this decomposition will be studied in the next section.

2.2 A new decomposition in the dissipative case

A map $(\theta, u) \in \mathbb{T} \times \mathcal{K}_\rho \mapsto \varphi_\theta(u)$ which is continuously differentiable w.r.t. θ coincides everywhere with its Fourier series, i.e.

$$\varphi_\theta(u) = \sum_{j \in \mathbb{Z}} e^{ij\theta} c_j(\varphi)(u), \quad \text{where} \quad c_j(\varphi)(u) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ij\theta} \varphi_\theta(u) \mathrm{d}\theta. \tag{2.10}$$

We define the shifted map $\widetilde{\varphi}$ by

$$\widetilde{\varphi}_\theta(u) = e^{i\theta\Lambda} \varphi_\theta(u). \tag{2.11}$$

Using these Fourier coefficients $(c_j)_{j \in \mathbb{Z}}$, we consider new maps by setting the change of variable $\Omega^{[n]}$ and the defect $\eta^{[n]}$, for $(\tau, u) \in \mathbb{R}_+ \times \mathcal{K}_{R_k}$,

$$\Omega_\tau^{[n]}(u) := \sum_{j \in \mathbb{Z}} e^{-j\tau} c_j\big(\widetilde{\Phi}^{[n]}\big)(u), \qquad \eta_\tau^{[n]}(u) := i \sum_{j \in \mathbb{Z}} e^{-j\tau} c_j\big(\widetilde{\delta}^{[n]}\big)(u). \tag{2.12}$$

These series are purely formal for now, and their convergence is demonstrated at the end of this subsection. Here $\widetilde{\Phi}^{[n]}$ and $\widetilde{\delta}^{[n]}$ are respectively the shifted change of variable and the shifted defect, with the shift given by (2.11). If there exists an index $j < 0$ and a vector $u \in \mathcal{K}_\rho$ such that $c_j(\widetilde{\Phi}^{[n]})(u) \neq 0$, then $\Omega_\tau^{[n]}(u)$ cannot be bounded uniformly for all $\tau \in \mathbb{R}_+$. We also define the flow $\Gamma^{[n]}$ by setting

$$\frac{\mathrm{d}}{\mathrm{d}t} \Gamma_t^{[n]}(u) = F^{[n]}\Big(\Gamma_t^{[n]}(u)\Big), \quad \text{where} \quad F^{[n]} = iG^{[n]}. \tag{2.13}$$

Note that we do not know the lifetime of any particular solution of the Cauchy problem $\partial_t v = F^{[n]}(v)$, $v(0) = v_0 \in \mathcal{K}_R$ yet.

Remark 2.5. *From the identity $g_\theta(\overline{u}) = -\overline{g_{-\theta}(u)}$, one can obtain the relations on the Fourier coefficients*

$$c_j(\Phi^{[n]})(\overline{u}) = \overline{c_j(\Phi^{[n]})(u)} \quad \textit{and} \quad c_j(\delta^{[n]})(\overline{u}) = -\overline{c_j(\delta^{[n]})(u)}. \tag{2.14}$$

Obviously, the same holds for $\widetilde{\Phi}^{[n]}$ *and* $\widetilde{\delta}^{[n]}$*, as the tilde operator simply shifts the indices of these coefficients component by component. This ensures that if* u *is in* $\mathbb{R}^d$ *then so are* $\Omega_\tau^{[n]}(u)$ *and* $\eta_\tau^{[n]}(u)$*. Similarly, if* u *is in* $\mathbb{R}^d$ *then* $F^{[n]}(u)$ *is in* $\mathbb{R}^d$*.*

For $\Phi^{[n]}$*, for instance, it is equivalent to* $\Phi_\theta^{[n]}(\overline{u}) = \overline{\Phi_{-\theta}^{[n]}(u)}$*, and can be shown by induction using (2.5). Indeed, for* $j = 0$*,* $c_j(\Phi^{[n]})(u) = u$ *(making the result obvious), and for* $j \neq 0$ *the change of variable satisfies* $c_j(\Phi^{[n+1]})(u) = \frac{1}{ij} c_j\left(T(\Phi^{[n]})\right)(u)$ *with* $T(\Phi^{[n]})$ *defined by (2.6). The following calculation is also valid for* $j = 0$*,*

$$\begin{aligned} c_j(g \circ \Phi^{[n]})(\overline{u}) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ij\theta} g_\theta \left(\Phi_\theta^{[n]}(\overline{u})\right) \mathrm{d}\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ij\theta} g_\theta \left(\overline{\Phi_{-\theta}^{[n]}(u)}\right) \mathrm{d}\theta \\ &= -\frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{e^{ij\theta} g_{-\theta}\left(\Phi_{-\theta}^{[n]}(u)\right)} \mathrm{d}\theta = -\overline{c_j(g \circ \Phi^{[n]})(u)}. \end{aligned}$$

From this, one gets $c_j\left(T(\Phi^{[n]})\right)(\overline{u}) = -\overline{c_j(g \circ \Phi^{[n]})(u)} - \overline{\partial_u c_j(\Phi^{[n]})(u)} \cdot \left(-\overline{c_0(g \circ \Phi^{[n]})(u)}\right) = -\overline{c_j\left(T(\Phi^{[n]})\right)(u)}$*, yielding the desired result.*

The micro part $w^{[n]}$ of the decomposition is the difference between the solution u^ε of (1.2) and the asymptotic approximation $\Omega_{t/\varepsilon}^{[n]} \circ \Gamma_t^{[n]} \circ \left(\Omega_0^{[n]}\right)^{-1}(u_0)$. Assuming that $\Omega^{[n]}$ and $\eta^{[n]}$ are well-defined (we will prove it in Theorem 2.8), it is necessary to show that the map $\eta^{[n]}$ can be characterized as a *defect* (similarly to $\delta^{(k)}$). Being a defect means that $\eta^{[n]}$ characterises the error of the approximation $\frac{\mathrm{d}}{\mathrm{d}t}\left[\Omega_{t/\varepsilon}^{[n]} \circ \Gamma_t^{[n]}\right] \approx -\frac{1}{\varepsilon}\Lambda\Gamma_t^{[n]} + f \circ \Omega_{t/\varepsilon}^{[n]} \circ \Gamma_t^{[n]}$. A straightforward computation yields

$$\eta_\tau^{[n]} = \sum_{j \in \mathbb{Z}} e^{-j\tau} c_j \left(\frac{i}{\varepsilon} \widetilde{\partial_\theta \Phi}^{[n]} + \widetilde{\partial_u \Phi}^{[n]} \cdot (iG^{[n]}) - i\widetilde{g \circ \Phi}^{[n]}\right) \tag{2.15}$$

where we can recognize $\widetilde{\partial_\theta \varphi} = \partial_\theta \widetilde{\varphi} - i\Lambda\widetilde{\varphi}$, $\widetilde{\partial_u \varphi} = \partial_u \widetilde{\varphi}$ and $i\widetilde{g \circ \varphi} = f \circ \widetilde{\varphi}$. The characterization as a defect requires the following result:

Lemma 2.6. *Let* ρ *and* r *be two radii such that* $0 \leq \rho < r \leq 2R$ *and let* ν *be a positive integer. We set* φ *a periodic map* $(\theta, u) \in \mathbb{T} \times \mathcal{K}_\rho \mapsto \varphi_\theta(u) \in \mathcal{K}_r$ *that is near-identity in the sense*

$$\forall (\theta, u) \in \mathbb{T} \times \mathcal{K}_\rho, \quad |\varphi_\theta(u) - u| \leq r - \rho$$

and that is continuously differentiable w.r.t. θ *for all* $u \in \mathcal{K}_\rho$*. Using the definitions of (2.10) and (2.11), assume that all the Fourier coefficients of negative index of the shifted map* $\widetilde{\varphi}$ *vanish. Then, setting* $D := \{\xi \in \mathbb{C},\ |\xi| < 1\}$ *and* $\overline{D}$ *its closure, the map* $(\xi, u) \in \overline{D} \times \mathcal{K}_\rho \mapsto \sum_{j \in \mathbb{Z}} \xi^j c_j\left(\widetilde{\varphi}\right)(u)$ *is well-defined with values in* $\mathcal{K}_r$*,* p*-times continuously differentiable. Furthermore, for all* $(\xi, u) \in \overline{D} \times \mathcal{K}_\rho$*, the following identity is satisfied*

$$f\left(\sum_{j \geq 0} \xi^j c_j(\widetilde{\varphi})(u)\right) = \sum_{j \in \mathbb{Z}} \xi^j c_j\left(f \circ \widetilde{\varphi}\right)(u)$$

and for all $j < 0$*,* $c_j\left(f \circ \widetilde{\varphi}\right)(u)$ *is identically zero. In particular the map* $(\tau, u) \in \mathbb{R}_+ \times \mathcal{K}_\rho \mapsto \sum_{j \in \mathbb{Z}} e^{-j\tau} c_j(\widetilde{\varphi})(u)$ *is well-defined with values in* $\mathcal{K}_r$*.*

Proof. Let us work at fixed $u \in \mathcal{K}_\rho$. By product $\widetilde{\varphi}$ is continuously differentiable w.r.t. θ, therefore the series of its Fourier coefficients is absolutely convergent. Furthermore, $\widetilde{\varphi}$ only has nonnegative modes by assumption, meaning the indices can be restricted to nonnegative values in the definition

$$\zeta : \xi \in \overline{D} \mapsto \zeta(\xi) := \sum_{j \in \mathbb{Z}} \xi^j c_j\left(\widetilde{\varphi}\right)(u).$$

As such, the function ζ is well-defined on $\overline{D}$ and is holomorphic on D.

Let us now show that it has values in $\mathcal{K}_r$. Because u is in $\mathcal{K}_\rho$, by Definition (2.2), we set $x \in X_0$ such that $\left|u - \begin{pmatrix} x \\ 0 \end{pmatrix}\right| \leq \rho_1 + \rho$. Using a triangle inequality in the definition of ζ yields

$$\left|\zeta(\xi) - \begin{pmatrix} x \\ 0 \end{pmatrix}\right| \leq \left|\xi^\Lambda u - \begin{pmatrix} x \\ 0 \end{pmatrix}\right| + \left|\sum_{j \geq 0} \xi^j c_j\left(\widetilde{\varphi - \mathrm{id}}\right)(u)\right| \tag{2.16}$$

where $\xi^\Lambda = (\xi^{\lambda_1}, \dots, \xi^{\lambda_d})^T$ if $\Lambda = \mathrm{Diag}(\lambda_1, \dots, \lambda_d)$ and by convention $\xi^0 = 1$ for all $\xi \in \mathbb{C}$. Because $\lambda_\alpha = 0$ for all $1 \leq \alpha \leq d_x$, and according to the maximum modulus principle,

$$\left|\xi^\Lambda u - \begin{pmatrix} x \\ 0 \end{pmatrix}\right| \leq \left|u - \begin{pmatrix} x \\ 0 \end{pmatrix}\right| \leq \rho, \quad \text{and} \quad \left|\sum_j \xi^j c_j(\widetilde{\varphi - \mathrm{id}})(u)\right| \leq \sup_{\theta \in \mathbb{T}} |\varphi_\theta(u) - u| \leq r - \rho.$$

The bound $\left|\zeta(\xi) - \begin{pmatrix} x \\ 0 \end{pmatrix}\right| \leq \rho_1 + r$ becomes obvious using (2.16), therefore by Definition (2.2), $\zeta(\xi)$ is in $\mathcal{K}_r$.

In turn, the function $\xi \mapsto \mathfrak{f}(\xi) := f(\zeta(\xi))$ is well-defined for all $\xi \in \overline{D}$, is continuous on this set, and is holomorphic on D. As such, it can be developed as a power series around $\xi = 0$. We write $(\mathfrak{f}_j)$ the coefficients of this power series such that for $\xi \ll 1$, $\mathfrak{f}(\xi) = \sum_j \xi^j \mathfrak{f}_j$. By Cauchy formula,

$$\mathfrak{f}_j = \frac{1}{2i\pi} \oint_{|\xi|=1} \xi^{-(j+1)} f\big(\zeta(\xi)\big) \mathrm{d}\xi = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ij\theta} f \circ \widetilde{\varphi}_\theta(u) \mathrm{d}\theta = c_j(f \circ \widetilde{\varphi})(u),$$

therefore $f\left(\sum_{j \geq 0} \xi^j c_j(\widetilde{\varphi})(u)\right) = \sum_{j \geq 0} \xi^j c_j\left(f \circ \widetilde{\varphi}\right)(u)$. For $j < 0$, the relation $c_j(f \circ \widetilde{\varphi})(u) = \frac{1}{2i\pi} \oint_{|\xi|=1} \xi^{-(j+1)} \mathfrak{f}(\xi) \mathrm{d}\xi$ still holds, therefore $c_j(f \circ \widetilde{\varphi})(u) = 0$ by Cauchy's integral theorem. □

Assuming now that $\widetilde{\Phi}^{[n]}$ satisfies the assumptions of Lemma 2.6 (this will be proved in Theorem 2.8), then from (2.15), we get

$$\eta_\tau^{[n]} = \frac{1}{\varepsilon}\left(\partial_\tau + \Lambda\right)\Omega_\tau^{[n]} + \partial_u \Omega_\tau^{[n]} \cdot F^{[n]} - f \circ \Omega^{[n]}. \tag{2.17}$$

This means that $\eta^{[n]}$ is indeed a defect, and this relation will later serve to prove that $w^{[n]}$ is of size $\mathcal{O}(\varepsilon^{n+1})$.

Before proceeding, given a radius $\rho \in [0, 2R]$ and a map $(\tau, u) \in \mathbb{R}_+ \times \mathcal{K}_\rho \mapsto \psi_\tau(u)$, let us introduce the norm

$$\|\psi\|_\rho := \sup_{(\tau,u) \in \mathbb{R}_+ \times \mathcal{K}_\rho} |\psi_\tau(u)|. \tag{2.18}$$

Lemma 2.7. *Given a radius $\rho \in [0, 2R]$ and an integer $\nu \in \mathbb{N}$, let φ be a periodic map $(\theta, u) \in \mathbb{T} \times \mathcal{K}_\rho \mapsto \varphi_\theta(u)$ that is analytic w.r.t. u, that is $(\nu+1)$-times continuously differentiable w.r.t. θ and that has vanishing Fourier coefficients for negative indices, i.e. for all $j < 0$, $c_j(\varphi)$ is identically zero. Then the associated dissipative map $(\tau, u) \in \mathbb{R}_+ \times \mathcal{K}_\rho \mapsto \psi_\tau(u)$ defined by*

$$\psi_\tau(u) := \sum_{j \in \mathbb{Z}} e^{-j\tau} c_j(\varphi)(u)$$

is well defined for $(\tau, u) \in \mathbb{R}_+ \times \mathcal{K}_\rho$, analytic w.r.t. u and ν-times continuously differentiable w.r.t. τ. Furthermore it respects the following bounds for $0 \leq \beta \leq \nu$,

$$\left\| \partial_\tau^\beta \psi_\tau \right\|_\rho \leq \| \partial_\theta^\beta \varphi \|_{\mathbb{T},\rho}$$

where the norm on ψ and its derivatives is defined by (2.18).

Proof. It is well-known that the Fourier series of φ and of its derivatives $\partial_\theta^\beta \varphi$ for $1 \leq \beta \leq \nu$ are absolutely convergent. Therefore $\psi_\tau(u)$ and $\partial_\tau^\beta \psi_\tau(u)$ are well-defined for (τ, u) in $\mathbb{R}_+ \times \mathcal{K}_\rho$ by

$$\partial_\tau^\beta \psi_\tau(u) = \sum_{j \geq 0} (-j)^\beta e^{-j\tau} c_j(\varphi)(u) = i^\beta \cdot \sum_{j \geq 0} e^{-j\tau} c_j \left(\partial_\theta^\beta \varphi \right)(u).$$

The absolute convergence also ensures that analyticity w.r.t. u is preserved, as an absolutely convergent series of holomorphic functions is holomorphic. We define $(\xi, u) \mapsto \zeta_\beta(\xi, u)$ the power series defined for all $\xi \in \mathbb{C}, |\xi| \leq 1$ and all $u \in \mathcal{K}_\rho$ by

$$\zeta_\beta(\xi, u) = \sum_{j \geq 0} \xi^j c_j \left(\partial_\theta^\beta \varphi \right)(u)$$

such that $\partial_\tau^\beta \psi_\tau(u) = i^\beta \, \zeta_\beta(e^{-\tau}, u)$. The maximum modulus principle ensures

$$\sup_{\tau \in \mathbb{R}_+} |\partial_\tau^\beta \psi_\tau(u)| \leq \sup_{|\xi| \leq 1} |\zeta_\beta(\xi, u)| = \sup_{|\xi| = 1} |\zeta_\beta(\xi, u)| \leq \| \partial_\theta^\beta \varphi \|_{\mathbb{T},\rho}$$

which is the desired result. □

Using the lemma's notations and assumptions, since ψ is ν-times continuously differentiable, we may also define the norm

$$\| \psi \|_{\rho,\nu} := \max_{0 \leq \beta \leq \nu} \| \partial_\tau^\beta \psi \|_\rho \tag{2.19}$$

with $\| \cdot \|_\rho$ defined by (2.18). This result can now be applied to maps $\Omega^{[n]}$ and $\eta^{[n]}$, by simply checking that the Fourier coefficients of the shifted maps $\widetilde{\Phi}^{[n]}$ and $\widetilde{\delta}^{[n]}$ vanish for negative indices. The shift is given by (2.11).

Theorem 2.8. *For n in $\mathbb{N}^*$, let us denote $r_n = R/n$ and $\varepsilon_n := r_n/16M$ with R and M defined in Property 2.3. For all $\varepsilon > 0$ such that $\varepsilon \leq \varepsilon_n$, the maps $(\tau, u) \mapsto \Omega_\tau^{[n]}(u)$, $u \mapsto F^{[n]}(u)$ and $(\tau, u) \mapsto \eta_\tau^{[n]}(u)$ given by (2.12) and (2.13) are well-defined on $\mathbb{R}_+ \times \mathcal{K}_R$*

and are analytic w.r.t. u. The change of variable $\Omega^{[n]}$ and the residue $\eta^{[n]}$ are both $(p+1)$-times continuously differentiable w.r.t. τ. Moreover the following bounds are satisfied for all $0 \le \nu \le p+1$,

$$(i) \quad \left\|\Omega^{[n]} - e^{-\tau\Lambda}\right\|_R \le 4\varepsilon M, \qquad (ii) \quad \left\|\partial_\theta^\nu\left[\Omega^{[n]} - e^{-\tau\Lambda}\right]\right\|_R \le 8\big(1+|||\Lambda|||\big)^\nu \varepsilon M\, \nu!$$

$$(iii) \quad \|F^{[n]}\|_R \le 2M \qquad (iv) \quad \|\eta_\tau^{[n]}(u)\|_{R,p+1} \le 2M\big(1+|||\Lambda|||\big)^{p+1}\left(2\mathcal{Q}_p\frac{\varepsilon}{\varepsilon_n}\right)^n$$

where $|||\cdot|||$ is the induced norm from $\mathbb{R}^d$ to $\mathbb{R}^d$, and $\mathcal{Q}_p$ is a p-dependent constant.

Proof. We show by induction that $\widetilde{\Phi}_\theta^{[n]}(u)$ and $\widetilde{\delta}_\theta^{[n]}(u)$ only have non-negative Fourier modes. To start off the induction, notice $\Phi^{[0]} = \mathrm{id}$, therefore $\widetilde{\Phi}_\theta^{[0]}(u) = e^{i\theta\Lambda}u$. Since Λ only has coefficients in $\mathbb{N}$, only positive modes are generated. Assuming for $0 \le k < n$ that $\widetilde{\Phi}^{[k]}$ has vanishing Fourier coefficients for negative indices, let us prove that $\widetilde{\Phi}^{[k+1]}$ does as well. By definition (2.5),

$$\widetilde{\Phi}_\theta^{[k+1]}(u) = e^{i\theta\Lambda}u + \varepsilon\int_0^\theta e^{i\theta\Lambda}T(\Phi^{[k]})_\sigma(u)\mathrm{d}\sigma - \varepsilon e^{i\theta\Lambda}\left\langle\int_0^\bullet T(\Phi^{[k]})_\sigma(u)\mathrm{d}\sigma\right\rangle$$

from which we gather that the only problematic term in the definition of $\widetilde{\Phi}^{[k+1]}$ is the integral

$$\int_0^\theta e^{i\theta\Lambda}T(\Phi^{[k]})_s\mathrm{d}s = \int_0^\theta e^{i(\theta-s)\Lambda}\widetilde{T(\Phi^{[k]})}_s\mathrm{d}s = \int_0^\theta e^{i(\theta-s)\Lambda}\left(-i\,f\circ\widetilde{\Phi}_s^{[k]} + \partial_u\widetilde{\Phi}_s^{[k]}\cdot G^{[k]}\right)\mathrm{d}s$$

where we used $e^{i\theta\Lambda}g_\theta \circ \Phi_\theta^{[k]} = -i\,f\circ\widetilde{\Phi}_\theta^{[k]}$. The convolution product of a periodic map $\theta \mapsto \varphi_\theta$ with $\theta \mapsto e^{i\theta\Lambda}$ generates only one new, nonnegative mode, which is $\theta \mapsto e^{i\theta\Lambda}$. By assumption and Theorem 2.4, Lemma 2.6 is applicable, therefore $f\circ\widetilde{\Phi}^{[k]}$ only involves nonnegative modes. In turn, the same goes for $\widetilde{T(\Phi^{[k]})}$ and then for $\widetilde{\Phi}^{[k+1]}$.

This being true, Lemma 2.7 is applicable, producing the desired bounds directly. The only relationship needed is

$$\|\partial_\theta^\nu\widetilde{\varphi}\|_\rho = \left\|\sum_{q=0}^\nu\binom{\nu}{q}(i\Lambda)^q e^{i\theta\Lambda}\,\partial_\theta^{\nu-q}\varphi\right\|_\rho \le (1+|||\Lambda|||)^\nu\|\varphi\|_{\rho,\nu}.$$

□

3 The micro-macro paradigm

In this section, we start by denoting $v^{[n]}(t) := \Gamma_t^{[n]} \circ \big(\Omega_0^{[n]}\big)^{-1}(u_0)$ and inject the decomposition

$$u^\varepsilon(t) = \Omega_{t/\varepsilon}^{[n]}\big(v^{[n]}(t)\big) + w^{[n]}(t) \tag{3.1}$$

into the original problem (1.2) in order to find a system on $v^{[n]}$ and $w^{[n]}$. The idea of the decomposition is that $v^{[n]}$ and $w^{[n]}$ are not stiff and can therefore be computed with *uniform accuracy*, i.e. the numerical error is independent of ε.

With definition (3.1), $w^{[n]}$ is of size $\mathcal{O}(\varepsilon^{n+1})$ and its derivatives are bounded uniformly up to order $n+1$. We shall prove this immediately. Then, we will prove that using explicit exponential Runge-Kutta of order $n+1$ to compute $v^{[n]}$ and $w^{[n]}$ generates an error of uniform order $n+1$ on u^ε.

3.1 Definition and properties of the micro-macro problem

From decomposition (3.1) we obtain the following system

$$\begin{cases} \partial_t v^{[n]}(t) = F^{[n]}(v^{[n]}), \\ \partial_t w^{[n]}(t) = -\dfrac{1}{\varepsilon}\Lambda\left(\Omega^{[n]}_{t/\varepsilon}(v^{[n]}) + w^{[n]}\right) + f\left(\Omega^{[n]}_{t/\varepsilon}(v^{[n]}) + w^{[n]}\right) - \dfrac{\mathrm{d}}{\mathrm{d}t}\Omega^{[n]}_{t/\varepsilon}(v^{[n]}), \end{cases}$$

with initial conditions $v^{[n]}(0) = \left(\Omega^{[n]}_0\right)^{-1}(u_0)$ and $w^{[n]}(0) = 0$. By definition and using identity (2.17),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Omega^{[n]}_{t/\varepsilon}(v^{[n]}(t)) &= \frac{1}{\varepsilon}\partial_\tau \Omega^{[n]}_{t/\varepsilon}(v^{[n]}) + \partial_u \Omega^{[n]}_{t/\varepsilon}(v^{[n]}) \cdot F^{[n]}(v^{[n]}) \\ &= -\frac{1}{\varepsilon}\Lambda\Omega^{[n]}_{t/\varepsilon}(v^{[n]}) + \eta^{[n]}_{t/\varepsilon}(v^{[n]}) + f\big(\Omega^{[n]}_{t/\varepsilon}(v^{[n]})\big). \end{aligned}$$

We finally get the micro-macro problem

$$\begin{cases} \partial_t v^{[n]}(t) = F^{[n]}(v^{[n]}), & \text{(3.2a)} \\ \partial_t w^{[n]}(t) = -\dfrac{1}{\varepsilon}\Lambda w^{[n]} + f\left(\Omega^{[n]}_{t/\varepsilon}(v^{[n]}) + w^{[n]}\right) - f\left(\Omega^{[n]}_{t/\varepsilon}(v^{[n]})\right) - \eta^{[n]}_{t/\varepsilon}(v^{[n]}). & \text{(3.2b)} \end{cases}$$

with initial conditions $v^{[n]}(0) = \left(\Omega^{[n]}_0\right)^{-1}(u_0)$, $w^{[n]}(0) = 0$. Assuming that the macro equation (3.2a) is well-posed, this can be written in a more convenient form,

$$\begin{cases} \partial_t v^{[n]} = F^{[n]}(v^{[n]}), \\ \partial_t w^{[n]} = -\dfrac{1}{\varepsilon}\Lambda w^{[n]} + L^{[n]}(t/\varepsilon, t, w^{[n]})w^{[n]} + S^{[n]}(t/\varepsilon, t), \end{cases} \tag{3.3}$$

where $L^{[n]}(\tau, t, w)\, w = f\left(\Omega^{[n]}_\tau \circ v^{[n]}(t) + w\right) - f\left(\Omega^{[n]}_\tau \circ v^{[n]}(t)\right)$ i.e.

$$\begin{aligned} & L^{[n]}(\tau, t, w) = \int_0^1 \partial_u f\left(\Omega^{[n]}_\tau \circ v^{[n]}(t) + \mu w\right) \mathrm{d}\mu, \\ \text{and} \qquad & S^{[n]}(\tau, t) = -\eta^{[n]}_\tau(v^{[n]}(t)). \end{aligned}$$

After showing that problem (3.2a) is well-posed, we shall use both formulations (3.2) and (3.3) interchangeably depending on the context.

Theorem 3.1. *For all $n \in \mathbb{N}^*$, let us define $r_n = R/n$ and $\varepsilon_n := r_n/16M$, with R and M given in (2.9). For all $\varepsilon \le \varepsilon_n$, Problem (3.2) is well-posed until some final time T_n independent of ε, and the following bounds are satisfied for all $t \in [0, T_n]$ and $0 \le \nu \le \min(n, p)$,*

$$\begin{aligned} &(i) \quad v^{[n]}(t) \in \mathcal{K}_R && (ii) \quad |w^{[n]}(t)| \le \frac{R}{4}\left(\frac{\varepsilon}{\varepsilon_n}\right)^{n+1} \\ &(iii) \quad |\partial_t^\nu E^{[n]}(t)| = \mathcal{O}(\varepsilon^{n-\nu}) && (iv) \quad \|\partial_t^{\nu+1} E^{[n]}\|_{L^1} = \mathcal{O}(\varepsilon^{n-\nu}) \end{aligned}$$

where $E^{[n]} = \partial_t w^{[n]} + \frac{1}{\varepsilon}\Lambda w^{[n]}$.

Proof. This proof is in several parts: first we show that problem (3.2a) is well-posed, and use this result to show that the bound on $w^{[n]}$ is satisfied, thereby also proving that (3.2b) is well-posed. Finally we focus on the bounds on $E^{[n]}$.

Let us set $\varphi(v) = u_0+v-\Omega_0^{[n]}(u_0+v)$. Using Theorem 2.8, if $|v| \leq R/4$ then $|\varphi(v)| \leq R/4$. By Brouwer fixed-point theorem, there exists v^* such that $\varphi(v^*) = v^*$, i.e. $u^* \in \mathcal{K}_{R/4}$ such that $\Omega_0^{[n]}(u^*) = u_0$. Therefore $v^{[n]}(0) := u^* \in \mathcal{K}_{R/4}$.

Given $t > 0$ and assuming $v^{[n]}(s) \in \mathcal{K}_R$ for all $s \in [0,t]$, one can bound $v^{[n]}(t)$ using Theorem 2.8:

$$\left|v^{[n]}(t) - v^{[n]}(0)\right| = \left|\int_0^t F^{[n]}\left(v^{[n]}(s)\right) \mathrm{d}s\right| \leq 2Mt.$$

Setting $T_v := \dfrac{3R}{8M}$ ensures $\left|v^{[n]}(t) - v^{[n]}(0)\right| \leq 3R/4$, meaning that for all $t \in [0, T_v]$, $v^{[n]}(t)$ exists and is in $\mathcal{K}_R$. Again from Theorem 2.8, we deduce $\Omega_\tau^{[n]}\big(v^{[n]}(t)\big) \in \mathcal{K}_{5R/4}$.

Focusing now on $w^{[n]}$ and assuming for all $s \in [0,t]$, $|w^{[n]}(s)| \leq R/4$, the linear term $L^{[n]}\big(\tau, s, w^{[n]}(s)\big)$ is bounded using a Cauchy estimate:

$$\left|L^{[n]}\big(\tau, s, w^{[n]}(s)\big)\right| \leq \|\partial_u f\|_{3R/2} \leq \frac{\|f\|_{2R}}{2R - \frac{3}{2}R} \leq \frac{2M}{R}$$

using a Cauchy estimate. The integral form then gives the bounds

$$\begin{aligned}\left|w^{[n]}(t)\right| &\leq \left|\int_0^t e^{\frac{s-t}{\varepsilon}\Lambda} L^{[n]}\big(s/\varepsilon, s, w^{[n]}(s)\big) w^{[n]}(s) \mathrm{d}s + \int_0^t e^{\frac{s-t}{\varepsilon}\Lambda} S^{[n]}(s/\varepsilon, s) \mathrm{d}s\right| \\ &\leq \int_0^t \frac{2M}{R}\left|w^{[n]}(s)\right| \mathrm{d}s + \left|\int_0^t e^{\frac{s-t}{\varepsilon}\Lambda} S^{[n]}(s/\varepsilon, s) \mathrm{d}s\right| \end{aligned} \tag{3.4}$$

We compute for each component separately, using (2.12) the definition of $\eta^{[n]}$ and (2.11) the definition of the shift operator, writing $\Lambda = \mathrm{Diag}(\lambda_1, \ldots, \lambda_d)$,

$$\eta_\tau^{[n]}(u) = \sum_{j \geq 0} e^{-j\tau} c_j\left(\widetilde{\delta}^{[n]}\right)(u) = \left(\sum_{j \geq -\lambda_k} e^{-(j+\lambda_k)\tau} c_j\left(\delta^{[n]}\right)(u)_k\right)_{1 \leq k \leq d}$$

from which we get, using $c_0(\delta^{[n]}) = \langle \delta^{[n]} \rangle = 0$,

$$\begin{aligned}\left|\int_0^t e^{\frac{s-t}{\varepsilon}\Lambda} S^{[n]}(s/\varepsilon, s) \mathrm{d}s\right| &= \left|\left(e^{-\lambda_k \frac{t}{\varepsilon}} \sum_{\substack{j+\lambda_k \geq 0 \\ j \neq 0}} \int_0^t e^{-j\frac{s}{\varepsilon}} c_j\Big(\delta^{[n]}\Big)(v^{[n]}(s))_k \mathrm{d}s\right)_{1 \leq k \leq d}\right| \\ &\leq \left|\left(\sum_{\substack{j+\lambda_k \geq 0 \\ j \neq 0}} \varepsilon e^{-\lambda_k \frac{t}{\varepsilon}} \left|\frac{1 - e^{-j\frac{t}{\varepsilon}}}{j}\right| \cdot \sup_{u \in \mathcal{K}_R} |c_j(\delta^{[n]})(u)_k|\right)_{1 \leq k \leq d}\right| \\ &\leq \varepsilon \cdot \left|\sum_{j \in \mathbb{Z}^*} \left(\sup_{u \in \mathcal{K}_R} \left|\frac{1}{j}\, c_j(\delta^{[n]})(u)_k\right|\right)_{1 \leq k \leq d}\right| \leq \varepsilon \cdot C \|\delta^{[n]}\|_{\mathbb{T},R,1}\end{aligned}$$

for some constant $C > 0$ and where $\|\cdot\|_{\mathbb{T},R,1}$ is given by Definition (2.8). We go from the first to the second inequality simply by bounding the difference of exponentials by 1. Using Theorem 2.4, there exists a constant $M_n > 0$ such that for all $t \in [0, T_v]$,

$$\left| \int_0^t e^{\frac{s-t}{\varepsilon}\Lambda} S^{[n]}(s/\varepsilon, s) \mathrm{d}s \right| \leq M_n \left(\frac{\varepsilon}{\varepsilon_n} \right)^{n+1}. \tag{3.5}$$

Using Gronwall's lemma in (3.4) with this inequality yields

$$|w^{[n]}(t)| \leq M_n \, e^{\frac{2M}{R}t} \left(\frac{\varepsilon}{\varepsilon_n} \right)^{n+1} \leq M_n \, e^{\frac{2M}{R}t}.$$

We now set $T_w > 0$ such that $M_n \, e^{\frac{2M}{R}T_w} \leq R/4$ (T_w may therefore depend on n, but does not depend on ε) and

$$T_n = \min(T_v, T_w).$$

This ensures the well-posedness of the solution of (3.3) on $[0, T_n]$ as well as the size of $w^{[n]}$.

Finally, the results on $E^{[n]}$ are a direct consequence of the bounds on the linear term

$$\sup_{\alpha+\beta+\gamma \leq p+1} \|\partial_\tau^\alpha \partial_t^\beta \partial_u^\gamma L^{[n]}\| < +\infty$$

and on the source term

$$\sup_{0 \leq \alpha+\beta \leq p} \|\partial_\tau^\alpha \partial_t^\beta S^{[n]}\|_{L^\infty} = \mathcal{O}(\varepsilon^n), \qquad \sup_{\substack{\beta \geq 1 \\ 1 \leq \alpha+\beta \leq p+1}} \|\partial_\tau^\alpha \partial_t^\beta S^{[n]}\|_{L^1} = \mathcal{O}(\varepsilon^{n+1}).$$

This stems directly from Cauchy estimates and Theorem 2.8.

□

Remark 3.2. *So far we have not discussed how to compute the initial condition $v^{[n]}(0)$. Setting $\varepsilon\phi^{[n]} = \Phi_0^{[n]} - \mathrm{id}$ and $v_k = v^{[n]}(0)$, by definition $\Omega_0^{[n]}(v_k) = \Phi_0^{[n]}(v_k) = u_0$, therefore using $v_k = v_{k-1} + \mathcal{O}(\varepsilon^k)$ and $\phi^{[n]} = \phi^{[k-1]} + \mathcal{O}(\varepsilon^k)$ (see [CLMV19] for details), it is easy to show*

$$v_k = u_0 - \varepsilon\phi^{[n]}(v_{k-1}) + \mathcal{O}(\varepsilon^{n+1}).$$

We can now define approached initial conditions for problem (3.3) iteratively

$$v_0 = u_0, \qquad v_{n+1} = u_0 - \varepsilon\phi^{[n+1]}(v_k), \quad \text{and} \quad w^{[n]}(0) = u_0 - \Omega_0^{[n]}(v_n) \tag{3.6}$$

which ensures $w^{[n]}(0) = \mathcal{O}(\varepsilon^{n+1})$, meaning our previous results are not jeopardised.

3.2 Uniform accuracy of numerical schemes

Using a classic scheme to solve Problem (3.2) cannot work due to the term $\frac{1}{\varepsilon}\Lambda w^{[n]}$. This is why we focus on exponential schemes, which render this term non-problematic (see [MZ09]). Furthermore, for these schemes the error bound involves the "modified" norm

$$|\, u \,|_\varepsilon = \left| u + \frac{1}{\varepsilon}\Lambda u \right|. \tag{3.7}$$

This norm is interesting because after a short time $t \geq \varepsilon \log(1/\varepsilon)$, the z-component of the solution u^ε of (1.2) is of size ε. This is due to the *center manifold theorem*, which states that there exists a invariant manifold $\mathcal{M}$ stable for (1.1) which can be written

$$\mathcal{M} = \{(x,z) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_z} \ : \ z = \varepsilon h^\varepsilon(x)\}$$

such that all solutions converge towards it exponentially quickly, i.e. there exists $\mu > 0$ independent of ε,

$$|z^\varepsilon(t) - \varepsilon h^\varepsilon(x^\varepsilon(t))| \leq Ce^{-\mu t/\varepsilon}.$$

Using the norm $|\cdot|_\varepsilon$ somewhat rescales z^ε (but not x^ε) by ε^{-1} such that studying the error in this norm can be seen as a sort of "relative" error. The following theorem uses known results on exponential Runge-Kutta schemes which can be found for instance in [HO05; HO04].

Theorem 3.3. *Under the assumptions of Theorem 3.1 and denoting $T_n \leq T$ a final time such that problem (3.2) is well-posed on $[0,T_n]$. Given $(t_i)_{i\in[\![0,N]\!]}$ a discretisation of $[0,T_n]$ of time-step $\Delta t := \max_i |\, t_{i+1} - t_i \,|$. computing an approximate solution (v_i, w_i) of (3.2) using an exponential Runge-Kutta scheme of order $q := \min(n,p)+1$ yields a uniform error of order q, i.e.*

$$\max_{0\leq i\leq N} \left| \, u^\varepsilon(t_i) - \Omega^{[n]}_{t_i/\varepsilon}(v_i) - w_i \, \right|_\varepsilon \leq C\Delta t^q \tag{3.8}$$

where C is independent of ε.

The left-hand side of this inequality involves $|\cdot|_\varepsilon$ and shall be called the modified error. It dominates the absolute error which uses $|\cdot|$.

Proof. The idea in this proof is to bound the errors on the macro part and micro part separately, using

$$\left| \, u^\varepsilon(t_i) - \Omega^{[n]}_{t_i/\varepsilon}(v_i) - w_i \, \right|_\varepsilon \leq \left| \, \Omega^{[n]}_{t_i/\varepsilon}\left(v^{[n]}(t_i)\right) - \Omega^{[n]}_{t_i/\varepsilon}(v_i) \, \right|_\varepsilon + \left| \, w^{[n]}(t_i) - w_i \, \right|_\varepsilon .$$

As the macro part $v^{[n]}$ involves no linear term, the scheme acts like any RK scheme on this part. Since $v^{[n]}$ and $F^{[n]}$ are non-stiff, it is obvious that the scheme is *uniformly* of order q, i.e.

$$\left| v^{[n]}(t_i) - v_i \right| \leq \Delta t^q \cdot t_i \cdot \|\partial_t^{q+1} v^{[n]}\|_{L^\infty}$$

using usual error bounds on RK schemes. The reader may notice that the absolute error involving $|\cdot|$ was used, not the modified error involving $|\cdot|_\varepsilon$. The results in [HO04] state that an exponential RK scheme of order q generates an error given by

$$\left| \, w^{[n]}(t_i) - w_i \, \right|_\varepsilon \leq C\Delta t^q \Big(\|\partial_t^{q-1} E^{[n]}\|_\infty + \|\partial_t^q E^{[n]}\|_{L^1} \Big). \tag{3.9}$$

The bounds on $E^{[n]} = \partial_t w^{[n]} + \frac{1}{\varepsilon}\Lambda w^{[n]}$ and its derivatives w.r.t. ε can be found in Theorem 3.1, rendering the computation of bounds on the error of the micro part straightforward. From Theorem 2.8.*(i)*, $\Omega^{[n]}_\tau(u) = e^{-\tau\Lambda}u + \mathcal{O}(\varepsilon)$, therefore the error on $\Omega^{[n]}_{t/\varepsilon}(v^{[n]})$ is of the form

$$\Omega^{[n]}_{t_i/\varepsilon}\big(v^{[n]}(t_i)\big) - \Omega^{[n]}(v_i) = e^{-t_i\Lambda/\varepsilon}\big(v^{[n]}(t_i) - v_i\big) + \varepsilon r_i$$

where $v^{[n]}(t_i) - v_i$ and r_i are of size $t_i \cdot \Delta t^q$. The error can therefore be bounded, denoting $|||\cdot|||$ the induced norm from $\mathbb{R}^d$ to $\mathbb{R}^d$,

$$\left| \Omega^{[n]}_{t_i/\varepsilon}\big(v^{[n]}(t_i)\big) - \Omega^{[n]}(v_i) \right|_\varepsilon \leq \left(1 + \left|\left|\left| \frac{t_i}{\varepsilon} \Lambda e^{-\frac{t_i}{\varepsilon}\Lambda} \right|\right|\right|\right) |v^{[n]}(t_i) - v_i| + (\varepsilon + |||\Lambda|||)\, |r_i|.$$

From this we get the desired result on u^ε. □

Remark 3.4. *Only exponential schemes are considered here rather than IMEX-BDF schemes which are sometimes preferred (as in [HS19]). The reason for this is twofold.*

First, the error bounds are generally better for these schemes. Indeed, an IMEX-BDF scheme of order q involves the L^1 norm of $\partial_t^{q+1} w^{[n]}$, which is worse than the L^1 norm of $\partial_t^q E^{[n]}$. The former is of size $\mathcal{O}(\varepsilon^{n-q})$ while the latter is of size $\mathcal{O}(\varepsilon^{n+1-q})$. This allows the use of schemes of order $n+1$ rather than n.

Second, because Λ is diagonal, the exponentials $e^{-\frac{\Delta t}{\varepsilon}\Lambda}$ are easy to compute. Therefore there is no computational drawback to exponential schemes.

4 Application to ODEs derived from suitably discretized PDEs

In this section, we present some tools to adapt our previously developed method to partial differential equations. This is done by studying two hyperbolic relaxation systems of the form

$$\begin{cases} \partial_t u + \partial_x \widetilde{u} = 0 \\ \partial_t \widetilde{u} + \partial_x u = \dfrac{1}{\varepsilon}(g(u) - \widetilde{u}) \end{cases}$$

for $g(u) = (\varepsilon - 1)\partial_x u$ (telegraph equation) and for $u \mapsto g(u)$ scalar (relaxed conservation law). These two problems may seem similar in theory, and the latter actually serves as a stepping stone to treat the former in [JPT98; JPT00], but we will be treat them quite differently in practice. Because our results may not be valid when using operators, we shall only be studying these problems *after* discretising it them, using either Fourier modes or finite volumes.

Even after discretization, it will be obvious that a direct application of the method is impossible, often because of the apparition of a Laplace operator with the "wrong" sign. The goal of this section is precisely to present possible workarounds to overcome the problems that appear. As such, the computation of maps $\Omega^{[n]}$, $F^{[n]}$ and $\eta^{[n]}$ used in (3.2) will not be detailed. Should the reader wish to see a more detailed and direct application of our method, they can find one in Subsection 5.1.

4.1 The telegraph equation

A commonly studied equation in kinetic theory is the one-dimensional Goldstein-Taylor model, also known as the telegraph equation (see [JPT98; LM08], for instance). It can be written, for $(t, x) \in [0, T] \times \mathbb{R}/2\pi\mathbb{Z}$

$$\begin{cases} \partial_t \rho + \partial_x j = 0, \\ \partial_t j + \dfrac{1}{\varepsilon}\partial_x \rho = -\dfrac{1}{\varepsilon} j, \end{cases} \tag{4.1}$$

where ρ and j represent the mass density and the flux respectively. Using Fourier transforms in x, it is possible to represent a function $\alpha(t,x)$ by

$$\alpha(t,x) = \sum_{k\in\mathbb{Z}} \alpha_k(t) e^{ikx}.$$

Considering a given frequency $k \in \mathbb{Z}$ the problem can be reduced to

$$\begin{cases} \partial_t \rho_k = -ikj_k, \\ \partial_t j_k = -\dfrac{1}{\varepsilon}\left(j_k + ik\rho_k\right). \end{cases}$$

Tackling this problem can easily lead to dead-ends, therefore we will guide the reader through our reasoning navigating some of these dead-ends. This will lead to micro-macro developments of orders 0 and 1. These struggles can be seen as limitations of our approach, however we show that with only slight tweaks, it is possible to obtain an error of uniform order 3 using a classic exponential RK scheme. We see this as an encouragement to keep working with this method.

In order to make a component $-\frac{1}{\varepsilon}z$ appear, it would be tempting to set $z_k = j_k + ik\rho_k$. This quantity would verify the following differential equation

$$\partial_t z_k = -\frac{1}{\varepsilon} z_k + k^2 z_k - ik^3 \rho_k.$$

Integrating this differential equation gives

$$z_k(t) = \exp\left(-\lambda \frac{t}{\varepsilon}\right) z_k(0) - ik^3 \int_0^t e^{(s-t)\lambda/\varepsilon} \rho_k(s) \mathrm{d}s. \tag{4.2}$$

where $\lambda = 1 - \varepsilon k^2$. Because $\varepsilon \in (0,1]$ and $k \in \mathbb{Z}$ should not be correlated, it seems obvious that λ can take any value in $(-\infty, 1)$. For λ negative, this equation is unstable and cannot be solved numerically using standard tools. To overcome this, we consider the stabilised change of variable instead

$$z_k = j_k + \frac{ik}{1+\kappa\varepsilon k^2}\ \rho_k$$

where κ is a positive constant which we shall calibrate as the study progresses. This is the same change of variable as before up to $\mathcal{O}(\varepsilon)$, but $ik\rho_k$ was regularised with an elliptic operator to help with high frequencies. The problem to solve becomes

$$\begin{cases} \partial_t \rho_k = -\dfrac{k^2}{1+\kappa\varepsilon k^2}\rho_k - ikz_k, \\ \partial_t z_k = -\dfrac{1}{\varepsilon} z_k + \dfrac{k^2}{1+\kappa\varepsilon k^2} z_k - \dfrac{ik^3}{1+\kappa\varepsilon k^2}\left(\kappa + \dfrac{1}{1+\kappa\varepsilon k^2}\right)\rho_k. \end{cases}$$

As in (4.2), the growth of z_k is given by $e^{-\lambda t/\varepsilon}$ if λ is defined by

$$\lambda = 1 - \frac{\varepsilon k^2}{1+\kappa\varepsilon k^2} \in \left(1 - \frac{1}{\kappa}, 1\right].$$

For stability reasons λ must be positive, therefore we will choose $\kappa \geq 1$.

Let us set $u_k = (\rho_k, z_k)^T$ and $\Lambda = \mathrm{Diag}(0,1)$ such that $\partial_t u_k = -\frac{1}{\varepsilon}\Lambda u_k + f(u_k)$ with

$$f(u) = \begin{pmatrix} -\dfrac{k^2}{1+\kappa\varepsilon k^2}u_1 - iku_2 \\ \dfrac{k^2}{1+\kappa\varepsilon k^2}u_2 - \dfrac{ik^3}{1+\kappa\varepsilon k^2}\left(\kappa + \dfrac{1}{1+\kappa\varepsilon k^2}\right)u_1 \end{pmatrix}.$$

In the upcoming study, we will usually prefer to write $f(\rho, z)$ rather than $f(u)$ so as to keep the distinction between both coordinates clear. Assuming $|k| \leq k_{\max}$, it is possible to bound $f(\rho_k, z_k)$ independently of k and of ε, allowing us to apply the method developed in this paper in order to approximate every ρ_k and j_k, and eventually $\rho(x,t)$ and $j(x,t)$. Note that no rigorous aspects of convergence in functional spaces are considered here – this will come in later papers. We will be omitting the index k going forward for the sake of clarity.

The micro-macro method is initialised by setting the change of variable $\Omega_\tau^{[0]}(\rho, z) = (\rho, e^{-\tau}z)^T$. The vector field followed by the macro part $v^{[0]}$ is $F^{[0]}$ given by

$$F^{[0]}(\rho, z) = \hat{k}^2 \begin{pmatrix} -\rho \\ z \end{pmatrix}.$$

with $\hat{k} = k\left(1+\kappa\varepsilon k^2\right)^{-\frac{1}{2}}$. This means that the macro variable $v^{[0]}(t)$ is given by

$$v^{[0]}(t) = \begin{pmatrix} e^{-\hat{k}^2 t} & 0 \\ 0 & e^{\hat{k}^2 t} \end{pmatrix} v^{[0]}(0).$$

Notice that the growth of $v_2^{[0]}(t)$ is in $e^{\hat{k}^2 t}$, which is akin to the heat equation in reverse time.[2] This is problematic, as it is possible for $\hat{k}$ to be quite big. For example with $k = 10, \kappa = 2$ and $\varepsilon = 10^{-2}$, one gets $e^{\hat{k}^2} \approx 3 \cdot 10^{14}$. In order to obtain the solution of (4.1), $u_k(t) = \Omega_{t/\varepsilon}^{[0]}\left(v^{[0]}(t)\right) + w^{[0]}(t)$, however, we are only interested in $\Omega_{t/\varepsilon}^{[0]}\left(v^{[0]}(t)\right)$ for the macro part, and $\eta_{t/\varepsilon}^{[0]}\left(v^{[0]}(t)\right)$ for the micro part, which only depend on $e^{-\frac{t}{\varepsilon}\Lambda}v^{[0]}(t)$ as can be seen in the upcoming expression of $\eta^{[0]}$ and using $\Omega_\tau^{[0]}(u) = e^{-\tau\Lambda}u$. This means that the interesting quantity is

$$e^{-\frac{t}{\varepsilon}\Lambda}v^{[0]}(t) = \begin{pmatrix} e^{-\hat{k}^2 t} & 0 \\ 0 & e^{-(1-\varepsilon\hat{k}^2)\frac{t}{\varepsilon}} \end{pmatrix} v^{[0]}(0). \tag{4.3}$$

Recognising $1 - \varepsilon\hat{k}^2 = \lambda > 0$ in this expression, it is obvious that $v_2^{[0]}$ is a decreasing function of time, therefore it is bounded uniformly for all t, k and ε. Because the exact computation of $e^{-\frac{t}{\varepsilon}\Lambda}v^{[0]}(t)$ is readily available, it is used during implementation, leaving only $w^{[0]}$ to be computed numerically using ERK schemes. Should the reader wish to conduct their own implementation, they should use the defect

$$\eta_\tau^{[0]}(\rho, z) = \begin{pmatrix} ike^{-\tau}z \\ \hat{k}^2\left(\kappa + \dfrac{1}{1+\kappa\varepsilon k^2}\right)ik\rho \end{pmatrix} = \eta_0^{[0]}(\rho, e^{-\tau}z).$$

[2] This problem does not appear for the oscillatory equivalent (2.1): A direct calculation yields $G^{[0]}(y) = i\hat{k}^2(y_1, -y_2)^T$, meaning both components of the macro part in y^ε oscillate.

By linearity of f, the micro variable $w^{[0]}$ follows the differential equation

$$\partial_t w^{[0]} = -\frac{1}{\varepsilon}\Lambda w^{[0]} + f(w^{[0]}) - \eta_0^{[0]}\left(e^{-\frac{t}{\varepsilon}\Lambda} v^{[0]}(t)\right), \qquad w^{[0]}(0) = 0.$$

The rescaled macro variable $e^{-\frac{t}{\varepsilon}\Lambda} v^{[0]}(t)$ is given by relation (4.3) with initial condition $v^{[0]}(0) = u(0) = (\rho_k(0), z_k(0))^T$.

Extending our development to order 1 is not trivial either. Direct application of iterations (2.5) yields

$$\Omega_\tau^{[1]}(\rho, z) = \begin{pmatrix} \rho + \varepsilon ike^{-\tau} z \\ z - \varepsilon\hat{k}^2\left(\kappa + \dfrac{1}{1+\kappa\varepsilon k^2}\right) ik\rho \end{pmatrix}$$

from which the vector field for the macro part is

$$F^{[1]}(\rho, z) = \hat{k}^2\left(1 + \varepsilon k^2\left(\kappa + \frac{1}{1+\kappa\varepsilon k^2}\right)\right)\begin{pmatrix} -\rho \\ z \end{pmatrix}.$$

Following the same reasoning as before, one should study the evolution of the z-component of the rescaled macro variable $e^{-\frac{t}{\varepsilon}\Lambda} v^{[1]}(t)$. This evolution is in $e^{-\widetilde{\lambda} t/\varepsilon}$ where $\widetilde{\lambda} = 1 - \varepsilon\hat{k}^2\left(1 + \varepsilon k^2\left(\kappa + \frac{1}{1+\kappa\varepsilon k^2}\right)\right)$. Studying $\widetilde{\lambda}$ as a function of εk^2 in $\mathbb{R}_+$ shows that it is negative for $\varepsilon k^2 > 1$, whatever the value of $\kappa \geq 1$.

To circumvent this, we replace ε by $\frac{\varepsilon}{1+\kappa\varepsilon k^2}$ in iterations (2.5). This adds terms of order ε^2 in the definition of $\Omega^{[1]}$ that do not modify any properties of the micro-macro development but it regularises the problem. Specifically, we define

$$\Omega_0^{[1]}(\rho, z) = \begin{pmatrix} \rho + \dfrac{\varepsilon}{1+\kappa\varepsilon k^2} ikz \\ z - \dfrac{\varepsilon}{1+\kappa\varepsilon k^2}\hat{k}^2\left(\kappa + \dfrac{1}{1+\kappa\varepsilon k^2}\right) ik\rho \end{pmatrix},$$

from which we get the vector field

$$F^{[1]}(\rho, z) = \hat{k}^2\left(1 + \varepsilon\hat{k}^2\left(\kappa + \frac{1}{1+\kappa\varepsilon k^2}\right)\right)\begin{pmatrix} -\rho \\ z \end{pmatrix}.$$

This time also, the identities $\Omega_\tau^{[1]}(u) = \Omega_0^{[1]}(e^{-\tau\Lambda}u)$ and $\eta_\tau^{[1]}(u) = \eta_0^{[1]}(e^{-\tau\Lambda}u)$ are satisfied, therefore the interesting variable is $e^{-\frac{t}{\varepsilon}\Lambda} v^{[1]}(t)$. The quantity dictating its growth is

$$\widetilde{\lambda} = 1 - \varepsilon\hat{k}^2\left(1 + \varepsilon\hat{k}^2\left(\kappa + \frac{1}{1+\kappa\varepsilon k^2}\right)\right)$$

which is positive for all $\varepsilon k^2 \in \mathbb{R}_+$ if and only if $\kappa \geq 2$. As with the development of order 0, the macro variable should be rescaled and computed exactly. The micro part $w^{[1]}$ is given by the differential equation

$$\partial_t w^{[1]} = -\frac{1}{\varepsilon}\Lambda w^{[1]} + f(w^{[1]}) - \eta_0^{[1]}\left(e^{-\frac{t}{\varepsilon}\Lambda} v^{[1]}(t)\right), \qquad w^{[1]}(0) = u_k(0) - \Omega_0^{[1]}\left(v^{[1]}(0)\right)$$

where, writing $\hat{I} = (1+\kappa\varepsilon k^2)^{-1}$,

$$\eta_0^{[1]}(\rho, z) = ik\cdot\hat{k}^2\left(\kappa + \hat{I}\left(2 + \varepsilon\hat{k}^2(\kappa + \hat{I})\right)\right)\begin{pmatrix} z \\ \hat{k}^2(\kappa+\hat{I})\rho \end{pmatrix}$$

$$\text{and} \quad v^{[1]}(0) = \begin{pmatrix} \rho_k(0) - \varepsilon \hat{I} i k z_k(0) \\ z_k(0) + \varepsilon \hat{k}^2 (\kappa + \hat{I}) i k \rho_k(0) \end{pmatrix}.$$

We approached the initial condition using Remark 3.2, but an exact computation of the exact initial condition $\left(\Omega_0^{[1]}\right)^{-1}(u_0)$ is possible, as the map $u \mapsto \Omega_0^{[1]}(u)$ is linear.

4.2 Relaxed conservation law

Our second test case is a hyperbolic problem for $(t, x) \in [0, T] \times \mathbb{R}/2\pi\mathbb{Z}$,

$$\begin{cases} \partial_t u + \partial_x \widetilde{u} = 0, \\ \partial_t \widetilde{u} + \partial_x u = \dfrac{1}{\varepsilon}(g(u) - \widetilde{u}), \end{cases} \tag{4.4}$$

with smooth initial conditions $u(0, x)$ and $\widetilde{u}(0, x)$. This is a stiffly relaxed conservation law, as presented in [JX95]. In order to proceed, we require the following condition to be met:

$$|g'(u)| < 1 \tag{4.5}$$

This is a known stability condition when deriving asymptotic expansions for this kind of problem.

We start by discretising this system in space with $N > 0$ points. Going forward, $(x_j)_{j \in \mathbb{Z}/N\mathbb{Z}}$ denotes a fixed uniform discretisation of $\mathbb{R}/2\pi\mathbb{Z}$, of mesh size $\Delta x := 2\pi/N$. We define the vectors $U = (u_j)_j, \widetilde{U} = (\widetilde{u}_j)_j$ and, given a vector α of size N, $g(\alpha) = (g(\alpha_j))_j$. For simplicity, $u_j(t)$ is the approximation of $u(t, x_j)$, and the same goes for $\widetilde{u}$. We denote D the matrix of centered finite differences and L the classic discrete Laplace operator, which is to say

$$D\alpha = \left(\frac{1}{2\Delta x}(\alpha_{j+1} - \alpha_{j-1})\right)_j \quad \text{and} \quad L\alpha = \left(\frac{1}{\Delta x^2}(\alpha_{j+1} - 2\alpha_j + \alpha_{j-1})\right)_j$$

Using an upwind scheme after diagonalising problem (4.4) yields

$$\begin{cases} \partial_t U + D\widetilde{U} - \dfrac{\Delta x}{2} L U = 0, \\ \partial_t \widetilde{U} + DU - \dfrac{\Delta x}{2} L\widetilde{U} = \dfrac{1}{\varepsilon}(g(U) - \widetilde{U}). \end{cases} \tag{4.6}$$

Setting $U_1 = U$ and $U_2 := \widetilde{U} - g(U_1)$, and neglecting the terms involving L for clarity, this problem becomes

$$\begin{cases} \partial_t U_1 = -D\big(U_2 + g(U_1)\big), \\ \partial_t U_2 = -\dfrac{1}{\varepsilon} U_2 + g'(U_1) D U_2 - T(U_1) \end{cases} \tag{4.7}$$

where we defined $T(U_1) := DU_1 - g'(U_1) D g(U_1)$. From this, our method can be applied, but precautions must be taken in order to avoid having to solve the heat equation in backwards time. Therefore we set

$$\Omega_\tau^{[1]}(U_1, U_2) = \begin{pmatrix} U_1 + \varepsilon(1 - 2\varepsilon D^2)^{-1} D U_2 \\ e^{-\tau} U_2 - \varepsilon T(U_1) \end{pmatrix}.$$

Similarly to the manipulations for the telegraph equation, we multiplied ε by $(I_N - 2\varepsilon D^2)^{-1}$, but this time only for the first component. Writing $\widetilde{D} = (I_N - 2\varepsilon D^2)^{-1} D$, the associated vector field is

$$F^{[1]}(U_1, U_2) = \begin{pmatrix} -Dg(U_1) + \varepsilon DT(U_1) \\ g'(U_1)DU_2 - \varepsilon T'(U_1)\widetilde{D}U_2 - \varepsilon^2 g''(U_1)\big(T(U_1), \widetilde{D}U_2\big) \end{pmatrix}.$$

As in Subsection 5.1, it is possible to obtain $\Omega^{[0]}$ and $F^{[0]}$ by simply neglecting the terms of order ε and above in the expressions above.

Remark 4.1. *Remember that for the telegraph equation, the macro variable $v^{[1]}(t)$ needed to be rescaled by $e^{-t\Lambda/\varepsilon}$. This is not the case here: In the limit $\Delta x \to 0$, the macro variable $v^{[1]} = (\overline{u}_1, \overline{u}_2)^T$ is given by*

$$\begin{cases} \partial_t \overline{u}_1 = -\partial_x \left[g(\overline{u}_1) - \varepsilon\big(1 - g'(\overline{u}_1)^2\big)\partial_x \overline{u}_1 \right], \\ \partial_t \overline{u}_2 = g'(\overline{u}_1)\partial_x \overline{u}_2 - \big(1 - g'(\overline{u}_1)^2\big) \cdot (1 - 2\varepsilon\partial_x^2)^{-1} \varepsilon \partial_x^2 \overline{u}_2 + \varepsilon \phi^\varepsilon(\overline{u}_1, \widetilde{D}\overline{u}_2) \end{cases}$$

with $\widetilde{D} = (1 - 2\varepsilon\partial_x^2)^{-1}\partial_x$ and $\phi^\varepsilon(u_1, u_2) = g''(u_1)\left(2g'(u_1) - \varepsilon(1 - g'(u_1)^2)\partial_x u_1\right) u_2$. The operator $(1 - 2\varepsilon\partial_x^2)^{-1}\varepsilon\partial_x^2$ is bounded, therefore $\overline{u}_2$ is well-defined. The equation on $\overline{u}_1$ is a well-known result. If ε was also relaxed in the U_2-component of $\Omega^{[1]}$, there might be no need for condition (4.5) but the result would be different.

Obtaining the defects of order 0 and 1 from these expressions presents no difficulty. For $\eta^{[1]}$, we separate here the U_1-component and the U_2-component for clarity.

$$\eta_\tau^{[0]}(U, W) = \begin{pmatrix} e^{-\tau} DW \\ T(U) \end{pmatrix},$$

$$\begin{aligned} \eta_0^{[1]}(U_1, U_2)_{U_1} = {} & D\big(g(U_1 + \varepsilon\widetilde{D}W) - g(U_1)\big) + (D - \widetilde{D})U_2 \\ & + \varepsilon\widetilde{D}\Big(g'(U_1)DW - \varepsilon T'(U_1)\widetilde{D}W - \varepsilon^2 g''(U_1)\big(T(U_1), \widetilde{D}W\big)\Big), \end{aligned} \tag{4.8a}$$

$$\begin{aligned} \eta_0^{[1]}(U_1, U_2)_{U_2} = {} & -\big(g'(U_1 + \varepsilon\widetilde{D}U_2) - g'(U_1)\big)DU_2 \\ & + T(U_1 + \varepsilon\widetilde{D}U_2) - T(U_1) - \varepsilon T'(U_1)\widetilde{D}U_2 \\ & + \varepsilon g'(U_1 + \varepsilon\widetilde{D}U_2)DT(U_1) - \varepsilon^2 g''(U_1)\big(\widetilde{D}U_2, T(U_1)\big) \\ & + \varepsilon T'(U_1)\left(Dg(U_1) - \varepsilon T(U_1)\right). \end{aligned} \tag{4.8b}$$

The values of $\eta_\tau^{[1]}(U_1, U_2)$ can be recovered using identity

$$\eta_\tau^{[1]}(U_1, U_2) = \eta_0^{[1]}(U_1, e^{-\tau}U_2).$$

Note that, using a given scheme, solving a single step is much more costly for the micro-macro problem than for the direct problem: Not only is the system size doubled, but the functions implicated necessitate more computing power to obtain a single value (especially $\eta^{[1]}$, as is apparent here). It is therefore plausible to think that our method is best for computing values during the transient phase, after which it is possible to solve the original problem with uniform accuracy.

5 Numerical simulations

In this section we shall demonstrate our results by confirming the theoretical convergence rates of exponential Runge-Kutta (ERK) schemes from [HO05]. We also use these schemes on the original problem (1.1), thereby exhibiting the problem of order reduction.

In the first subsection we study a toy model with some non-linearity that can be found in [CCS16], for which we compute the micro-macro expansion up to order 2. In the second subsection, we showcase the results of uniform convergence for the partial differential equations of Section 4. For these, the exact solution shall not take into account the error in space, i.e. it will be the solution to the discretized problem. Finally in the third subsection, we present a surprising numerical result of order gain for problems near equilibrium, and discuss future works.

5.1 Oscillating toy problem

We first study an "oscillating" problem presented in [CCS16]

$$\begin{cases} \dot{x} = (1-z)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x \\ \dot{z} = -\dfrac{1}{\varepsilon} z + x_1^2 x_2^2 \end{cases} \tag{5.1}$$

with initial conditions $x_0 = (0.1, 0.7)^T$ and $z_0 = 0.05$, and final time $T = 1$. This is of the form $\partial_t u = -\frac{1}{\varepsilon}\Lambda u + f(u)$ when setting

$$\Lambda = \mathrm{Diag}(0,0,1) \qquad \text{and} \qquad f(u) = \begin{pmatrix} -(1-u_3)u_2 \\ (1-u_3)u_1 \\ (u_1u_2)^2 \end{pmatrix}.$$

Here we will sometimes prefer to write $f(x,z)$ so as to distinguish the x and z component. In order to apply our method, we set

$$g_\theta(u) = -ie^{-i\theta\Lambda} f\left(e^{i\theta\Lambda}u\right) = -i\begin{pmatrix} -u_2 + e^{i\theta}u_2u_3 \\ u_1 - e^{i\theta}u_1u_3 \\ e^{-i\theta}(u_1u_2)^2 \end{pmatrix}.$$

By construction, $\Phi_\theta^{[0]}(u) = u$ therefore $G^{[0]}(u) = \langle g \rangle(u) = (-u_2, u_1, 0)^T$. One then gets the change of variable at order 1 using (2.5),

$$\Phi_\theta^{[1]}(u) = \begin{pmatrix} u_1 - \varepsilon e^{i\theta}u_2u_3 \\ u_2 + \varepsilon e^{i\theta}u_1u_3 \\ u_3 + \varepsilon e^{-i\theta}(u_1u_2)^2 \end{pmatrix}, \quad \text{yielding} \quad G^{[1]}(u) = -i\begin{pmatrix} -\left(1-\varepsilon(u_1u_2)^2\right)u_2 \\ \left(1-\varepsilon(u_1u_2)^2\right)u_1 \\ 2\varepsilon u_1u_2u_3(u_1^2-u_2^2) \end{pmatrix}.$$

We remind the reader of the definition $G^{[1]} = \langle g \circ \Phi^{[1]} \rangle$. In order to compute the defect $\delta^{[1]}$ and the change of variable of the next order $\Phi^{[2]}$, one must compute the difference $g \circ \Phi^{[1]} - \partial_u \Phi^{[1]} \cdot G^{[1]}$, denoted $T(\Phi^{[1]})$ which is

$$T(\Phi^{[1]})_\theta(u) = -i \cdot \begin{pmatrix} e^{i\theta}u_3\left(u_2 + \varepsilon e^{i\theta}u_1u_3 + 2\varepsilon^2 u_1u_2^2(u_1^2-u_2^2)\right) \\ -e^{i\theta}u_3\left(u_1 - \varepsilon e^{i\theta}u_2u_3 - 2\varepsilon^2 u_1^2u_2(u_1^2-u_2^2)\right) \\ e^{-i\theta}\left(U_0 + \varepsilon U_1 + \varepsilon^2 U_2\right) \end{pmatrix}$$

where for clarity we defined

$$U_0 = \left(u_1^2 + \varepsilon^2 e^{2i\theta}(u_2u_3)^2\right)\left(u_2^2 + \varepsilon^2 e^{2i\theta}(u_1u_3)^2\right),$$

$$U_1 = -2u_1u_2(u_1^2 - u_2^2)\left(1 - \varepsilon(u_1u_2)^2 + \varepsilon e^{3i\theta}u_3^3\right) \quad \text{and} \quad U_2 = -(2u_1u_2u_3)^2.$$

Note that U_0, U_1 and U_2 depend on both ε and θ. The defect $\delta^{[1]}$ is defined as $\frac{1}{\varepsilon}\partial_\theta\Phi^{[1]} - T(\Phi^{[1]})$ and can easily be obtained from the previous formula.

We truncate terms of order ε^2 and above in $T(\Phi^{[1]})$ (which will not impact results of uniform accuracy) and integrate it following formula (2.5) to compute the expansion of order 2: the change of variable of $\Phi^{[2]}$ and the vector field $G^{[2]}$. We then identify the Fourier coefficients of $e^{i\theta\Lambda}\Phi^{[2]}$ to obtain $\Omega^{[2]}$, and we obtain $F^{[2]}$ from $G^{[2]}$. This yields

$$\Omega_\tau^{[2]}(x,z) = \begin{pmatrix} x_1 - \varepsilon e^{-\tau}x_2z - \frac{1}{2}\varepsilon^2 e^{-2\tau}z^2x_1 \\ x_2 + \varepsilon e^{-\tau}x_1z - \frac{1}{2}\varepsilon^2 e^{-2\tau}z^2x_2 \\ z + \varepsilon(x_1x_2)^2 - 2\varepsilon^2 x_1x_2(x_1^2 - x_2^2) \end{pmatrix},$$

$$F^{[2]}(x,z) = \begin{pmatrix} x_2(-1 + \varepsilon(x_1x_2)^2 - 2\varepsilon^2 x_1x_2(x_1^2 - x_2^2)) \\ x_1(1 - \varepsilon(x_1x_2)^2 + 2\varepsilon^2 x_1x_2(x_1^2 - x_2^2)) \\ 2\varepsilon z x_1x_2(x_1^2 - x_2^2) \end{pmatrix}.$$

The defect $\eta^{[2]}$ is obtained using relation (2.17) or by computing $\delta^{[2]}$ and identifying the Fourier coefficients.

Remark 5.1. *It is easy to find an approximation of the center manifold $x \mapsto \varepsilon h^\varepsilon(x)$ by taking the limit $\tau \to \infty$ of the z-component of $\Omega^{[k]}$. For example here*

$$\varepsilon h^\varepsilon(x) = \varepsilon(x_1x_2)^2 - 2\varepsilon^2 x_1x_2(x_1^2 - x_2^2) + \mathcal{O}(\varepsilon^3).$$

This coincides with the results in [CCS16].

We remind the reader that the problem that is solved at times $(t_i)_{0\le i\le N}$ is

$$\begin{cases} \partial_t v^{[k]}(t) = F^{[k]}(v^{[k]}), \\ \partial_t w^{[k]}(t) = -\dfrac{1}{\varepsilon}\Lambda w^{[k]} + f\left(\Omega^{[k]}_{t/\varepsilon}(v^{[k]}) + w^{[k]}\right) - f\left(\Omega^{[k]}_{t/\varepsilon}(v^{[k]})\right) - \eta^{[k]}_{t/\varepsilon}(v^{[k]}), \end{cases}$$

with $k = 1, 2$. This yields vectors $(v_i) \approx (v^{[k]}(t_i))$ and $(w_i) \approx (w^{[k]}(t_i))$, from which an approximation $u_i \approx u^\varepsilon(t_i)$ is then obtained by setting $u_i = \Omega^{[k]}_{t_i/\varepsilon}(v_i) + w_i$. Initial conditions $v^{[k]}(0)$ and $w^{[k]}(0)$ are computed using Remark 3.2.

The difference $f\left(\Omega^{[2]}_{t/\varepsilon}(v^{[2]}) + w^{[2]}\right) - f\left(\Omega^{[2]}_{t/\varepsilon}(v^{[2]})\right)$ is computed using

$$f(x + \widetilde{x}, z + \widetilde{z}) - f(x,z) = \begin{pmatrix} -(1-z)\widetilde{x}_2 + (x_2 + \widetilde{x}_2)\widetilde{z} \\ (1-z)\widetilde{x}_1 - (x_1 + \widetilde{x}_1)\widetilde{z} \\ \left(x_1x_2 + (x_1 + \widetilde{x}_1)(x_2 + \widetilde{x}_2)\right)\left(x_1\widetilde{x}_2 + \widetilde{x}_1x_2 + \widetilde{x}_1\widetilde{x}_2\right) \end{pmatrix}$$

in order to avoid rounding errors due to the size difference between u and $\widetilde{u}$.

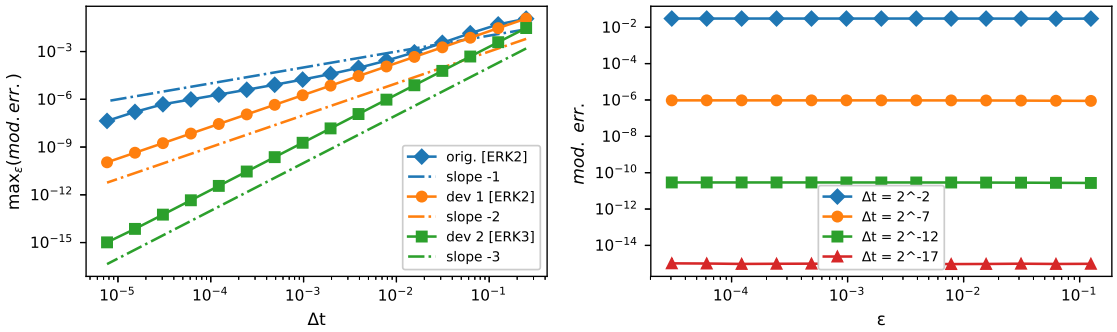

Figure 1: *Oscillating case: On the left, maximum error on ε (for $\varepsilon = 2^{-k}$ with k spanning $\{3, \dots, 15\}$) as a function of Δt when using exponential RK schemes (abbr. ERK) of different orders. On the right, the error as a function of ε when solving the micro-macro problem of order 2 using ERK3.*

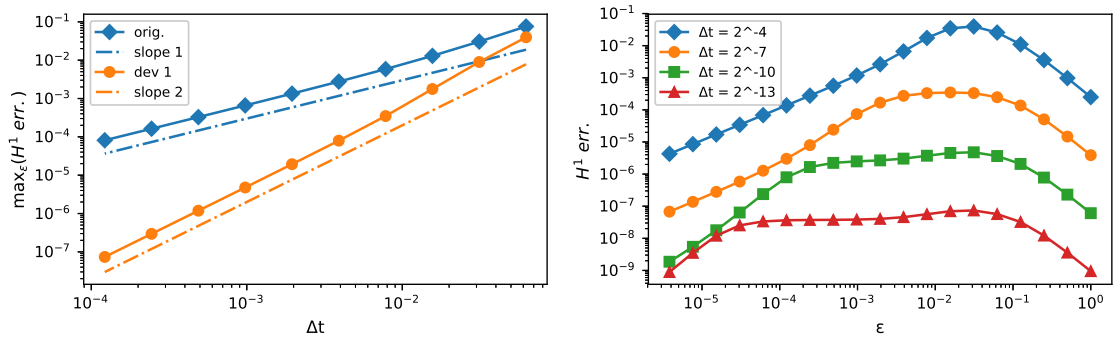

Figure 2: *Telegraph equation: Absolute H^1 error on the solution of (4.1) computed by an ERK3 scheme. Supremum on ε as a function of Δt (left) and evolution of this error as a function of ε for the 1st-order development (right).*

Figure 1 showcases the phenomenon of order reduction when solving the original problem (5.1): Despite using a scheme of order 2, the error depends of ε in such a way that, at fixed Δt, there exists no constant C such that the error is bounded by $C\Delta t^2$ for all ε. However there exists C such that the error is bounded by $C\Delta t$. This phenomenon of order reduction is discussed in [HO05].

In that case, we cannot say that the error is of *uniform* order 1, as this would require the error to be independent of ε. However, this is the case when solving the micro-macro problem, as can be seen on the right-hand side of the figure for a development of order 2. Furthermore, the theoretical orders of convergence from Theorem 3.3 are confirmed. Indeed, using a scheme of order 2 (resp. 3) on the micro-macro problem of order 1 (resp. 2) generates a uniform error of the expected order of convergence, with no order reduction.

5.2 Partial differential equations

The telegraph equation *(see Subsection 4.1 for details)*

Implementations are conducted using $\kappa = 2$, space frequencies are bounded by $k_{\max} := 12$, and initial data is $\rho(0, x) = e^{\cos(x)}$, $j(0, x) = \frac{1}{2}\cos^3(x)$. Results can be seen in Figure 2 when using a scheme of order 3. When solving the original problem, some order reduction is observed, from 3 to 1. Here the convergence is not uniform, as it varies with ε when

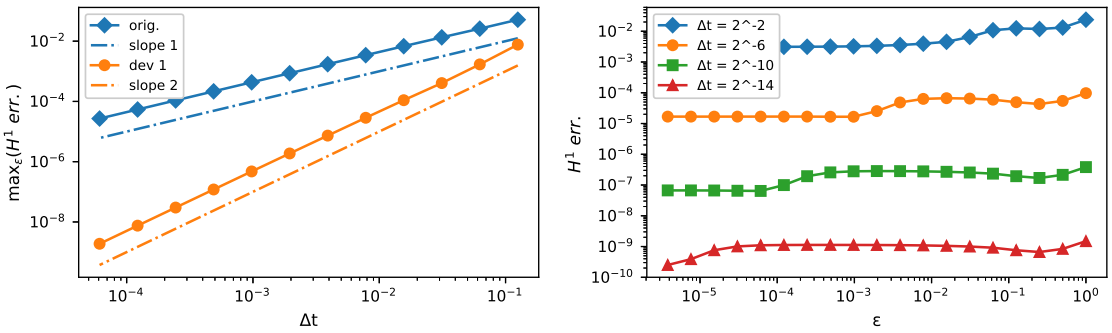

Figure 3: *Relaxed Burgers-type problem: Maximum modified H^1 error (for ε spanning 1 to 2^{-18} using an ERK3 scheme as a function of Δt (left), and H^1 error as a function of ε for the micro-macro problem of order 1 (right).*

fixing Δt, but this is an artefact due to the exact solving of the macro part: The bounds presented in Theorem 3.3 are *at worst*, and the relationship between the error bound and the stiffness of the linear operator is rather complex when using exponential RK schemes (again, see [HO05] for details). It is known that these schemes have properties of asymptotic preservation (shown in [DP11] for instance), which explains the error variations with ε.

Relaxed conservation law *(see Subsection 4.2 for details)*

For our tests, following [HS19], we consider $g(u) = bu^2$ with $b = 0.2$. Simulations run to a final time $T = 0.25$ and the mesh size is fixed: $N = 16$. Initial data is $u(0,x) = \frac{1}{2}e^{\sin(x)}$ and $\widetilde{u}(0,x) = \cos(x)$. The reference solution was computed up to a precision 10^{-12} using an ERK2 scheme. Convergence results are presented in Figure 3, confirming theoretical results once more.

It should be said again that our approach does not study the error in space, only in time. For instance, the relationship between the error bound and the grid size is not considered. Further studies will be conducted, especially considering CFL conditions, L^2 and H^1 norms, and computational costs.

5.3 Near-equilibrium results

If one chooses an initial condition $z^\varepsilon(0) = 0$ in (1.1), then it is close to the center manifold up to $\mathcal{O}(\varepsilon)$, and Problem (1.2) can be solved with uniform accuracy of order 2 but only when considering the absolute error $|\cdot|$, not the modified error $|\cdot|_\varepsilon$ from (3.7). The same behaviour is observed for the telegraph equation when setting $j(0,x) = -\partial_x\rho(0,x)$, meaning $z = \mathcal{O}(\varepsilon)$. This would theoretically mean that we need to push the micro-macro developments up to order 2 if we want to improve the order of convergence. However, this is not the case: uniform accuracy of order 3 is obtained from a development of order 1 for all test cases. This "order gain" also propagates to our micro-macro development of order 2 for the oscillating toy problem. These results can be seen in Figure 4 and will be studied in future works.

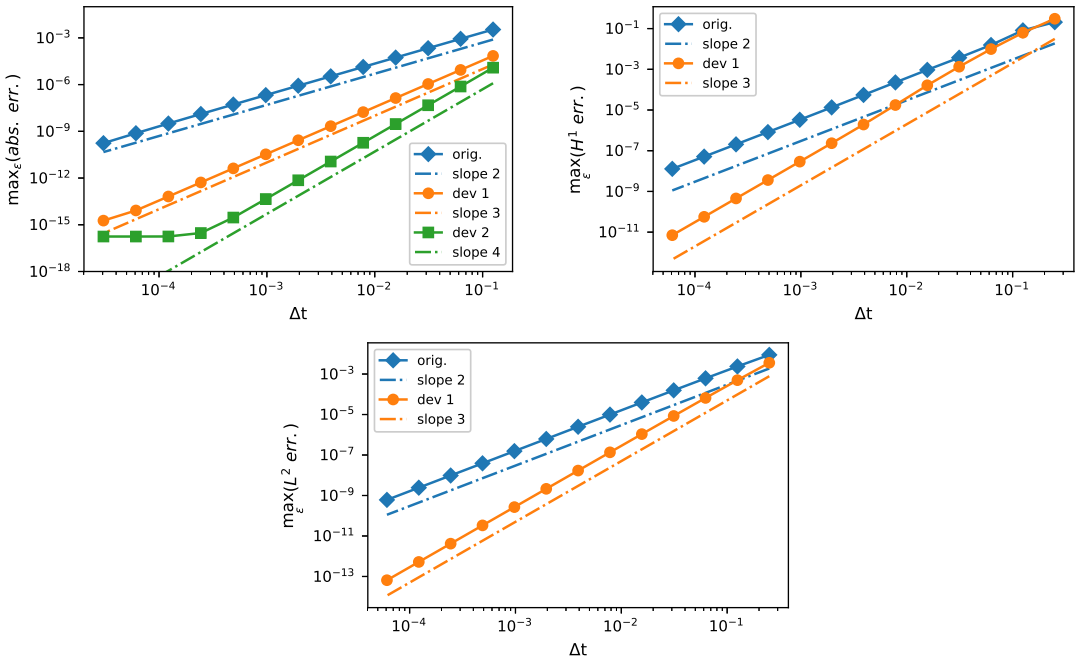

Figure 4: *In reading order, errors when solving the oscillating toy problem, the telegraph equation and the relaxed conservation law. All systems start near equilibrium and are solved with exponential Runge-Kutta schemes of the observed order of convergence.*

References

[ACM99] Georgios Akrivis, Michel Crouzeix, and Charalambos Makridakis. "Implicit-explicit multistep methods for quasilinear parabolic equations". In: *Numerische Mathematik* 82.4 (1999), pp. 521–541.

[ARW95] Uri M Ascher, Steven J Ruuth, and Brian TR Wetton. "Implicit-explicit methods for time-dependent partial differential equations". In: *SIAM Journal on Numerical Analysis* 32.3 (1995), pp. 797–823.

[AP96] Pierre Auger and Jean-Christophe Poggiale. "Emergence of population growth models: fast migration and slow growth". In: *Journal of Theoretical Biology* 182.2 (1996), pp. 99–108.

[Car82] Jack Carr. *Applications of centre manifold theory*. Vol. 35. Applied Mathematical Sciences. Springer-Verlag New York, 1982.

[CCMM15] François Castella, Philippe Chartier, Florian Méhats, and Ander Murua. "Stroboscopic Averaging for the Nonlinear Schrödinger Equation". In: *Foundations of Computational Mathematics* 15.2 (Apr. 2015), pp. 519–559.

[CCS15] Francois Castella, Philippe Chartier, and Julie Sauzeau. "Analysis of a time-dependent problem of mixed migration and population dynamics". In: *arXiv preprint arXiv:1512.01880* (2015).

[CCS16] François Castella, Philippe Chartier, and Julie Sauzeau. "A formal series approach to the center manifold theorem". In: *Foundations of Computational Mathematics* (2016), pp. 1–38.

[CLMV19] Philippe Chartier, Mohammed Lemou, Florian Méhats, and Gilles Vilmart. "A New Class of Uniformly Accurate Numerical Schemes for Highly Oscillatory Evolution Equations". In: *Foundations of Computational Mathematics* (2019).

[DP11] Giacomo Dimarco and Lorenzo Pareschi. "Exponential Runge–Kutta methods for stiff kinetic equations". In: *SIAM Journal on Numerical Analysis* 49.5 (2011), pp. 2057–2077.

[GHM94] Günther Greiner, JAP Heesterbeek, and Johan AJ Metz. "A singular perturbation theorem for evolution equations and time-scale arguments for structured population models". In: *Canadian applied mathematics quarterly* 3.4 (1994), pp. 435–459.

[HW96] Ernst Hairer and Gerhard Wanner. *Solving ordinary differential equations II. Stiff and Differential-Algebraic Problems*. Springer Berlin Heidelberg, 1996.

[HO04] Marlis Hochbruck and Alexander Ostermann. "Exponential Runge–Kutta methods for parabolic problems". In: *Applied Numerical Mathematics* 53.2-4 (2004), pp. 323–339.

[HO05] Marlis Hochbruck and Alexander Ostermann. "Explicit exponential Runge–Kutta methods for semilinear parabolic problems". In: *SIAM Journal on Numerical Analysis* 43.3 (2005), pp. 1069–1090.

[HS19] Jingwei Hu and Ruiwen Shu. *On the uniform accuracy of implicit-explicit backward differentiation formulas (IMEX-BDF) for stiff hyperbolic relaxation systems and kinetic equations*. 2019. arXiv: 1912.00559v1 [math.NA].

[HR07] Willem Hundsdorfer and Steven J Ruuth. "IMEX extensions of linear multistep methods with general monotonicity and boundedness properties". In: *Journal of Computational Physics* 225.2 (2007), pp. 2016–2042.

[Jin99] Shi Jin. "Efficient asymptotic-preserving (AP) schemes for some multiscale kinetic equations". In: *SIAM Journal on Scientific Computing* 21.2 (1999), pp. 441–454.

[JPT98] Shi Jin, Lorenzo Pareschi, and Giuseppe Toscani. "Diffusive relaxation schemes for multiscale discrete-velocity kinetic equations". In: *SIAM Journal on Numerical Analysis* 35.6 (1998), pp. 2405–2439.

[JPT00] Shi Jin, Lorenzo Pareschi, and Giuseppe Toscani. "Uniformly accurate diffusive relaxation schemes for multiscale transport equations". In: *SIAM Journal on Numerical Analysis* 38.3 (2000), pp. 913–936.

[JX95] Shi Jin and Zhouping Xin. "The relaxation schemes for systems of conservation laws in arbitrary space dimensions". In: *Communications on pure and applied mathematics* 48.3 (1995), pp. 235–276.

[LM08] Mohammed Lemou and Luc Mieussens. "A new asymptotic preserving scheme based on micro-macro formulation for linear kinetic equations in the diffusion limit". In: *SIAM Journal on Scientific Computing* 31.1 (2008), pp. 334–368.

[MZ09] Stefano Maset and Marino Zennaro. "Unconditional stability of explicit exponential Runge-Kutta methods for semi-linear ordinary differential equations". In: *Mathematics of computation* 78.266 (2009), pp. 957–967.

[Per68] Lawrence M Perko. "Higher order averaging and related methods for perturbed periodic and quasi-periodic systems". In: *SIAM Journal on Applied Mathematics* 17.4 (1968), pp. 698–724.

[Sak90] Kunimochi Sakamoto. "Invariant manifolds in singular perturbation problems for ordinary differential equations". In: *Proceedings of the Royal Society of Edinburgh Section A: Mathematics* 116.1-2 (1990), pp. 45–78.

[SAAP00] Eva Sánchez, Ovide Arino, Pierre Auger, and Rafael Bravo de la Parra. "A singular perturbation in an age-structured population model". In: *SIAM Journal on Applied Mathematics* 60.2 (2000), pp. 408–436.

[Vas63] Adelaida Borisovna Vasil'eva. "Asymptotic behaviour of solutions to certain problems involving non-linear differential equations containing a small parameter multiplying the highest derivatives". In: *Russian Mathematical Surveys* 18.3 (1963), p. 13.