MAJORANT SERIES FOR THE N-BODY PROBLEM

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ABSTRACT. As a follow-up of a previous work of the authors, this work considers *uniform global time-renormalization functions* for the *gravitational* N-body problem. It improves on the estimates of the radii of convergence obtained therein by using a completely different technique, both for the solution to the original equations and for the solution of the renormalized ones. The aforementioned technique which the new estimates are built upon is known as *majorants* and allows for an easy application of simple operations on power series. The new radii of convergence so-obtained are approximately doubled with respect to our previous estimates. In addition, we show that *majorants* may also be constructed to estimate the local error of the *implicit midpoint rule* (and similarly for Runge-Kutta methods) when applied to the time-renormalized N-body equations and illustrate the interest of our results for numerical simulations of the solar system.

1. Introduction

We are concerned here with the solution of the N-body problem: considering N masses m_i, $i = 1, \ldots, N$, moving in a three-dimensional space under the influence of gravitational forces, Newton's law describe the evolution of their *positions* q_i and *velocities* v_i for $i = 1, \ldots, N$ through the equations

$$m_i \frac{d^2 q_i}{dt^2} = \sum_{j \neq i} \frac{G m_i m_j}{\|q_j - q_i\|^3}(q_j - q_i), \tag{1}$$

where G is the gravitational constant and $\|q_j - q_i\|$ is the distance between q_i and q_j in the euclidean norm of $\mathbb{R}^3$.

The N-body equations are of great importance in physics and celestial mechanics in particular and the question of representing its solution in the form of a convergent series has been a long-standing problem. For an introduction on the historical and practical aspects of the subject, we refer to [2]. To make a long story short, let us just recall that the problem was solved for $N = 3$ by Karl Fritiof Sundman [11] and Qiu-Dong Wang for the $N \geq 3$-case [12] in the 1990s.

As an intermediate step to obtain a representation of each solution of (1) in terms of a series expansion convergent for all t in its maximal interval of existence, both authors rewrite the N-body equations (1) in terms of a new independent variable τ related to the physical time t by

$$\frac{d\tau}{dt} = s(q(t))^{-1}, \quad \tau(0) = 0,$$

for an appropriate *time-renormalization function* $s(q)$ depending on the positions $q = (q_1, \dots, q_N)$. In [2], we proposed new time-renormalization functions for the purpose of simulating the N-body problem with constant stepsizes without degrading the accuracy of the computed trajectories. In contrast with previously known functions [11, 12], ours depend not only on positions but also on velocities (up to our knowledge, for the first time).

Our time-renormalized equations were derived by considering estimates of the domain of existence of the holomorphic extension of maximal solutions of the N-body problem (1) to the complex domain. Noticeably, these global time-renormalizations were shown to be *uniform* in the sense that the solution of the time-renormalized equations in the fictitious time τ can be extended *analytically* to the strip

$$\{\tau \in \mathbb{C} \ : \ |\mathrm{Im}(\tau)| \leq \beta\}$$

for some $\beta > 0$ independent of the initial conditions $(q^0, v^0) \in \mathbb{R}^{6N}$ (provided that $q_i^0 \neq q_j^0$ for all $i \neq j$) and the masses m_i, $1 \leq i \leq N$.

Our main contribution in this paper is to explicitly construct *majorants* for the expansions of the solution of the N-body problem as a series in powers of either t or τ. We furthermore construct majorants for the expansion of discrete solutions of the time-renormalized equations in series of powers of the step-size τ .

Although not fully resorting to *Geometric Numerical Integration*, our work is related to this special issue as it paves the way for the development of geometric integrators for the N-body problem. As a matter of fact, constant step-size integration in physical time is not feasible for trajectories with close approaches (such as encountered in gravitational problems), as it would require unaffordable computations. In contrast, uniform global time-renormalization allows to numerically integrate efficiently the N-body problem with fixed step-sizes, as usually required for the geometric numerical integration of Hamiltonian systems.

Generally speaking, a power series $f = \sum_{k\geq 0} f_k\, t^k \in \mathbb{R}^n[[t]]$ is said to be *majored* by $\bar{f} = \sum_{k\geq 0} \bar{f}_k\, t^k \in \mathbb{R}_+[[t]]$ if, for all $k \in \mathbb{N}$,

$$\|f_k\| \leq \bar{f}_k$$

where $\|\cdot\|$ is the euclidean norm in $\mathbb{R}^n$ and we then write

$$f \trianglelefteq \bar{f}.$$

The application of the technique of majorant equations goes back to the proof of Cauchy-Kovalevskaya theorem [3, 10, 6] (see also [5] for a specific application to ordinary differential equations) and has several advantages in our context:

- it allows for an easy estimate of the radius of convergence of the series f;
- simple rules on majorant series apply to the usual operations on power series, such as addition, multiplication, derivation and integration;
- when used for the time-renormalized N-body equations, it leads to improved estimates of the value of β;
- it can be used to analyze numerical discretizations of the time-renormalized equations, and in particular, to obtain bounds for their local errors.

We conclude this introductory section with the outline of the article. In Section 2 we will define and derive the main rules that apply to majorants. We then construct majorants for the power expansions of the solutions of (1) in Section 3, and accordingly, majorants for the power expansions of the solutions of renormalized equations in Section 4. In Section 5 we give a majorant series for the implicit mid-point rule discretization (and more generally for arbitrary Runge-Kutta discretizations,) which leads to uniform bounds for the local errors. Alternative time-renormalization functions are proposed in Section 6. In Section 7 we illustrate our results for the numerical simulation of a 15-body model of the solar system.

2. Basic properties of majorants

We denote by $\mathbb{R}^n[[t]]$ the set of *formal* power series in t with coefficients in $\mathbb{R}^n$. Given $f = \sum_{k\geq 0} f_k\, t^k \in \mathbb{R}^n[[t]]$, we denote

$$f' = \sum_{k\geq 0} (k+1)\, f_{k+1}\, t^k \in \mathbb{R}^n[[t]],$$

and

$$\int f = \int_0^t \left(\sum_{k\geq 0} f_k\, s^k \right) ds = \sum_{k\geq 1} \frac{1}{k} f_{k-1}\, t^k \in \mathbb{R}^n[[t]].$$

Definition 1. *Given $f = \sum_{k\geq 0} f_k\, t^k \in \mathbb{R}^n[[t]]$ and $\bar{f} = \sum_{k\geq 0} \bar{f}_k\, t^k \in \mathbb{R}[[t]]$, we say that f is* majored *by $\bar{f}$ and we write*

$$f \trianglelefteq \bar{f}$$

if, for all $k \in \mathbb{N}$,

$$\|f_k\| \leq \bar{f}_k$$

where $\|\cdot\|$ is the euclidean norm in $\mathbb{R}^n$.

Remark 1. *If $f \trianglelefteq \bar{f}$, then the coefficients of the majorant series $\bar{f}$ are necessarily non-negative, that is, $\bar{f} \in \mathbb{R}_+[[t]]$, where $\mathbb{R}_+ = \{x \in \mathbb{R} \ : \ x \geq 0\}$.*

Proposition 1. *Let $f, g \in \mathbb{R}^n[[t]]$, $h \in \mathbb{R}[[t]]$ and $\bar{f}, \bar{g}, \bar{h} \in \mathbb{R}_+[[t]]$. Then*

$$f \trianglelefteq \bar{f} \text{ and } \bar{f} \trianglelefteq \bar{g} \Longrightarrow f \trianglelefteq \bar{g}, \tag{2}$$

$$f \trianglelefteq \bar{f} \text{ and } g \trianglelefteq \bar{g} \Longrightarrow f + g \trianglelefteq \bar{f} + \bar{g}, \tag{3}$$

$$f \trianglelefteq \bar{f} \text{ and } h \trianglelefteq \bar{h} \Longrightarrow f\, h \trianglelefteq \bar{f}\, \bar{h}, \tag{4}$$

$$f \trianglelefteq \bar{f} \text{ and } g \trianglelefteq \bar{g} \Longrightarrow \langle f, g \rangle \trianglelefteq \bar{f}\, \bar{g}, \tag{5}$$

$$f \trianglelefteq \bar{f} \Longrightarrow \|f\|^2 \trianglelefteq \bar{f}^2, \tag{6}$$

$$f \trianglelefteq \bar{f} \Longrightarrow f' \trianglelefteq \bar{f}', \tag{7}$$

$$f \trianglelefteq \bar{f} \Longrightarrow \int f \trianglelefteq \int \bar{f}. \tag{8}$$

Proof. The assertions (2) and (3) are immediate consequences of Definition 1. The assertion (4) follows from the Cauchy product of series: if $f = \sum_{l \geq 0} t^l f_l$ and $h = \sum_{l \geq 0} t^l h_l$, then

$$hf = \sum_{l \geq 0} t^l \sum_{k=0}^{l} h_k f_{l-k}$$

so that

$$\|(hf)_l\| \leq \sum_{k=0}^{l} \|h_k f_{l-k}\| \leq \sum_{k=0}^{l} |h_k| \|f_{l-k}\| \leq \sum_{k=0}^{l} \bar{h}_k \bar{f}_{l-k} = (\bar{h}\bar{f})_l.$$

As for (5), we write

$$f = \sum_{k \geq 0} t^k f_k \quad \text{and} \quad g = \sum_{l \geq 0} t^l g_l$$

so that

$$\langle f, g \rangle = \sum_{l \geq 0} t^l \sum_{k=0}^{l} \langle f_k, g_{l-k} \rangle.$$

Now, by Cauchy-Schwartz inequality, we have

$$|\langle f_k, g_{l-k} \rangle| \leq \|f_k\| \|g_{l-k}\|$$

so that

$$|(\langle f, g \rangle)_l| \leq \sum_{k=0}^{l} |\langle f_k, g_{l-k} \rangle| \leq \sum_{k=0}^{l} \|f_k\| \|g_{l-k}\| \leq \sum_{k=0}^{l} \bar{f}_k \bar{g}_{l-k} = (\bar{f}\bar{g})_l.$$

Obviously, (6) follows from (5).

As for (7) and (8), we have

$$f' = \sum_{k\geq 0^*} t^{k-1} k f_k \quad \text{and} \quad \int f = \sum_{k\geq 0} t^{k+1} \frac{1}{k+1} f_k$$

so that

$$\forall k \geq 0, \quad |f'_k| = (k+1)|f_{k+1}| \leq (k+1)|\bar{f}_{k+1}| = |(\bar{f})'_k|$$

and

$$\forall k \geq 1, \quad \left(\int f\right)_k = \frac{1}{k}|f_{k-1}| \leq \frac{1}{k}|\bar{f}_{k-1}| = \left(\int \bar{f}\right)_k.$$

□

Proposition 2. *If $f = 1 + \sum_{l\geq 1} t^l f_l \in \mathbb{R}[[t]]$, then, for any $\nu \in \mathbb{R}$, $p = (f)^\nu = 1 + \sum_{k\geq 1} t^k p_k \in \mathbb{R}[[t]]$, where*

$$\forall k \geq 1, \quad p_k = \frac{1}{k}\sum_{j=0}^{k-1}((k-j)\,\nu - j) f_{k-j}\, p_j. \tag{9}$$

Morevover, for any $\nu < 0$, we have

$$f \trianglelefteq \bar{f} \Longrightarrow f^\nu \trianglelefteq (2-\bar{f})^\nu. \tag{10}$$

Proof. We first observe that $p_0 = 1$ and

$$p'(t) f(t) = \nu f^\nu(t) f'(t)$$

implies (9). This very same formula for $\bar{p} = (2-\bar{f})^\nu$ now gives

$$\bar{p}_k = \frac{1}{k}\sum_{j=0}^{k-1}((k-j)\,\nu - j)(-\bar{f}_{k-j})\bar{p}_j = \frac{1}{k}\sum_{j=0}^{k-1}(j-(k-j)\,\nu)\bar{f}_{k-j}\bar{p}_j$$

and for $\nu < 0$, it is clear that $\bar{p}_k \geq 0$ and furthermore, under the assumption $f \trianglelefteq \bar{f}$, that

$$|p_k| \leq \frac{1}{k}\sum_{j=0}^{k-1}|(k-j)\,\nu - j||f_{k-j}||p_j| \leq \frac{1}{k}\sum_{j=0}^{k-1}|(k-j)\,\nu - j|\;\bar{f}_{k-j}\;|p_j|$$

which, by an induction argument, implies relation (10). □

3. N-body problem: Equations in physical time

Let us consider a Newtonian N-body gravitational problem,

$$\begin{aligned} \frac{dq_i}{dt} &= v_i, \\ \frac{dv_i}{dt} &= g_i(q_1, \ldots, q_N), \end{aligned} \tag{11}$$

with

$$g_i(q_1, \ldots, q_N) = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{G\, m_j}{||q_i - q_j||^3}(q_j - q_i) \tag{12}$$

for $i = 1, \ldots, N$ and where each $q_i \in \mathbb{R}^3$ represents the coordinates of the i-th body and $v_i \in \mathbb{R}^3$ its velocity.

Since the right-hand side of (11) is smooth provided that $q_i \neq q_j$ for all $1 \leq i < j \leq N$, the equations (11) supplemented with the initial conditions

$$q_i(0) = q_i^0, \quad v_i(0) = v_i^0, \quad i = 1, \ldots, N, \tag{13}$$

admit a unique formal solution as series in powers of t (that is, $q_i \in \mathbb{R}^3[[t]]$, $v_i \in \mathbb{R}^3[[t]]$, $i = 1, \ldots, N$) for regular initial values, that is, provided that

$$q_i^0 \neq q_j^0 \quad \text{for all} \quad 1 \leq i < j \leq N. \tag{14}$$

We denote $q = (q_1, \ldots, q_N)$ and $v = (v_1, \ldots, v_N)$. For later use, we also denote

$$\mu(q, v) = \max_{1 \leq i < j \leq N} \frac{\|v_i - v_j\|}{\|q_i - q_j\|}, \quad \nu(q) = \max_{1 \leq i < j \leq N} \frac{M_{ij}(q)}{\|q_i - q_j\|}, \tag{15}$$

where for $1 \leq i < j \leq N$,

$$K_i(q) = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{G\, m_j}{\|q_i - q_j\|^2}, \quad M_{ij}(q) = K_i(q) + K_j(q). \tag{16}$$

Lemma 1. *Consider the power series expansion*

$$(q, v) = (q_1, \ldots, q_N, v_1, \ldots, v_N) \in \mathbb{R}^{6N}[[t]]$$

of the solution of (11)–(13) with (14). If the power series

$$\rho = 1 + \sum_{k \geq 1} \rho_k\, t^k \in \mathbb{R}_+[[t]] \tag{17}$$

is a majorant of $\frac{q_i-q_j}{\|q_i^0-q_j^0\|}$ *for all* $i \neq j$, *i.e.*

$$\forall 1 \leq i < j \leq N, \quad q_i - q_j \trianglelefteq \|q_i^0 - q_j^0\|\, \rho, \tag{18}$$

then,

$$\forall 1 \leq i \leq N, \quad g_i(q) \trianglelefteq K_i(q^0) \frac{\rho}{(2-\rho^2)^{3/2}}.$$

Proof. Using (18) in combination with (6) and $\bar{f} = \|q_i^0 - q_j^0\|\rho$ implies

$$\forall (i,j) \in \{1,\dots,N\}^2, \quad \|q_i - q_j\|^2 \trianglelefteq \|q_i^0 - q_j^0\|^2 \rho^2,$$

i.e.

$$\forall i \neq j, \quad \frac{\|q_i - q_j\|^2}{\|q_i^0 - q_j^0\|^2} \trianglelefteq \rho^2.$$

Upon using (10) with $\nu = -3/2$ we obtain

$$\forall i \neq j, \quad \frac{\|q_i^0 - q_j^0\|^3}{\|q_i - q_j\|^3} \trianglelefteq (2-\rho^2)^{-3/2},$$

which, in combination with (18) and using (4), leads to

$$\forall i \neq j, \quad \frac{q_j - q_i}{\|q_i - q_j\|^3} \trianglelefteq \frac{1}{\|q_j^0 - q_i^0\|^2} \frac{\rho}{(2-\rho^2)^{3/2}}$$

and finally to the result by multiplying both sides by $h = \bar{h} = Gm_j$ (using again (4)) and then summing over all $j \neq i$ (using (3)) . □

Lemma 2. *Let $\mathcal{A}$ be the set of power series of the form (17) satisfying (18), and consider the operator*

$$\begin{array}{rcl} \Phi : \mathcal{A} & \to & \mathbb{R}[[t]] \\ \rho & \mapsto & 1 + \mu(q^0, v^0)\, t + \nu(q^0) \int\int \frac{\rho}{(2-\rho^2)^{3/2}}. \end{array} \tag{19}$$

Then $\Phi(\mathcal{A}) \subset \mathcal{A}$.

Proof. Owing to Lemma 1, we then have

$$\forall 1 \leq i \leq N, \quad g_i(q) \trianglelefteq K_i(q^0) \frac{\rho}{(2-\rho^2)^{3/2}},$$

so that, by virtue of inequality (8)

$$\forall 1 \leq i \leq N, \quad \int (g_i(q) - g_j(q)) \trianglelefteq M_{ij}(q^0) \int \frac{\rho}{(2-\rho^2)^{3/2}}.$$

and

$$\forall 1 \leq i \leq N, \quad \int\int (g_i(q) - g_j(q)) \trianglelefteq M_{ij}(q^0) \int\int \frac{\rho}{(2-\rho^2)^{3/2}}.$$

Now, from the integral form of the second equation of (11) we have

$$v_i - v_j = v_i^0 - v_j^0 + \int (g_i(q) - g_j(q))$$

which translates into

$$v_i - v_j \trianglelefteq \|v_i^0 - v_j^0\| + M_{ij}(q^0) \int \frac{\rho}{(2-\rho^2)^{3/2}}.$$

Similarly, the integral form of the first equation of (11) leads to

$$q_i - q_j = q_i^0 - q_j^0 + t\,(v_i^0 - v_j^0) + \int\int (g_i(q) - g_j(q)),$$

which in turn implies that, for $1 \le i < j \le N$,

$$\begin{aligned} q_i - q_j &\trianglelefteq \|q_i^0 - q_j^0\| + \|v_i^0 - v_j^0\| t + M_{ij}(q^0) \int\int \frac{\rho}{(2-\rho^2)^{3/2}} \\ &\trianglelefteq \|q_i^0 - q_j^0\| \left(1 + \mu(q^0, v^0)\, t + \nu(q^0) \int\int \frac{\rho}{(2-\rho^2)^{3/2}}\right), \end{aligned}$$

that is, $\rho \in \mathcal{A}$. □

Theorem 1. *The power series expansion*

$$(q, v) = (q_1, \dots, q_N, v_1, \dots, v_N) \in \mathbb{R}^{6N}[[t]]$$

of the solution (11)–(13) with (14), satisfies (18), where $\rho \in \mathbb{R}_+[[t]]$ is the unique power series solution of the following initial value problem

$$\rho'' = \nu(q^0)\, \frac{\rho}{(2-\rho^2)^{3/2}}, \quad \rho_0 = 1, \quad \rho_0' = \mu(q^0, v^0). \tag{20}$$

Furthermore, for each $i = 1, 2, \dots, N$,

$$q_i - q_i^0 - t\, v_i^0 \trianglelefteq \min_{1 \le j \le N} \|q_i^0 - q_j^0\| \sum_{k=2}^{\infty} \rho_k\, t^k. \tag{21}$$

Proof. Consider

$$\rho^{[0]} = 1 + \sum_{k \ge 1} \rho_k^{[0]}\, t^k \in \mathbb{R}_+[[t]]$$

such that, for all $k \ge 1$,

$$\rho_k^{[0]} = \max_{i \ne j} \frac{\|(q_i - q_j)_k\|}{\|q_i^0 - q_j^0\|}.$$

Here, $(q_i - q_j)_k \in \mathbb{R}^3$ denotes the coefficient for t^k of $(q_i - q_j) \in \mathbb{R}^3[[t]]$, that is to say

$$\frac{1}{k!}\,\frac{d^k}{dt^k}(q_i(t) - q_j(t))\Big|_{t=0}.$$

Lemma 2 implies that

$$\forall m \geq 1, \quad \forall 1 \leq i < j \leq N, \quad q_i - q_j \trianglelefteq \|q_i^0 - q_j^0\|\,\rho^{[m]} \tag{22}$$

where $\rho^{[m]} = \Phi(\rho^{[m-1]})$ for $m \geq 1$. Clearly, the operator Φ satisfies the following property [1]: for each $k \geq 1$, $(\Phi(\rho))_k$ is a polynomial of the coefficients ρ_l for $l \leq k-2$.

The sequence $\{\rho^{[m]}\}_{m\in\mathbb{N}}$ converges towards a limit

$$\rho^{[\infty]} = 1 + \sum_{k\geq 1} \rho_k^{[\infty]}\, t^k \in \mathbb{R}_+[[t]]$$

in the sense that for each index $k \geq 0$, the sequence $\{\rho_k^{[m]}\}_{m\in\mathbb{N}}$ is ultimately constant, i.e., there exists $m_k \geq 1$ such that $\rho_k^{[m]} = \rho_k^{[\infty]}$ for all $m \geq m_k$. Indeed, assume that this is not the case. Let k be the smallest index $l \geq 2$ such that the sequence $\{\rho_l^{[m]}\}_{m\in\mathbb{N}}$ is not ultimately constant. Since $\rho_k^{[m]} = (\Phi(\rho^{[m-1]}))_k$ is a polynomial of the coefficients $\rho_l^{[m-1]}$ for $l \leq k-2$ and the sequences $\{\rho_l^{[m]}\}_{m\in\mathbb{N}}$ (for each for $l \leq k-2$) are ultimately constant, we get a contradiction. This limit is the unique solution of the fixed point equation

$$\rho^{[\infty]} = \Phi(\rho^{[\infty]})$$

and it is thus clear from (22) that estimate (18) holds for $\rho = \rho^{[\infty]}$, the solution of

$$\rho = 1 + \mu(q^0, v^0)\, t + \nu(q^0) \int\!\!\int \frac{\rho}{(2-\rho^2)^{3/2}}, \tag{23}$$

or in other words, the unique power series solution of (20).

Finally, (21) follows from applying Lemma 1 to

$$q_i = q_i^0 + t\, v_i^0 + \int\!\!\int g_i(q)$$

and taking into account (23), which leads to

$$q_i - q_i^0 - t\, v_i^0 \trianglelefteq \frac{K_i(q^0)}{\nu(q^0)} \sum_{k=2}^{\infty} \rho_k\, t^k \trianglelefteq \min_{1\leq j\leq N} \|q_i^0 - q_j^0\| \sum_{k=2}^{\infty} \rho_k\, t^k.$$

□

[1]Such an operator is called a *Noetherian* operator in [5].

Clearly, the solution $\rho(t)$ of (20) is $\rho(t) = 1+\lambda(t\sqrt{\mu_0^2+\nu_0})$ where $\lambda(t)$ is the solution of

$$\lambda'' = (1-\eta_0)\frac{1+\lambda}{(1-2\lambda-\lambda^2)^{3/2}}, \quad \lambda(0)=0, \quad \lambda'(0)=\sqrt{\eta_0}, \tag{24}$$

with

$$\eta_0 = \frac{\mu_0^2}{\mu_0^2+\nu_0}, \quad \mu_0 = \max_{1\leq i<j\leq N} \frac{\|v_i^0 - v_j^0\|}{\|q_i^0-q_j^0\|}, \quad \nu_0 = \max_{1\leq i<j\leq N} \frac{M_{ij}(q^0)}{\|q_i^0-q_j^0\|}.$$

Since $\mu_0 \geq 0$ and $\nu_0 > 0$, we have that $0 \leq \eta_0 < 1$. The right-hand side of that second order differential equation being analytic at $\lambda = 0$, the power series expansion of the solution $\lambda(t)$ of (24) is convergent for small enough $t \in \mathbb{R}$.

Next, we give a formula for the radius of convergence $r(\eta_0)$ of the power series expansion of the solution $\lambda(t)$ of (24) for $\eta_0 \in (0,1)$.

Proposition 3. *For $\eta_0 \in (0,1)$, let f be the function*

$$f(\lambda) = \left(\eta_0 + 2\,(1-\eta_0)\,\left((1-2\lambda-\lambda^2)^{-1/2}-1\right)\right)^{-1/2}. \tag{25}$$

The differential equation (24) has an analytic solution $t \mapsto \lambda(t)$ defined on the disk $D_{r(\eta_0)}(0)$ with

$$r(\eta_0) = \int_0^{\sqrt{2}-1} f(\sigma) d\sigma. \tag{26}$$

Proof. The right-hand side of (24) being holomorphic in $\lambda = 0$, it has an holomorphic solution $t \mapsto \lambda(t)$ in a neighbourhood of the origin $0 \in \mathbb{C}$. Let $R > 0$ be the radius of convergence of the series expansion

$$\lambda(t) = \sum_{k=1}^{\infty} \lambda_k\, t^k \tag{27}$$

of $\lambda(t)$. Clearly, $\lambda(\rho) < \sqrt{2}-1$ for $0 < \rho < R$; otherwise, it should exist $0 < \rho < R$ such that $\lambda(\rho) = \sqrt{2}-1$, which is incompatible with $\lambda''(\rho) = h(\sqrt{2}-1)$. Given that the function

$$h : \sigma \mapsto \frac{1+\sigma}{(1-2\sigma-\sigma^2)^{3/2}}$$

has a series expansion around $\sigma = 0$ with real positive coefficients, we have that

$$\forall k \geq 1, \quad \lambda_k \geq 0. \tag{28}$$

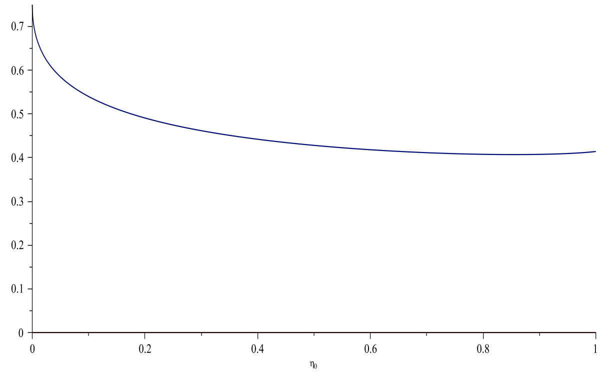

FIGURE 1. Radius of convergence $r(\eta_0)$ as a function of $\eta_0 \in [0,1)$

Hence,

$$\forall n \geq 1, \quad \lambda^{[n]}(t) := \sum_{k=1}^{n} \lambda_k \, \rho^k < \sqrt{2}-1$$

provided that $0 \leq \rho < R$, and thus $\lambda^{[n]}(R) \leq \sqrt{2}-1$, which implies, thanks to (28) that (27) is convergent for $t = R$ and that $\lambda(R) < \sqrt{2}-1$. In turn, this implies that (27) is also convergent for all $t^* \in \mathcal{B}(R) := \{t \in \mathbb{C} \;:\; |t| = R\}$ and that

$$|\lambda(t^*)| \leq \lambda(R) \leq \sqrt{2}-1.$$

Actually, $\lambda(R) = \sqrt{2}-1$. Indeed, if $\lambda(R) < \sqrt{2}-1$, then $h(\sigma)$ is holomorphic for all $\sigma = \lambda(t^*)$, $t^* \in \mathcal{B}(R)$. Consequently, $\lambda(t)$ is holormorphic in $\mathcal{B}(R)$ and hence in the closed disk $\{t \in \mathbb{C} \;:\; |t| \leq R\}$, which contradicts the assumption that R is the radius of convergence of (27) (i.e., the distance to the nearest singularity of $\lambda(t)$).

If $\eta_0 > 0$, multiplying both sides of equation (24) by $2\lambda'$ and applying the operator $\int$ on both sides we obtain the first-order differential equation

$$f(\lambda)\,\lambda' = 1, \quad \lambda(0) = 0, \tag{29}$$

so that, for $0 \leq \rho < R$, $F(\lambda(\rho)) = \rho$, where

$$F(\lambda) = \int_0^{\lambda} f(\sigma) d\sigma.$$

By continuity of F, we finally have that $R = F(\lambda(R)) = F(\sqrt{2}-1)$.

□

In view of Theorem 1, we conclude that the power series expansion

$$(q, v) = (q_1, \ldots, q_N, v_1, \ldots, v_N) \in \mathbb{R}^{6N}[[t]]$$

of the solution of (11)–(13) with (14) is convergent for all $t \in \mathbb{R}$ such that

$$|t| < \frac{r(\eta_0)}{\sqrt{\mu_0^2 + \nu_0}},$$

where

$$\eta_0 := \frac{\mu_0^2}{\mu_0^2 + \nu_0} \in (0, 1].$$

Hence, we get as a corollary of Theorem 1 the following result:

Corollary 1. *The solution of (11)–(13) with (14) admits an holomorphic extension as a function of the complex time t in the disk*

$$\left\{ t \in \mathbb{C} \ : \ |t| < \frac{r(\eta_0)}{\sqrt{\mu_0^2 + \nu_0}} \right\}.$$

Remark 2. *In* [2]*, the statement in Corollary 1 was proven with a different approach, and a disk of different radius, namely*

$$\sup_{0<\sigma<\sqrt{2}-1} 2\sigma \left(\max_{1\leq i<j\leq N} \left(\frac{\|v_i^0 - v_j^0\|}{\|q_i^0 - q_j^0\|} + \sqrt{\frac{\|v_i^0 - v_j^0\|^2}{\|q_i^0 - q_j^0\|^2} + \kappa(\sigma)\, \frac{M_{ij}(q^0)}{\|q_i^0 - q_j^0\|}} \right) \right)^{-1},$$

where

$$\kappa(\sigma) := \frac{2\,\sigma\,(1+\sigma)}{(1 - 2\sigma - \sigma^2)^{3/2}}.$$

If the maxima

$$\mu_0 = \max_{1\leq i<j\leq N} \frac{\|v_i^0 - v_j^0\|}{\|q_i^0 - q_j^0\|} \quad \text{and} \quad \nu_0 = \max_{1\leq i<j\leq N} \frac{M_{ij}(q^0)}{\|q_i^0 - q_j^0\|}$$

are attained at a common pair of indices (i, j) (which is typically the case, specially in binary close encounters), then this coincides with

$$\sup_{0<\sigma<\sqrt{2}-1} 2\sigma \left(\mu_0 + \sqrt{\mu_0^2 + \kappa(\sigma)\, \nu_0} \right)^{-1} = \frac{\hat{r}(\eta_0)}{\sqrt{\mu_0^2 + \nu_0}},$$

where

$$\hat{r}(\eta_0) := \sup_{0<\sigma<\sqrt{2}-1} 2\sigma \left(\sqrt{\eta_0} + \sqrt{\eta_0 + \kappa(\sigma)\,(1 - \eta_0)} \right)^{-1}.$$

One can check that $\hat{r}(\eta_0) \leq r(\eta_0)$ for all $\eta_0 \in [0, 1]$. For instance,

- *if* $\eta_0 = 1/2$ *(corresponding to the case* $\nu_0 = \mu_0^2$*), then* $\gamma = -1+\sqrt{2}$ *and*

$$r(1/2) = F(\lambda) = \int_0^{\gamma(1/2)} \frac{1}{\sqrt{1/2 - 1 + \frac{1}{\sqrt{1-2\sigma-\sigma^2}}}} d\sigma \approx 0.42812819,$$

while

$$\hat{r}(1/2) = \frac{2\sqrt{2}\sigma}{1+\sqrt{1+\kappa(\sigma)}} \approx 0.25796556.$$

- *if* $\eta_0 \to 1$*, then* $\gamma(\eta_0) \to -1+\sqrt{2}$ *and*

$$r(\eta_0) \to \int_0^{\gamma(\eta_0)} 1 d\sigma = \gamma(\eta_0) = -1+\sqrt{2},$$

which coincides with

$$\hat{r}(0) = \sup_{0<\sigma<\sqrt{2}-1} \sigma = -1+\sqrt{2}.$$

Remark 3. *We have numerically checked that*

$$\frac{\sqrt{2}-1}{\mu_0 + \sqrt{\nu_0/3}} < \frac{r(\eta_0)}{\sqrt{\mu_0^2+\nu_0}} < \frac{0.48}{\mu_0+\sqrt{\nu_0/3}}.$$

4. The N-body problem: Time-renormalized equations

Consider an N-body problem described with time-renormalized equations

$$\begin{aligned} \frac{dQ_i}{d\tau} &= s(Q,V)\, V_i, \qquad i = 1,\dots,N, \\ \frac{dV_i}{d\tau} &= s(Q,V)\, g_i(Q), \quad i = 1,\dots,N, \end{aligned} \tag{30}$$

where

$$g_i(Q) = \sum_{j\neq i} \frac{G\, m_j}{\|Q_i - Q_j\|^3}(Q_j - Q_i).$$

For smooth time-renormalization functions $S(Q,V)$, the solution of (30) supplemented with initial conditions

$$Q_i(0) = q_i^0, \quad V_i(0) = v_i^0, \quad i = 1,\dots,N, \tag{31}$$

admits a unique formal power series expansion ($Q_i \in \mathbb{R}^3[[\tau]]$, $V_i \in \mathbb{R}^3[[\tau]]$, $i = 1,\dots,N$) provided that (14) holds. We want to choose a real analytic function $S(Q,V)$ in such a way that the radius of convergence of such power series expansions are uniformly bounded from below by a positive constant β for all regular initial values and all values of the masses.

As a candidate for $S(Q,V)$, we choose a real-analytic function such that $s(q^0,v^0)$ is, up to a constant factor, a lower bound of the radius of convergence of the series expansion in powers of the physical time t of the solution $(q(t),v(t))$ of the N-body problem (11)–(13). Hence, based on Corollary 1, we determine the function $S(Q,V)$ as the real-analytic lower bound of

$$\left(\max_{1\leq i<j\leq N}\left(\frac{\|V_i-V_j\|}{\|Q_i-Q_j\|}\right)^2+\max_{1\leq i<j\leq N}\frac{M_{ij}(Q)}{\|Q_i-Q_j\|}\right)^{-1/2}$$

obtained by replacing the two ∞-norms by 1-norms, that is,

$$s(Q,V)=\left(\sum_{1\leq i<j\leq N}\frac{\|V_i-V_j\|^2}{\|Q_i-Q_j\|^2}+\sum_{1\leq i<j\leq N}\frac{M_{ij}(Q)}{\|Q_i-Q_j\|}\right)^{-1/2}. \tag{32}$$

This is precisely the time-renormalization function (32) introduced in [2].

Lemma 3. *Consider the power series expansion*

$$(Q,V)=(Q_1,\dots,Q_N,V_1,\dots,V_N)\in\mathbb{R}^{6N}[[\tau]]$$

of the solution of the equations (30) supplemented with the initial conditions (31) with (14). If the power series

$$\xi=1+\sum_{k\geq 1}\xi_k\,\tau^k\in\mathbb{R}_+[[\tau]],\quad \zeta=\sum_{k\geq 1}\zeta_k\,\tau^k\in\mathbb{R}_+[[\tau]] \tag{33}$$

are such that, for $1\leq i<j\leq N$

$$Q_i-Q_j\trianglelefteq\|q_i^0-q_j^0\|\,\xi, \tag{34}$$

$$V_i-V_j\trianglelefteq\|v_i^0-v_j^0\|+s(q^0,v^0)\,M_{ij}(q^0)\,\zeta, \tag{35}$$

then, for $1\leq i<j\leq N$

$$g_i(Q)\trianglelefteq K_i(q^0)\frac{\xi}{(2-\xi^2)^{3/2}}.$$

and

$$s(Q,V)\trianglelefteq s(q^0,v^0)\,(2-\chi(\xi,\zeta))^{-1/2},$$

where

$$\chi(\xi,\zeta)=(2-\xi^2)^{-1}\,(2\zeta+\zeta^2+(2-\xi^2)^{-1/2}). \tag{36}$$

Proof. Proceeding as in the proof of Lemma 1, we obtain

$$\forall 1 \le i < j \le N, \quad \frac{1}{\|Q_i - Q_j\|^2} \trianglelefteq \frac{1}{\|q_i^0 - q_j^0\|^2} (2 - \xi^2)^{-1}; \tag{37}$$

then

$$\forall 1 \le i < j \le N, \quad \frac{1}{\|Q_i - Q_j\|^3} \trianglelefteq \frac{1}{\|q_i^0 - q_j^0\|^3} (2 - \xi^2)^{-3/2}$$

and eventually, for $1 \le i \le N$,

$$g_i(Q) \trianglelefteq K_i(q^0) \frac{\xi}{(2 - \xi^2)^{3/2}}.$$

Similarly, we get

$$\forall 1 \le i < j \le N, \quad \frac{1}{\|Q_i - Q_j\|} \trianglelefteq \frac{1}{\|q_i^0 - q_j^0\|} (2 - \xi^2)^{-1/2} \tag{38}$$

so that by summing over j and using relation (3) of Proposition 1, we get

$$K_i(Q) \trianglelefteq K_i(q^0)\,(2 - \xi^2)^{-1}. \tag{39}$$

Now, from assumption (35) and inequality (5) of Proposition 1, we have

$$\begin{aligned} \|V_i - V_j\|^2 &\trianglelefteq \left(\|v_i^0 - v_j^0\| + s(q^0, v^0)\, M_{ij}(q^0)\, \zeta\right)^2 \\ &\trianglelefteq \|v_i^0 - v_j^0\|^2 + \|q_i^0 - q_j^0\|\, M_{ij}(q^0)\,(2\zeta + \zeta^2), \end{aligned} \tag{40}$$

where we have further applied the inequalities

$$s(q^0, v^0) \le \frac{\|q_i^0 - q_j^0\|}{\|v_i^0 - v_j^0\|} \quad \text{and} \quad s(q^0, v^0)^2 \le \frac{\|q_i^0 - q_j^0\|}{M_{ij}(q^0)}. \tag{41}$$

Combining (37) and (40) through inequality (4) of Proposition 1, we get

$$\frac{\|V_i - V_j\|^2}{\|Q_i - Q_j\|^2} \trianglelefteq \frac{\|v_i^0 - v_j^0\|^2}{\|q_i^0 - q_j^0\|^2} (2 - \xi^2)^{-1} + \frac{M_{ij}(q^0)}{\|q_i^0 - q_j^0\|} (2\zeta + \zeta^2)\,(2 - \xi^2)^{-1}.$$

Similarly, combining (38) and (39), we obtain

$$\frac{M_{ij}(Q)}{\|Q_i - Q_j\|} \trianglelefteq \frac{M_{ij}(q^0)}{\|q_i^0 - q_j^0\|} (2 - \xi^2)^{-1}\,(2 - \xi^2)^{-1/2}.$$

Finally, consider

$$s_A(Q, V) = \sum_{1 \le i < j \le N} \frac{\|V_i - V_j\|^2}{\|Q_i - Q_j\|^2},$$

$$s_B(Q, V) = \sum_{1 \le i < j \le N} \frac{M_{ij}(Q)}{\|Q_i - Q_j\|}.$$

so that

$$s(Q,V) = (s_A(Q,V) + s_B(Q,V))^{-1/2}.$$

Then,

$$\begin{aligned} s_A(Q,V) + s_B(Q) &\trianglelefteq \left(s_A(q^0,v^0) + s_B(q^0)\,(2\zeta + \zeta^2 + (2-\xi^2)^{-1/2})\right)\,(2-\xi^2)^{-1}, \\ &\leq \left(s_A(q^0,v^0) + s_B(q^0)\right)\,\chi(\xi,\zeta), \end{aligned}$$

where $\chi(\xi,\zeta)$ is given in terms of ξ and ζ by (36), and it follows from (10) with $\nu = -1/2$ that

$$s(Q,V) \trianglelefteq s(q^0,v^0)\,(2-\chi(\xi,\zeta))^{-1/2}.$$

□

Lemma 4. *Let $\mathcal{B}$ be the set of $(\xi,\zeta) \in (1+\tau\mathbb{R}[[\tau]]) \times \tau\mathbb{R}[[\tau]]$ satisfying (34)–(35), and consider the operator*

(42)

$$\begin{aligned} \Psi: \quad (1+\tau\mathbb{R}[[\tau]]) \times \tau\mathbb{R}[[\tau]] &\to \mathbb{R}[[\tau]] \times \mathbb{R}[[\tau]] \\ (\xi,\zeta) &\mapsto \left(1 + \int (2-\chi(\xi,\zeta))^{-1/2}\,(1+\zeta), \int \frac{(2-\chi(\xi,\zeta))^{-1/2}\,\xi}{(2-\xi^2)^{3/2}}\right). \end{aligned}$$

Then $\Psi(\mathcal{B}) \subset \mathcal{B}$.

Proof. Recall that

$$Q_i - Q_j = q_i^0 - q_j^0 + \int s(Q,V)\,(V_i - V_j), \tag{43}$$

$$V_i - V_j = v_i^0 - v_j^0 + \int s(Q,V)\,\left(g_i(Q) - g_j(Q)\right). \tag{44}$$

If $(\xi,\zeta) \in \mathcal{B}$, then (43), Lemma 3, and (41) imply that, for $1 \leq i < j \leq N$

$$\begin{aligned} Q_i - Q_j &\trianglelefteq \|q_i^0 - q_j^0\| + s(q^0,v^0)\,\|v_i^0 - v_j^0\| \int (2-\chi(\xi,\zeta))^{-1/2} \\ &\quad + s(q^0,v^0)^2\,M_{ij}(q^0) \int (2-\chi(\xi,\zeta))^{-1/2}\,\zeta \\ &\trianglelefteq \|q_i^0 - q_j^0\| \left(1 + \int (2-\chi(\xi,\zeta))^{-1/2}\,(1+\zeta)\right). \end{aligned}$$

Similarly, (44) and Lemma 3 imply that

$$V_i - V_j \trianglelefteq \|v_i^0 - v_j^0\| + s(q^0,v^0)\,M_{ij}(q^0) \int \frac{(2-\chi(\xi,\zeta))^{-1/2}\,\xi}{(2-\xi^2)^{3/2}}.$$

Hence, we conclude that $\Psi(\xi,\zeta) \in \mathcal{B}$. □

Theorem 2. *The power series representation*

$$(Q,V) = (Q_1,\dots,Q_N,V_1,\dots,V_N) \in \mathbb{R}^{6N}[[\tau]]$$

of the solution of equations (30) supplemented with the initial conditions (31) with (14) satisfies (34)–(35), where $(\xi,\zeta) \in \mathbb{R}_+[[\tau]] \times \mathbb{R}_+[[\tau]]$ *is the power series solution of the following initial value problem*

$$\begin{aligned} \xi' &= (1+\zeta)\,(2-\chi(\xi,\zeta))^{-1/2}, \quad \xi(0)=1, \\ \zeta' &= \frac{\xi\,(2-\chi(\xi,\zeta))^{-1/2}}{(2-\xi^2)^{3/2}}, \quad \zeta(0)=0, \end{aligned} \tag{45}$$

where $\chi(\xi,\zeta)$ *is given as a function of* (ξ,ζ) *by (36).*

Proof. Our proof of Theorem 2 mimics the proof of Theorem 1. We begin by considering the sequence $\{\xi^{[m]},\zeta^{[m]}\}_{m\in\mathbb{N}}$, where $(\xi^{[m]},\zeta^{[m]}) = \Psi(\xi^{[m-1]},\zeta^{[m-1]})$ for $m \geq 1$ and

$$\xi^{[0]} = 1 + \sum_{k\geq 1} \xi_k^{[0]}\,\tau^k \in \mathbb{R}_+[[\tau]], \quad \zeta^{[0]} = \sum_{k\geq 1} \zeta_k^{[0]}\,\tau^k \in \mathbb{R}_+[[\tau]]$$

is such that for $1 \leq i < j \leq N$ and $m \geq 1$,

$$\xi_m^{[0]} = \frac{\|(Q_i - Q_j)_m\|}{\|q_i^0 - q_j^0\|} \quad \text{and} \quad \zeta_m^{[0]} = \frac{\|(V_i - V_j)_m\|}{s(q^0,v^0)\,M_{ij}(q^0)}.$$

Clearly, $(\xi^{[0]},\zeta^{[0]}) \in \mathcal{B}$, so that by Lemma 4, $(\xi^{[m]},\zeta^{[m]}) \in \mathcal{B}$ for $m \geq 1$. Proceeding as in the proof of Theorem 1, one concludes that the sequence $\{\xi^{[m]},\zeta^{[m]}\}_{m\in\mathbb{N}}$ converges (in the sense of each coefficients of the two series are ultimately constant) towards a limit $(\xi^{[\infty]},\zeta^{[\infty]}) \in \mathcal{B}$, which is the unique solution of the fixed point equation

$$(\xi^{[\infty]},\zeta^{[\infty]}) = \Psi(\xi^{[\infty]},\zeta^{[\infty]}).$$

We thus have that estimates (34)–(35) hold for $(\xi,\zeta) = (\xi^{[\infty]},\zeta^{[\infty]})$ the solution of

$$\left\{ \begin{array}{rcl} \xi &=& 1 + \int (2-\chi(\xi,\zeta))^{-1/2}\,(1+\zeta), \\ \zeta &=& \int \frac{(2-\chi(\xi,\zeta))^{-1/2}\,\xi}{(2-\xi^2)^{3/2}} \end{array} \right.$$

(where $\chi(\xi,\zeta)$ is given in terms of ξ and ζ by (36)), or in other words, the unique power series solution of (45). □

Theorem 3. *Under the assumptions of Theorem 2, for* $1 \leq i \leq N$,

$$\begin{aligned} V_i - v_i^0 &\trianglelefteq s(q^0,v^0)\,K_i(q^0)\,\zeta, \\ Q_i - q_i^0 &\trianglelefteq \max\left(s(q^0,v^0)\,\|v_i^0\|, s(q^0,v^0)^2\,K_i(q^0)\right)\,(\xi - 1). \end{aligned}$$

Proof. The following majorants for $Q_i - q_i^0$ and $V_i - v_i^0$ can be obtained from $Q_i - q_i^0 = \int s(Q,V)V_i$ and $V_i - v_i^0 = \int s(Q,V)g_i(Q)$ respectively by virtue of Lemma 3,

$$V_i - v_i^0 \trianglelefteq s(q^0,v^0)\,K_i(q^0) \int \frac{\xi\,(2-\chi(\xi,\zeta))^{-1/2}}{(2-\xi^2)^{3/2}} = s(q^0,v^0)\,K_i(q^0)\,\zeta,$$

$$\begin{aligned} Q_i - q_i^0 &\trianglelefteq s(q^0,v^0)\,\|v_i^0\| \int (2-\chi(\xi,\zeta))^{-1/2} + s(q^0,v^0)^2\,K_i(q^0) \int \zeta\,(2-\chi(\xi,\zeta))^{-1/2} \\ &\trianglelefteq \max\left(s(q^0,v^0)\,\|v_i^0\|, s(q^0,v^0)^2\,K_i(q^0)\right)\,(\xi-1). \end{aligned}$$

□

Proposition 4. *The radius of convergence R of the power series solution (ξ,ζ) of (45) is given by*

$$R = G(\xi) = \int_0^{v_+} g(\sigma)d\sigma \approx 0.0839968103939379, \tag{46}$$

where

$$g(\sigma) = 2\,\frac{1}{(\sigma^2+2\,\sigma+2)^2}\sqrt{-\frac{3\,\sigma^6+18\,\sigma^5+50\,\sigma^4+80\,\sigma^3+76\,\sigma^2+40\,\sigma-8}{\sigma^4+4\,\sigma^3+8\,\sigma^2+8\,\sigma+2}}$$

and

$$\begin{aligned} v_+ &= -1 + \frac{\sqrt{502+18\sqrt{777}-5\,\left(251+9\sqrt{777}\right)^{2/3}+8\sqrt[3]{251+9\sqrt{777}}}}{3\sqrt[3]{251+9\sqrt{777}}} \\ &\approx 0.149902575567304. \end{aligned}$$

Proof. The solution (ξ,ζ) of (45), where $\chi = \chi(\xi,\zeta)$ is given by (36), can be computed alternatively as follows: obtain ζ as the initial value problem

$$\zeta' = (1+\gamma)^2\sqrt{\frac{1+4\gamma+2\gamma^2}{2-\chi}}, \quad \zeta(0) = 0, \tag{47}$$

where $\gamma = \zeta + \zeta^2/2$ and $\chi = (1+\gamma)^2(1+3\,\gamma)$, and then

$$\xi = \frac{\sqrt{1+4\,\gamma+2\,\gamma^2}}{\gamma+1}. \tag{48}$$

Indeed, it is straightforward to check that $(2-\xi^2)^{-1/2} = \gamma+1$ holds for the solution of (45), which implies that $\chi = (1+\gamma)^2(1+3\,\gamma)$ and

$$\xi = \sqrt{2-(\gamma+1)^{-2}} = \frac{\sqrt{1+4\,\gamma+2\,\gamma^2}}{\gamma+1}.$$

Being majorant series by construction, both ζ and ξ have expansions in powers of t with real positive coefficients. Using the same argument for ζ as for λ in Proposition 3, we can show that equation (47) has an analytic solution $\zeta(\tau)$ on the disk $D_R(0)$, where

$$R = \int_0^{v_+} g(\sigma) d\sigma,$$

$$g(\sigma) = 2 \frac{1}{(\sigma^2 + 2\sigma + 2)^2} \sqrt{- \frac{3\sigma^6 + 18\sigma^5 + 50\sigma^4 + 80\sigma^3 + 76\sigma^2 + 40\sigma - 8}{\sigma^4 + 4\sigma^3 + 8\sigma^2 + 8\sigma + 2}},$$

and

$$v_+ := \sup_{\tau \in D_R(0)} |\zeta(\tau)|$$

is the root of $\sigma^4 + 4\sigma^3 + 8\sigma^2 + 8\sigma + 2$ with smallest modulus. As the right-hand side of equation (48) is also analytic on $D_{v_+}(0)$ as a function of ξ, the other component $\xi(t)$ of the solution of equation (45) is well-defined and analytic on the same disk $D_R(0)$. $\square$

In view of Theorem 1, we conclude that the power series expansion

$$(Q, V) = (Q_1, \ldots, Q_N, V_1, \ldots, V_N) \in \mathbb{R}^{6N}[[\tau]]$$

of the solution of (30)–(31) with (14) is convergent for all $\tau \in (-R, R)$. Hence, we get as a corollary of Theorem 2 the following result, originally proven in [2] with $R = 0.0444443$.

Corollary 2. *The solution of (30)–(31) with (14) admits an holomorphic extension as a function of the complex time τ in the strip*

$$\{\tau \in \mathbb{C} \ : \ |\mathrm{Im}(\tau)| < R = 0.0839968103939379\}. \tag{49}$$

Remark 4. *As pointed out in* [2]*, Corollary 2 implies that the solution of (30) with regular initial values admits a globally convergent series expansion in powers of a new variable σ, related to τ with the conformal mapping*

$$\tau \mapsto \sigma = \frac{\exp(\frac{\pi}{2R}\tau) - 1}{\exp(\frac{\pi}{2R}\tau) + 1}.$$

that maps the strip (49) into the unit disk. This is closely related to Sundman's result [11] *for the 3-body problem as well as Wang's results* [12] *for the general case of N-body problems. It is worth emphasizing that, in contrast with both Sundman's and Wang's solutions, our approach remains valid in the limit where $\min_{1 \le i \le N} m_i/M \to 0$ with $M = \sum_{1 \le i \le N} m_i$.*

5. Discretization of the time-renormalized N-body equations

We now consider the implicit mid-point rule discretization of the equations (30). The implicit midpoint rule gives, for small enough values of τ, an approximation of the solution of the initial value problem (30)–(31). We want to obtain majorants of the power series expansions in powers of τ of the local errors of the implicit midpoint approximation of the solution of (30)–(31).

Let

$$(\tilde{Q},\tilde{V}) = (\tilde{Q}_1,\dots,\tilde{Q}_N,\tilde{V}_1,\dots,\tilde{V}_N) \in \mathbb{R}^{6N}[[\tau]]$$

be the power series expansion of the implicit midpoint approximation, and consider $(\hat{Q},\hat{V}) = \frac{1}{2}\,(q^0+\tilde{Q}, v^0+\hat{V})$. Then, for $1 \le i \le N$ it holds that

$$\begin{aligned} \hat{Q}_i &= q_i^0 + \frac{\tau}{2}\, s(\hat{Q},\hat{V})\hat{V}_i, \\ \hat{V}_i &= v_i^0 + \frac{\tau}{2}\, s(\hat{Q},\hat{V})g_i(\hat{Q}), \end{aligned} \tag{50}$$

and

$$\begin{aligned} \tilde{Q}_i &= q_i^0 + \tau\, s(\hat{Q},\hat{V})\hat{V}_i, \\ \tilde{V}_i &= v_i^0 + \tau\, s(\hat{Q},\hat{V})g_i(\hat{Q}). \end{aligned} \tag{51}$$

Lemma 5. *Let $\hat{\mathcal{B}}$ be the set of $(\xi,\zeta) \in (1+\tau\mathbb{R}[[\tau]]) \times \tau\mathbb{R}[[\tau]]$ such that for $1 \le i < j \le N$*

$$\begin{aligned} \hat{Q}_i - \hat{Q}_j &\trianglelefteq \|q_i^0 - q_j^0\|\,\xi, \\ \hat{V}_i - \hat{V}_j &\trianglelefteq \|v_i^0 - v_j^0\| + s(q^0,v^0)\,M_{ij}(q^0)\,\zeta, \end{aligned}$$

and consider the operator

$$\begin{aligned} \widehat{\Psi}:\ (1+\tau\mathbb{R}[[\tau]]) \times \tau\mathbb{R}[[\tau]] &\to \mathbb{R}[[\tau]] \times \mathbb{R}[[\tau]] \\ (\xi,\zeta) &\mapsto \left(1 + \tfrac{\tau}{2}\,(2-\chi(\xi,\zeta))^{-1/2}\,(1+\zeta),\ \tfrac{\tau}{2}\,\tfrac{(2-\chi(\xi,\zeta))^{-1/2}\,\xi}{(2-\xi^2)^{3/2}}\right). \end{aligned}$$

Then $\widehat{\Psi}(\hat{\mathcal{B}}) \subset \hat{\mathcal{B}}$.

Proof. We only sketch the proof as it is analogous to that of Lemma 4. Expressing relative positions and velocities between bodies, we have for all $1 \le i < j \le N$

$$\begin{aligned} \hat{Q}_i - \hat{Q}_j &= q_i^0 - q_j^0 + \frac{\tau}{2}\, s(\hat{Q},\hat{V})\left(\hat{V}_i - V_j\right), \\ \hat{V}_i - \hat{V}_j &= v_i^0 - v_j^0 + \frac{\tau}{2}\, s(\hat{Q},\hat{V})\left(g_i(\hat{Q}) - g_j(\hat{Q})\right) \end{aligned}$$

so that, upon using Lemma 3 with (Q, V) replaced by $(\hat{Q}, \hat{V})$, we immediately obtain

$$\hat{Q}_i - \hat{Q}_j \trianglelefteq \|q_i^0 - q_j^0\| + \frac{\tau}{2}\, s(q^0, v^0)(2 - \chi(\xi,\zeta))^{-1/2}\Big(\|v_i^0 - v_j^0\| + s(q^0, v^0) M_{ij}(q^0)\Big),$$

$$\hat{V}_i - \hat{V}_j \trianglelefteq \|v_i^0 - v_j^0\| + \frac{\tau}{2}\, s(q^0, v^0)(2 - \chi(\xi,\zeta))^{-1/2}\left(K_i(q^0) + K_j(q^0)\right)\right) \frac{\xi}{(2-\xi^2)^{1/2}}.$$

It then follows from the bounds in (41) that

$$\hat{Q}_i - \hat{Q}_j \trianglelefteq \|q_i^0 - q_j^0\| \left(1 + \frac{\tau}{2}\,(2 - \chi(\xi,\zeta))^{-1/2}(1+\xi)\right),$$

$$\hat{V}_i - \hat{V}_j \trianglelefteq \|v_i^0 - v_j^0\| + s(q^0, v^0) M_{ij}(q^0) \left(\frac{\tau}{2}\,(2 - \chi(\xi,\zeta))^{-1/2} \frac{\xi}{(2-\xi^2)^{1/2}}\right).$$

This proves that $\hat{\Psi}(\mathcal{B}) \subset \mathcal{B}$. □

Theorem 4. *For $1 \le i < j \le N$,*

$$\hat{Q}_i - \hat{Q}_j \trianglelefteq \|q_i^0 - q_j^0\|\, \hat{\xi}, \tag{52}$$

$$\hat{V}_i - \hat{V}_j \trianglelefteq \|v_i^0 - v_j^0\| + s(q^0, v^0)\, M_{ij}(q^0)\, \hat{\zeta}, \tag{53}$$

where $(\hat{\xi}, \hat{\zeta}) \in (1 + \mathbb{R}[[\tau]]) \times \tau\mathbb{R}[[\tau]]$ is the unique fixed point of $\widehat{\Psi}$.

Proof. Again, we only sketch the proof, which is similar to the proof of Theorem 2. The operator $\hat{\Psi}$ being clearly Noetherian, the sequence

$$(\xi^{[m]}, \zeta^{[m]}) = \hat{\Psi}\left(\xi^{[m-1]}, \zeta^{[m-1]}\right), \quad m = 1, \ldots,$$

where

$$\xi^{[0]} = 1 + \sum_{k\ge 1} \xi_k^{[0]}\, \tau^k \in \mathbb{R}_+[[\tau]] \quad \text{and} \quad \zeta^{[0]} = \sum_{k\ge 1} \zeta_k^{[0]}\, \tau^k \in \mathbb{R}_+[[\tau]]$$

with

$$\xi_m^{[0]} = \frac{\|(\hat{Q}_i - \hat{Q}_j)_m\|}{\|q_i^0 - q_j^0\|} \quad \text{and} \quad \zeta_m^{[0]} = \frac{\|(\hat{V}_i - \hat{V}_j)_m\|}{s(q^0, v^0)\, M_{ij}(q^0)},$$

converges in $\mathcal{B}$ to a limit $(\xi^{[\infty]}, \zeta^{[\infty]})$ (owing to previous lemma). This limit is the unique series in powers of τ satisfying the equations

$$\hat{\xi} = 1 + \frac{\tau}{2}\,(2 - \chi(\hat{\xi}, \hat{\zeta}))^{-1/2}\,(1 + \hat{\zeta}), \tag{54}$$

$$\hat{\zeta} = \frac{\tau}{2}\, \frac{(2 - \chi(\hat{\xi}, \hat{\zeta}))^{-1/2}\, \hat{\xi}}{(2 - \hat{\xi}^2)^{3/2}}, \tag{55}$$

satisfying $(\hat{\xi}(\tau), \hat{\zeta}(\tau))\Big|_{\tau=0} = (1, 0)$. □

Remark 5. *It is not difficult to obtain the following relation*

$$\hat{\zeta} = \frac{1}{2}\left(\left(1 - \frac{4\hat{\xi}(1-\hat{\xi})}{(2-\hat{\xi}^2)^{3/2}}\right)^{1/2} - 1\right)$$

which in turn, can be substituted into (54) for instance, to obtain an algebraic equation involving only $\hat{\xi}$ and τ. It is then possible to solve this equation numerically in order to estimate the radius of convergence of the series $\hat{\xi}(\tau)$.

Next theorem can be proven along the same lines as Theorem 3.

Theorem 5. *For $1 \leq i \leq N$,*

$$\begin{aligned} \tilde{V}_i - v_i^0 &\trianglelefteq s(q^0, v^0)\, K_i(q^0)\, \hat{\zeta}, \\ \tilde{Q}_i - q_i^0 &\trianglelefteq \max\left(s(q^0, v^0)\, \|v_i^0\|, s(q^0, v^0)^2\, K_i(q^0)\right)\, (\hat{\xi} - 1). \end{aligned}$$

Remark 6. *We have numerically estimated the radius of convergence $\hat{R}$ of the power series $\hat{\xi}$ and $\hat{\zeta}$ to obtain $\hat{R} \approx 0.094790093$.*

We now consider the discretization of (30) by a s-stage Runge-Kutta scheme with Butcher tableau

$$\begin{array}{|c} A \\ \hline b^T \end{array} \tag{56}$$

where $A \in \mathbb{R}^{s\times s}$ and $b \in \mathbb{R}^s$. An straightforward generalization of Lemma 5 and Theorem 4 to Runge-Kutta schemes allows proving the following generalization of Theorem 5.

Theorem 6. *Let*

$$(\tilde{Q}, \tilde{V}) = (\tilde{Q}_1, \ldots, \tilde{Q}_N, \tilde{V}_1, \ldots, \tilde{V}_N) \in \mathbb{R}^{6N}[[\tau]]$$

be the power series expansion of the approximation of the solution $(Q(\tau), V(\tau))$ of the initial value problem (30)–(31) obtained by applying one step of the Runge-Kutta scheme with Butcher tableau (56). For $1 \leq i \leq N$,

$$\tilde{V}_i - v_i^0 \trianglelefteq s(q^0, v^0)\, K_i(q^0)\, \|b\|_\infty \sum_{k=1}^{\infty} \hat{\zeta}_k\, (2\, \|A\|_\infty\, \tau)^k,$$

$$\tilde{Q}_i - q_i^0 \trianglelefteq \max\left(s(q^0, v^0)\, \|v_i^0\|, s(q^0, v^0)^2\, K_i(q^0)\right)\, \|b\|_\infty \sum_{k=1}^{\infty} \hat{\xi}_k\, (2\, \|A\|_\infty\, \tau)^k.$$

As a corollary of Theorems 3 and 6, the local error of the application of one step of length $h \in \{\tau \in \mathbb{R} \ : \ |\tau| < \hat{R}/(2\,\|A\|_\infty)\}$ can be bounded as follows.

Corollary 3. *If the Runge-Kutta scheme is of order $p \geq 1$, then for $1 \leq i \leq N$,*

$$\|\tilde{V}_i - v_i^0\| \leq s(q^0, v^0)\, K_i(q^0)\, \|b\|_\infty \sum_{k=p+1}^{\infty} (\hat{\zeta}_k - \zeta_k)\,(2\,\|A\|_\infty\,\tau)^k,$$

$$\|\tilde{Q}_i - q_i^0\| \leq \max\left(s(q^0, v^0)\,\|v_i^0\|, s(q^0, v^0)^2\, K_i(q^0)\right)\, \|b\|_\infty \sum_{k=p+1}^{\infty} (\hat{\xi}_k - \xi_k)\,(2\,\|A\|_\infty\,\tau)^k.$$

6. Alternative time-renormalization functions

Clearly, (32) is not the umique globally defined time-reparametrization function that is uniform in the sense that any solution of (30) admits an holomorphic extension as a function of the complex time τ in a strip $\{\tau \in \mathbb{C} \ : \ |\mathrm{Im}(\tau)| < R\}$. As shown in [2], a computationally less complex alternative to function (32) is

$$s(q,v) = \left(\sum_{1\leq i<j\leq N} \left(\frac{||v_i - v_j||}{||q_i - q_j||} \right)^2 + A(q) \sum_{1\leq i<j\leq N} \frac{1}{\|q_i - q_j\|} \right)^{-1/2} \tag{57}$$

where

$$A(q) = \sum_{1\leq i<j\leq N} \frac{G\,(m_i + m_j)}{\|q_i - q_j\|^2}.$$

The key observation is that

$$\forall\, 1 \leq i < j \leq N, \quad M_{ij}(q) := K_i(q) + K_j(q) \leq A(q),$$

so that all proofs of Sect. 4 and 5 remain valid with (57) instead of (32).

Recall that we chose function (32) so that it is, up to a constant factor, a real-analytic lower bound of the estimate of the radius of convergence given in Corollary 1. Actually, in the proofs of Sections 4 and 5, this fact it is not strictly required. In fact, the essential ingredients of our proofs are the inequalities (41). Hence, we may determine $s(q, v)$ as a real-analytic lower bound of

$$\left(\max\left(\max_{1\leq i<j\leq N} \frac{\|v_i - v_j\|}{\|q_i - q_j\|}, \max_{1\leq i<j\leq N} \sqrt{\frac{M_{ij}(q)}{\alpha\,\|q_i - q_j\|}} \right) \right)^{-1}, \tag{58}$$

with some $\alpha > 0$, so that

$$s(q,v) \leq \frac{\|q_i - q_j\|}{\|v_i - v_j\|} \quad \text{and} \quad s(q,v)^2 \leq \frac{\alpha\,\|q_i - q_j\|}{M_{ij}(q)}. \tag{59}$$

By replacing the ∞-norm of the vector with $(N-1)N$ components in (58) by its $2p$-norm (for some positive integer p) and $M_{ij}(q)$ by $A(q)$, we arrive at

$$s(q,v) = \left(\sum_{1\leq i<j\leq N} \frac{\|v_i - v_j\|^{2p}}{\|q_i - q_j\|^{2p}} + A(q)^p \sum_{1\leq i<j\leq N} \frac{1}{(\alpha\, \|q_i - q_j\|)^p} \right)^{-\frac{1}{2p}}. \tag{60}$$

A time-renormalization that does not depend on the velocities can be derived from (60) by bounding $\|v_i - v_j\|$ in terms of the absolute value of the potential energy

$$U(q) = G \sum_{1\leq i<j\leq N} \frac{m_i\, m_j}{\|q_i - q_j\|}$$

and the total energy $E_0 = \frac{1}{2}\sum_{i=1}^N m_i\, \|v_i\|^2 - U(q)$. More precisely,

$$\|v_i - v_j\| \leq \sqrt{2}\,(m_i^{-1/2} + m_j^{-1/2})(E_0 + U(q)),$$

which leads to

$$\begin{aligned} s(q,E_0) := \Bigg(&(E_0 + U(q))^p \sum_{1\leq i<j\leq N} \frac{4\,(m_i^{-1/2} + m_j^{-1/2})^{2p}}{\|q_i - q_j\|^{2p}} \\ &+ \frac{A(q)^p}{\alpha^p} \sum_{1\leq i<j\leq N} \frac{1}{\|q_i - q_j\|^p} \Bigg)^{-\frac{1}{2p}}. \end{aligned} \tag{61}$$

All the results in Section 4 (resp. Section 5)) can be proven for the two alternative functions (60) and (61), with different majorant series ξ, ζ (resp. $\hat{\xi}, \hat{\zeta}$) depending on the prescribed parameters α and p, having different radius of convergences R (resp. $\hat{R}$). Note however that (61) is no longer valid in the limit when one of the masses vanishes, and hence it is not expected to perform well with too small mass ratios.

In practice, we suggest to consider $p = 1$ or $p = 2$, and $\alpha = 3$. That choice for α is motivated by Remark 3.

7. Numerical experiment

In order to illustrate the application of a Runge-Kutta scheme to time-renormalized N-body problems, we consider the a 15-body model of the Solar System that includes

- the Sun,
- the Earth-Moon binary considered as mass point centered at its barycenter,
- the remaining seven planets and Pluto, and

- the five main bodies of the asteroid belt: Ceres, Pallas, Vesta, Iris and Bamberga.

We consider the initial values at Julian day (TDB) 2440400.5 (the 28th of June of 1969), obtained form the DE430 ephemerides [4], renormalized so that the center of mass of the 15 bodies is at rest, and run the numerical integrations for 20000 years. Several close approaches between some of the asteroids occur in that interval of time.

We have applied the 16th order implicit Runge-Kutta method of collocation type with Gauss-Legendre nodes, implemented with fixed point iteration as described in [1]. In particular, we have performed our numerical experiments in the Julia programming language [8], using the Julia package IRKGaussLegendre.jl [7] integrated in SciML/DifferentialEquations.jl [9].

We have first integrated the problem in the equations with physical time (11) with a time-step Δt of 8 days. The local errors in positions for each of the bodies (except for the sun) are displayed in the left plot in Figure 2. We observe that for most of the steps, the local error is dominated by Mercury's error. Occasionally the errors of two asteroids become considerably larger than Mercury's error, due to a close approach. The highest spike of the local error occur after 10338 years, and is due to a close approach between Pallas and Vesta.

We then integrate the problem in the time-renormalized equations (30) with renormalization function (60) with $\alpha = 3$ and $p = 2$. (We have also tried with $p = 1$ but it has a poorer performance in that example). For a fair comparison with the integration in physical time, we chose the step-size in τ in such a way that the same number of steps (and approximately the same CPU time) is required in both cases.

We observe in the right plot in Figure 2 that there are no spikes of local errors due to close approaches, and that the local errors of Mercury are smaller than those in the integration in physical time. However, the local errors of the rest of the bodies (except for pairs of asteroids in a close approach) become larger for the integration in the time-renormalized equations. In order to understand that, notice that the local errors in physical time of the outer planets are considerably smaller than those of Mercury, because the dominant terms of accelerations of the outer planets are relatively smoother than those of Mercury. In the time-renormalized equations, the comparatively highly oscillatory motion of Mercury is inherited through the time-renormalized function by the equations of all of the bodies, leading to local errors in positions of similar size for Mercury and the rest of the bodies.

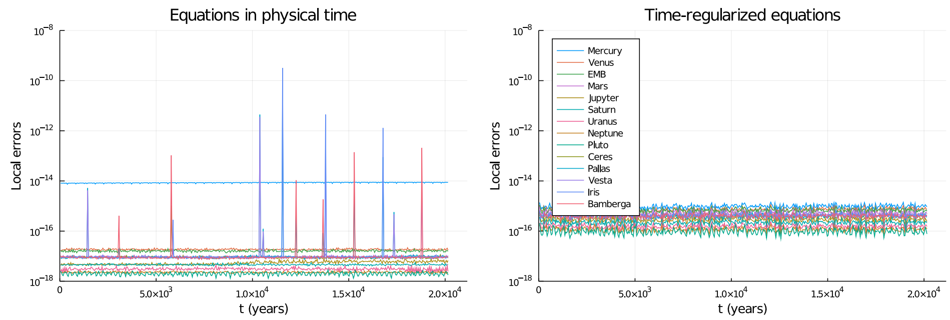

FIGURE 2. Local errors in position: left plot, integration in physical time with 920000 time-steps of size $\Delta t = 8$; right plot, integration in time-renormalized equations with 920000 time-steps of size $\Delta\tau = 0.7196076352409821$.

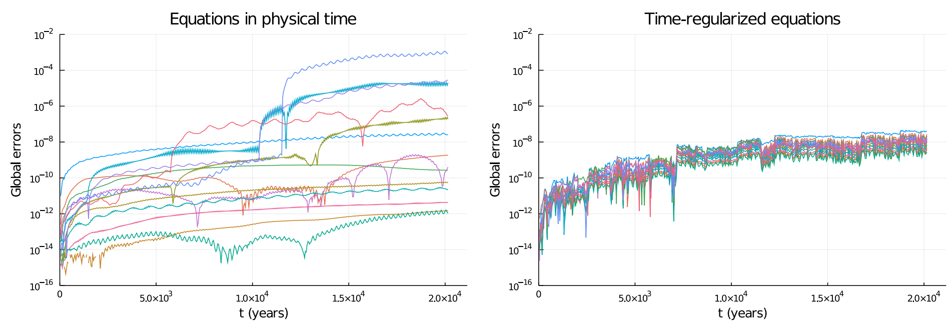

FIGURE 3. Global errors in position: left plot, integration in physical time with 920000 time-steps of size $\Delta t = 8$; right plot, integration in time-renormalized equations with 920000 time-steps of size $\Delta\tau = 0.7196076352409821$.

In Figure 3 the evolution of global errors is compared for the two numerical integrations. The large errors in the positions of the asteroids due to close approaches are not present in the integration of the time-renormalized equations.

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