Operational Semantics

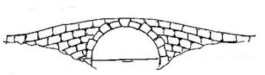
Simon Castellan
https://sicastel.gitlabpages.inria.fr/m2-sos

SOS, Master Recherche Science Informatique, U. Rennes 1

2021-2022

A mostly wrong history of bridge building

1. **Once upon a time:** bridge-building recipes



2. **Nowadays**: Maths and physics to the rescue.

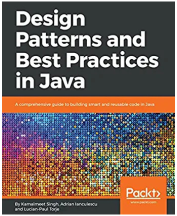
$$\text{Sketch} \xrightarrow{\text{Physics}} \text{Mathematical Models}$$

Calculuations: does it stand? is it resistant?

- What is a mathematical representation of a bridge?
- How do you go from the sketch to the model?
- How do you check for safety (it stands)?
- How do you check against attacks (ie. different scenario)?

Meanwhile, in computer science

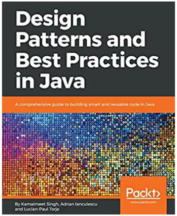
1. We have the recipes:



2. What about the fancy science?

Meanwhile, in computer science

1. We have the recipes:

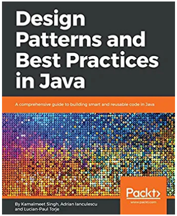


2. What about the fancy science? In its infancy!

$$\begin{tikzcd} \boxed{\text{Program}} \arrow[r, "\textbf{Semantics}"] & \boxed{\text{Mathematical Models}} \arrow[loop below, "\text{Formal methods}"] \end{tikzcd}$$

Meanwhile, in computer science

1. We have the recipes:



2. What about the fancy science? In its infancy!

$$\begin{tikzcd} \boxed{\text{Program}} \arrow[r, "\textbf{Semantics}"] & \boxed{\text{Mathematical Models}} \arrow[loop below, "\text{Formal methods}"] \end{tikzcd}$$

Semantics can make **formal properties** about programs:

$$\text{“The program } P \text{ computes factorial”} \rightsquigarrow \forall n \in \mathbb{N}, [\![P]\!](n) = n!.$$

The tradeoff of semantics

A model is a mathematical *point of view*: there are **many** models.

$$\boxed{\text{Abstraction}} \xleftrightarrow{\text{compromise}} \boxed{\text{Accuracy wrt reality}}$$

More abstract Easier to reason with, to prove properties.

More accurate Can state more properties.

Example 1 (Spectre)

Spectre is a recent attack using the branch prediction of processors.

(No current models cannot state the property: “This program is resistant to the Spectre attack”.)

In this course

Two parts:

1. **Semantics**: How can we model program execution?

2. **Analysis & Security**: How can we use these models to guarantee the correction or certain security properties?

In this course

Two parts:

1. **Semantics**: How can we model program execution?
    1. Case study on a idealised language, *While*: different methods of giving semantics
    2. A more concrete example: semantics of an idealised Java Virtual Machine
    3. Semantics of idealised functional languages: λ-calculus
2. **Analysis & Security**: How can we use these models to guarantee the correction or certain security properties?

Lecture 1: Semantics of *While*

Outline

A problem

```
int fact(int n) {
  int result = 1;
  while(n > 1) {
    result = n * result;
    n = n - 1;
  }
  return result;
}
```

A problem

```
int fact(int n) {
  int result = 1;
  while(n > 1) {
    result = n * result;
    n = n - 1;
  }
  return result;
}
```

How to show that this program is correct?

A problem

```
int fact(int n) {
  int result = 1;
  while(n > 1) {
    result = n * result;
    n = n - 1;
  }
  return result;
}
```

How to show `fact`(n) always evaluates to $n!$?

A problem

```
int fact(int n) {
  int result = 1;
  while(n > 1) {
    result = n * result;
    n = n - 1;
  }
  return result;
}
```

1. What is fact(n) ?
2. How does it evaluate/compute?

A problem

```
int fact(int n) {
  int result = 1;
  while(n > 1) {
    result = n * result;
    n = n - 1;
  }
  return result;
}
```

1. What is fact(n) ? $\rightsquigarrow$ **Formal language**
2. How does it evaluate/compute? $\rightsquigarrow$ **Semantics**

Idealising a language

Real life programming languages are **complex**.
$\rightsquigarrow$ The specification of the C language: 500 pages.

Idealising a language

Real life programming languages are **complex**.
$\rightsquigarrow$ The specification of the C language: 500 pages.

In semantics, we start a small set of features and grow over time.

Idealising a language

Real life programming languages are **complex**.
$\rightsquigarrow$ The specification of the C language: 500 pages.

In semantics, we start a small set of features and grow over time.

The standard starting point, **imperative programming**:

- Variables (of type `int`)
- Assignments of arithmetic expressions (involving variables) to variables
- Conditionals on boolean expressions derived from variables
- While statements

We create an **idealised language** that combines this features, *While*.

The *While* language

$n \in \mathbf{Num} \qquad x \in \mathbf{Var}$ — integers and variables

$a \in \mathbf{Aexp}$ — arithmetic expressions

$$a \quad ::= \quad n \mid x \mid a_1 + a_2 \mid \ldots$$

$b \in \mathbf{Bexp}$ — boolean expressions

$$\begin{aligned} b \quad ::= \quad & \texttt{true} \mid \texttt{false} \mid a_1 = a_2 \mid a_1 > a_2 \\ & \mid \texttt{not}\ b \mid b_1\ \texttt{and}\ b_2 \mid \ldots \end{aligned}$$

$c \in \mathbf{Cmd}$ — commands (i.e. programs)

$$\begin{aligned} c \quad ::= \quad & x \texttt{ := } a \mid \texttt{skip} \mid c_1 \texttt{ ; } c_2 \\ & \mid \texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2 \mid \texttt{while}\ b\ \texttt{do}\ c \end{aligned}$$

This definition is called a **BNF grammar**:

- Different syntactic categories
- Certain basic categories are assumed: numbers, variables.
- Valid programs are described by **abstract syntax trees** (ASTs);

Example 2 (Factorial in *While*)

```
result:=1; while n>1 do (result := n * result; n := n -1).
```

Different semantics of *While*

We now would like to prove that our factorial program is correct:

1. It always terminates
2. If `n` is equal to $k \in \mathbb{N}$ before the execution, at the end $\texttt{result} = k!$

Different semantics of *While*

We now would like to prove that our factorial program is correct:

1. It always terminates
2. If $\texttt{n}$ is equal to $k \in \mathbb{N}$ before the execution, at the end $\texttt{result} = k!$

Different families of models based on **memory states** $\sigma \in \textbf{Var} \to \mathbb{Z}$:

- **Denotational**: programs become functions on memory states.

$$[\![P]\!] = \sigma \mapsto \sigma[\texttt{n} \mapsto 1, \texttt{result} \mapsto \sigma(\texttt{n})!]$$

Different semantics of *While*

We now would like to prove that our factorial program is correct:

1. It always terminates
2. If $\texttt{n}$ is equal to $k \in \mathbb{N}$ before the execution, at the end $\texttt{result} = k!$

Different families of models based on **memory states** $\sigma \in \mathbf{Var} \to \mathbb{Z}$:

- **Denotational**: programs become functions on memory states.

$$[\![P]\!] = \sigma \mapsto \sigma[\texttt{n} \mapsto 1, \texttt{result} \mapsto \sigma(\texttt{n})!]$$

- **Axiomatic**: programs become memory predicate transformers.

$$\{\texttt{n} = k\}P\{\texttt{n} = 1 \textbf{ and } \texttt{result} = k!\}$$

Different semantics of *While*

We now would like to prove that our factorial program is correct:

1. It always terminates
2. If `n` is equal to $k \in \mathbb{N}$ before the execution, at the end $\texttt{result} = k!$

Different families of models based on **memory states** $\sigma \in \mathbf{Var} \to \mathbb{Z}$:

- **Denotational**: programs become functions on memory states.

$$[\![P]\!] = \sigma \mapsto \sigma[\texttt{n} \mapsto 1, \texttt{result} \mapsto \sigma(\texttt{n})!]$$

- **Axiomatic**: programs become memory predicate transformers.

$$\{\texttt{n} = k\}P\{\texttt{n} = 1 \textbf{ and } \texttt{result} = k!\}$$

- **Operational**: programs are executed by a mathematical machine.

$$(P, \sigma) \to^* \sigma[\texttt{n} \mapsto 1, \texttt{result} \mapsto \sigma(\texttt{n})!]$$

Different semantics of *While*

We now would like to prove that our factorial program is correct:

1. It always terminates
2. If `n` is equal to $k \in \mathbb{N}$ before the execution, at the end $\texttt{result} = k!$

Different families of models based on **memory states** $\sigma \in \mathbf{Var} \to \mathbb{Z}$:

- **Denotational**: programs become functions on memory states.

$$[\![P]\!] = \sigma \mapsto \sigma[\texttt{n} \mapsto 1, \texttt{result} \mapsto \sigma(\texttt{n})!]$$

- **Axiomatic**: programs become memory predicate transformers.

$$\{\texttt{n} = k\}P\{\texttt{n} = 1 \textbf{ and } \texttt{result} = k!\}$$

- **Operational**: programs are executed by a mathematical machine.

$$(P, \sigma) \to^* \sigma[\texttt{n} \mapsto 1, \texttt{result} \mapsto \sigma(\texttt{n})!]$$

Actually the statements are wrong . . . Why?

Different semantics of *While*

We now would like to prove that our factorial program is correct:

1. It always terminates
2. If n is equal to $k \in \mathbb{N}$ before the execution, at the end $\texttt{result} = k!$

Different families of models based on **memory states** $\sigma \in \mathbf{Var} \to \mathbb{Z}$:

- **Denotational**: programs become functions on memory states.

$$[\![P]\!] = \sigma \mapsto \sigma[\texttt{n} \mapsto 1, \texttt{result} \mapsto \sigma(\texttt{n})!]$$

- **Axiomatic**: programs become memory predicate transformers.

$$\{\texttt{n} = k\}P\{\texttt{n} = 1 \textbf{ and } \texttt{result} = k!\}$$

- **Operational**: programs are executed by a mathematical machine.

$$(P, \sigma) \to^* \sigma[\texttt{n} \mapsto 1, \texttt{result} \mapsto \sigma(\texttt{n})!]$$

Actually the statements are wrong . . . Why?

This course: focus on operational semantics.

Outline

Interpretation of values

Define the sets and functions used to describe the meaning of expressions.

- Values : integers and booleans

$$\mathbb{Z} \qquad\qquad \mathbb{B} = \{\mathbf{tt}, \mathbf{ff}\}$$

- Interpretation function for constants

$$\mathcal{N} \in \mathbf{Num} \rightarrow \mathbb{Z}$$

- A memory state or environment, σ maps variables to values

$$\sigma \in \mathbf{State} = \mathbf{Var} \rightarrow \mathbb{Z}$$

Reading the value of variable x in σ $\qquad \sigma(x)$
Updating σ by setting a new value v for x $\qquad \sigma' = \sigma[x \mapsto v]$

- Semantics of expressions

$$\mathcal{A} \in \mathbf{Aexp} \rightarrow \mathbf{State} \rightarrow \mathbb{Z} \qquad\qquad \mathcal{B} \in \mathbf{Bexp} \rightarrow \mathbf{State} \rightarrow \mathbb{B}$$

Expressions denote functions from states to integer values.
Notation: $\mathcal{A}(a)(\sigma)$ traditionally written $\mathcal{A}[\![a]\!]\sigma$.

Arithmetic expressions

Remember: The set of arithmetic expressions is defined **inductively**

$$a \in \mathbf{Aexp} ::= n \mid x \mid a_1 + a_2 \mid \ldots$$

$\mathcal{A}[\![.]\!]$ is defined by induction on the definition of **Aexp**, following the structure of expressions.

$$\begin{array}{lcl} \mathcal{A} \in \mathbf{Aexp} \to \mathbf{State} \to \mathbb{Z} & & \\ \mathcal{A}[\![n]\!]\sigma & = & \mathcal{N}[\![n]\!] \\ \mathcal{A}[\![x]\!]\sigma & = & \sigma(x) \\ \mathcal{A}[\![a_1 + a_2]\!]\sigma & = & \mathcal{A}[\![a_1]\!]\sigma + \mathcal{A}[\![a_2]\!]\sigma \end{array}$$

Note: $+$ is the syntactic operator, $+$ is the sum operator defined on integers.

The semantics is **compositional**: *the meaning of a syntactic construction is defined from the meaning of its constituent parts.*

Boolean expressions

Similarly, define $\mathcal{B}[\![.]\!]$ by induction on the definition of **Bexp**.

$$
\begin{array}{lcl}
\multicolumn{3}{l}{\mathcal{B} \in \textbf{Bexp} \to \textbf{State} \to \mathbb{B}} \\
\mathcal{B}[\![\, \texttt{true} \,]\!]\sigma & = & \textbf{tt} \\
\mathcal{B}[\![\, \texttt{false} \,]\!]\sigma & = & \textbf{ff} \\
\mathcal{B}[\![a_1 \texttt{=} a_2]\!]\sigma & = & \mathcal{A}[\![a_1]\!]\sigma = \mathcal{A}[\![a_2]\!]\sigma \\
\mathcal{B}[\![a_1 \texttt{<} a_2]\!]\sigma & = & \mathcal{A}[\![a_1]\!]\sigma < \mathcal{A}[\![a_2]\!]\sigma \\
\mathcal{B}[\![\, \texttt{not}\ b]\!]\sigma & = & \neg(\mathcal{B}[\![b]\!]\sigma) \\
\mathcal{B}[\![b_1\ \texttt{and}\ b_2]\!]\sigma & = & \mathcal{B}[\![b_1]\!]\sigma \wedge \mathcal{B}[\![b_2]\!]\sigma
\end{array}
$$

where $\neg$, $\wedge$, $=$ are operators defined on booleans and integers.

Proof technique

The set of arithmetic expressions **Aexp** is defined **inductively**

$$a ::= n \mid x \mid a_1 + a_2 \mid \ldots$$

Structural induction

To prove a property $\mathcal{P}$ of all arithmetic expressions:

1. Base cases: show the property for each atomic expression
2. Inductive cases : show the property for each composite expression, under the hypothesis that it holds on its constituent parts.

Formally, the induction principle for arithmetic expressions is :

$$\left.\begin{array}{ll} & \forall n \in \mathbf{Num}, \mathcal{P}(n) \\ \wedge & \forall x \in \mathbf{Var}, \mathcal{P}(x) \\ \wedge & \forall a_1, a_2 \in \mathbf{Aexp}, \mathcal{P}(a_1) \wedge \mathcal{P}(a_2) \Rightarrow \mathcal{P}(a_1 + a_2) \end{array}\right\} \Rightarrow \forall a \in \mathbf{Aexp}, \mathcal{P}(a)$$

Vocabulary: the above $\mathcal{P}(a_1)$ and $\mathcal{P}(a_2)$ are called the **induction hypotheses**.

Exercises

Exercise 2.1

Let $\sigma \in \textbf{State}$ *and* $x \in \textbf{Var}$ *such that* $\sigma(x) = 3$. *Show that* $\mathcal{B}[\![\texttt{not}(\texttt{x} = \texttt{1})]\!]\sigma = \textbf{tt}$.

Exercise 2.2

We extend the language **Aexp** *with the unary minus operator and the construction* $-a$. *Extend the semantic function* $\mathcal{A}$ *to give a compositional semantics for this construction.*

Exercise 2.3

We extend the language **Bexp** *with the construction* b_1 `or` b_2.

- *Extend the semantic function* $\mathcal{B}$ *to give a compositional semantics for this construction.*
- *Prove that for all* $\overline{b}$ *belonging to the extended language there exists a* b *belonging to the original language such that:*

$$\mathcal{B}[\![b]\!] = \mathcal{B}[\![\overline{b}]\!]$$

Outline

Operational semantics

References

- G. Winskel, *The Formal Semantics of Programming Languages* MIT Press, 1993 (chapters 2 and 3)
- H.R. Nielson and F. Nielson, *Semantics with Applications - A Formal Introduction*, Wiley 1992. (chapter 2)
- G. Plotkin, *A Structural Approach to Operational Semantics*, Technical Report, Aarhus University, 1981.
- G. Kahn, *Natural Semantics*, In Proc. of the Symposium on Theoretical Aspects of Computer Science, LNCS 247, pp. 22–39, Springer-Verlag, 1987.

Operational semantics

Describe how the execution of *While* programs is done, operationally.

The operational semantics of a language is defined by an abstract machine, formalised as a **transition system**.

Transition system

A transition system is a triple $(\Gamma, T, \rightsquigarrow)$ where

- Γ is a set of **configurations** (states of the machine)
- $T \subseteq \Gamma$ is a set of **final** configurations
- $\rightsquigarrow \subseteq \Gamma \times \Gamma$ is a **transition relation**

Two main styles of definitions for the transition relation:

- **Small-step semantics** *Structural Operational Semantics* (SOS)
  Relation $\rightarrow$ describes all intermediate, individual steps
- **Big-step semantics** *Natural semantics* (NS)
  Relation $\Downarrow$ describes how to obtain the final result of computation

Transition systems: some definitions

Transition system

A transition system is a triple $(\Gamma, T, \rightsquigarrow)$ where

- Γ is a set of **configurations**
- $T \subseteq \Gamma$ is a set of **final** configurations
- $\rightsquigarrow \subseteq \Gamma \times \Gamma$ is a **transition relation**

A transition system $(\Gamma, T, \rightsquigarrow)$ is said

- **deterministic** when relation $\rightsquigarrow$ is functional

$$\gamma \rightsquigarrow \gamma_1 \text{ and } \gamma \rightsquigarrow \gamma_2 \text{ implies } \gamma_1 = \gamma_2$$

- **non-blocking** when relation $\rightsquigarrow$ is total on $\Gamma \setminus T$

$$\text{for all } \gamma \in \Gamma \setminus T, \text{ there exists } \gamma' \text{ such that } \gamma \rightsquigarrow \gamma'$$

The notion of program **execution** will be defined on top of $\rightsquigarrow$.

Transition systems for *While*: configurations

To run a *While* program, we need a command $c \in \mathbf{Cmd}$, and a state $\sigma \in \mathbf{State}$.

For *While*, configurations are defined:

- $\Gamma = \{(c, \sigma) \mid c \in \mathbf{Cmd}, \sigma \in \mathbf{State}\} \cup \mathbf{State}$
- Final configurations : $T = \mathbf{State}$

So, either :

- $(c, \sigma) \rightsquigarrow (c', \sigma')$
  execution of c has not terminated, and (c', σ') is left to execute
- or $(c, \sigma) \rightsquigarrow \sigma'$
  execution of c has terminated in the final configuration σ'

Next slides: define two transition relations, following the structure of *While* commands

Small-step transition relation

$$\mathbf{Cmd} \ni c ::= x \texttt{ := } a \mid \texttt{skip} \mid c_1 \texttt{ ; } c_2 \mid \texttt{if } b \texttt{ then } c_1 \texttt{ else } c_2 \mid \texttt{while } b \texttt{ do } c$$

Easy for atomic commands:

- Executing skip terminates in 1 step and doesn't modify the state. For all possible σ, we have $(\texttt{skip}, \sigma) \to \sigma$
- Executing an assignment terminates in 1 step, and updates the state. For all possible σ, x, and a, we have $(x \texttt{ := } a, \sigma) \to \sigma[x \mapsto \mathcal{A}[\![a]\!]\sigma]$

For compound commands, like sequences ?

$$(c_1 \texttt{ ; } c_2, \sigma) \to ???$$

Intuitively, we have to first execute c_1 in small-step.

- The transition relation needs to be defined inductively!

Small-step transition relation, inductively

Inductively defined relations are usually formalised by a **rule system**.

A rule is of the form :

$$\textsc{RuleName} \qquad \textit{if} \ldots (\textit{side conditions}) \frac{\textit{premise}_1 \quad \ldots \quad \textit{premise}_n}{\text{conclusion}}$$

where $\textit{premise}_i$ and *conclusion* are elements of the relation being defined.[1]

It reads: "If $\textit{premise}_1$ and $\ldots \textit{premise}_n$, and if side conditions are satisfied, then *conclusion*". Premises must be, in turn, justified by rules.

- the conclusion holds whenever there is a **finite derivation tree** whose leaves are axioms of the system.

For the transition relation $\rightsquigarrow$, rules are of the form:

$$\textsc{RuleName} \qquad \textit{if} \ldots (\textit{side conditions}) \frac{\gamma_0 \rightsquigarrow \gamma_0' \quad \ldots \quad \gamma_i \rightsquigarrow \gamma_i'}{\gamma_j \rightsquigarrow \gamma_j'}$$

[1] A rule with no premise is called an axiom.

Structural operational semantics (SOS)

Rule system defining the small-step transition relation.

Precisely: these are rule **schemas**, to be instantiated on particular commands and states.

$$\text{ASSIG}\ \frac{}{(x \mathbin{\texttt{:=}} a, \sigma) \to \sigma[x \mapsto \mathcal{A}[\![a]\!]\sigma]} \qquad \text{SKIP}\ \frac{}{(\,\texttt{skip}\,, \sigma) \to \sigma}$$

$$\text{SEQ1}\ \frac{(c_1, \sigma) \to \sigma'}{(c_1 \texttt{ ; } c_2, \sigma) \to (c_2, \sigma')} \qquad \text{SEQ2}\ \frac{(c_1, \sigma) \to (c_1', \sigma')}{(c_1 \texttt{ ; } c_2, \sigma) \to (c_1' \texttt{ ; } c_2, \sigma')}$$

$$\text{IFT}\quad \textit{if}\ \mathcal{B}[\![b]\!]\sigma = \mathbf{tt}\ \frac{}{(\,\texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2, \sigma) \to (c_1, \sigma)}$$

$$\text{IFE}\quad \textit{if}\ \mathcal{B}[\![b]\!]\sigma = \mathbf{ff}\ \frac{}{(\,\texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2, \sigma) \to (c_2, \sigma)}$$

$$\text{WHI}\ \frac{}{(\,\texttt{while}\ b\ \texttt{do}\ c, \sigma) \to (\,\texttt{if}\ b\ \texttt{then}\ (c \texttt{ ; } \texttt{while}\ b\ \texttt{do}\ c)\ \texttt{else}\ \texttt{skip}\,, \sigma)}$$

Small-step executions and semantics

A small-step **execution** of a *While* command is a sequence of configurations

$$\gamma_0, \ldots, \gamma_p, \ldots \text{ such that, for each } i, \gamma_i \to \gamma_{i+1}$$

We write :

- $\to^*$ Reflexive and transitive closure of $\to$: finite number of transitions
- $\to^+$ Transitive closure of $\to$: finite, non-zero number of transitions
- $\to^i$ Exactly i transitions

Execution of (c, σ) is said

- to **terminate** iff there exists σ' such that $(c, \sigma) \to^* \sigma'$
- to **loop** iff there exists an infinite transition sequence starting from (c, σ)

Exercises

Exercise 3.1 (In class)

Show that for all σ *with* $\sigma(n) \geqslant 1$*:*

$$(P, \sigma) \rightarrow^* \sigma'$$

with $\sigma'(\texttt{result}) = \sigma(n)!$ *where* P *is the factorial program.*

Exercise 3.2 (At home)

*Give an SOS to the arithmetic expressions (***Aexp***) of the While language. Is your corresponding transition system deterministic? Explain why.*

Outline

Forgetting the intermediate steps

Our semantics allows to view commands as state transformers:

Definition 3

Command c turns state σ into state σ' when $(c, \sigma) \to^* \sigma'$.
We write $\langle c, \sigma \rangle \Downarrow \sigma'$.

Can we define the relation $\langle c, \sigma \rangle \Downarrow \sigma'$ directly (by induction)?

$\rightsquigarrow$ Yes: it is called **natural semantics**.

Natural (or big-step) semantics (NS)

Rule system defining the big-step transition relation.

Focuses on final state reached: no elementary computation step described. So, the transition relation is such that $\Downarrow \subseteq (\mathbf{Cmd} \times \mathbf{State}) \times \mathbf{State} \subseteq \Gamma \times T$

$$\text{ASSIG}\ \frac{}{(x := a, \sigma) \Downarrow \sigma[x \mapsto \mathcal{A}[\![a]\!]\sigma]} \qquad \text{SKIP}\ \frac{}{(\texttt{skip}, \sigma) \Downarrow \sigma}$$

$$\text{SEQ}\ \frac{(S_1, \sigma) \Downarrow \sigma' \qquad (S_2, \sigma') \Downarrow \sigma''}{(S_1\ ;\ S_2, \sigma) \Downarrow \sigma''}$$

$$\text{IFT} \qquad \textit{if}\ \mathcal{B}[\![b]\!]\sigma = \mathbf{tt}\ \frac{(S_1, \sigma) \Downarrow \sigma'}{(\texttt{if}\ b\ \texttt{then}\ S_1\ \texttt{else}\ S_2, \sigma) \Downarrow \sigma'}$$

$$\text{IFE} \qquad \textit{if}\ \mathcal{B}[\![b]\!]\sigma = \mathbf{ff}\ \frac{(S_2, \sigma) \Downarrow \sigma'}{(\texttt{if}\ b\ \texttt{then}\ S_1\ \texttt{else}\ S_2, \sigma) \Downarrow \sigma'}$$

$$\text{WHI1} \qquad \textit{if}\ \mathcal{B}[\![b]\!]\sigma = \mathbf{tt}\ \frac{(S, \sigma) \Downarrow \sigma' \qquad (\texttt{while}\ b\ \texttt{do}\ S, \sigma') \Downarrow \sigma''}{(\texttt{while}\ b\ \texttt{do}\ S, \sigma) \Downarrow \sigma''}$$

$$\text{WHI2} \qquad \textit{if}\ \mathcal{B}[\![b]\!]\sigma = \mathbf{ff}\ \frac{}{(\texttt{while}\ b\ \texttt{do}\ S, \sigma) \Downarrow \sigma}$$

Big-step executions and semantics

A big-step execution of a *While* command is simply a **derivable** $(c, \sigma) \Downarrow \sigma'$

Execution of (c, σ) is said

- to **terminate** iff there exists σ' such that $(c, \sigma) \Downarrow \sigma'$
- to **loop/block** iff there is no state σ' such that $(c, \sigma) \Downarrow \sigma'$

Semantics of commands: partial function $\mathcal{S}_{nat} \in \textbf{Cmd} \to \textbf{State} \hookrightarrow \textbf{State}$

$$\mathcal{S}_{nat}[\![c]\!]\sigma = \sigma' \qquad \textit{if } (c, \sigma) \Downarrow \sigma'$$

Commands c_1 and c_2 are **semantically equivalent** iff

$$\forall \sigma, \sigma'.\ (c_1, \sigma) \Downarrow \sigma' \Leftrightarrow (c_2, \sigma) \Downarrow \sigma'$$

Exercises

Exercise 4.1

Show that the NS semantics of the factorial program gives the expected behaviour.

Exercise 4.2 (At home)

The While language is extended with the construction `repeat` S `until` b. *Extend the NS accordingly.*

Proof technique associated with NS

Induction principle for derivation trees

$$\dfrac{\dfrac{\vdots}{P_1} \qquad \dfrac{}{P_2}}{P}$$

1. Prove the property for the the axioms of the rule system
2. For each rule, prove the property for the conclusion of the rule, **under the hypothesis** that the property holds for each of the premises, and that side conditions are satisfied.

Intuition: the property is proved:

- to hold for the leaves of the tree,
- and to propagate to any possible derivable conclusion.

Exercises

Exercise 4.3 (At home)

Prove that the NS of While is deterministic.

Exercise 4.4 (At home)

Prove that $c_1\ \texttt{;}\ (c_2\ \texttt{;}\ c_3)$ *and* $(c_1\ \texttt{;}\ c_2)\ \texttt{;}\ c_3$ *are semantically equivalent.*
Hint: induction is not necessary here.

Exercise 4.5 (At home ⋆)

Prove that

$$\texttt{while}\ b\ \texttt{do}\ c$$

and

$$\texttt{if}\ b\ \texttt{then}\ (c\ \texttt{;}\ \texttt{while}\ b\ \texttt{do}\ c)\ \texttt{else}\ \texttt{skip}$$

are semantically equivalent.
Hint: induction is not necessary here.

Outline

An equivalence of two semantics

Theorem

For all c and all σ, *we have* $\langle c, \sigma \rangle \rightarrow^* \sigma'$ *iff* $\langle c, \sigma \rangle \Downarrow \sigma'$.

The theorem is a direct consequence of the following two lemmas:

Lemma 4

For all command c and states σ, σ'

$$(c, \sigma) \Downarrow \sigma' \Rightarrow (c, \sigma) \rightarrow^* \sigma'$$

Lemma 5

For all command c and states σ, σ'

$$(c, \sigma) \rightarrow^k \sigma' \Rightarrow (c, \sigma) \Downarrow \sigma'$$

Proof of Lemma 4

Goal: for all command c and states σ, σ' $(c, \sigma) \Downarrow \sigma' \Rightarrow (c, \sigma) \rightarrow^* \sigma'$.

By induction on the derivation tree of $(S, \sigma) \Downarrow \sigma'$.

Case $(x := a, \sigma) \Downarrow \sigma[x \mapsto \mathcal{A}[\![a]\!]\sigma]$

Immediate from the SOS axiom $(x := a, \sigma) \rightarrow \sigma[x \mapsto \mathcal{A}[\![a]\!]\sigma]$

Case

$$\frac{(S_1, \sigma) \Downarrow \sigma' \qquad (S_2, \sigma') \Downarrow \sigma''}{(S_1 \; ; \; S_2, \sigma) \Downarrow \sigma''}$$

Thus

$(S_1, \sigma) \rightarrow^* \sigma'$ and $(S_2, \sigma') \rightarrow^* \sigma''$ *(by induction hypothesis)*

$(S_1 \; ; \; S_2, \sigma) \rightarrow^* (S_2, \sigma')$

$(S_1 \; ; \; S_2, \sigma) \rightarrow^* \sigma''$ *(by composition of transition sequences)*

Proof of Lemma 4

Case

$$\text{WHI1} \qquad b/c\ \mathcal{B}[\![b]\!]\sigma = \mathbf{tt}\ \frac{(S,\sigma) \Downarrow \sigma' \qquad (\,\mathtt{while}\ b\ \mathtt{do}\ S, \sigma') \Downarrow \sigma''}{(\,\mathtt{while}\ b\ \mathtt{do}\ S, \sigma) \Downarrow \sigma''}$$

The induction hypothesis gives us that

$$(S,\sigma) \to^* \sigma' \qquad \text{and} \qquad (\,\mathtt{while}\ b\ \mathtt{do}\ S, \sigma') \to^* \sigma''$$

According to the SOS, we have the following derivation:

$$\begin{aligned}(\,\mathtt{while}\ b\ \mathtt{do}\ S, \sigma) &\to (\,\mathtt{if}\ b\ \mathtt{then}\ (S\ \mathtt{;}\ \mathtt{while}\ b\ \mathtt{do}\ S)\ \mathtt{else}\ \mathtt{skip}\,, \sigma) \\ &\to (S\ \mathtt{;}\ \mathtt{while}\ b\ \mathtt{do}\ S, \sigma)\end{aligned}$$

Composing the transition sequences, we obtain

$$(\,\mathtt{while}\ b\ \mathtt{do}\ S, \sigma) \to^* \sigma''$$

Other cases same idea (exercise)

Proof of Lemma 5

Goal: for all S, σ, k, σ', $(S, \sigma) \rightarrow^k \sigma' \Rightarrow (S, \sigma) \Downarrow \sigma'$.

Proceed by induction on the length of the transition sequence of $(S, \sigma) \rightarrow^k \sigma'$:

- If $k = 0$ then (S, σ) and σ' should be identical. Vacuously holds.
- Otherwise, suppose the lemma holds for all $k \leqslant k_0$ and prove it for a sequence of length $k_0 + 1$.
  We proceed by case analysis on the command S :

  Case $x := a$. This command reduces in one step to a final state (so $k_0 = 0$) by SOS axiom ASSIG. Result then follows from NS axiom ASSIG.

Proof of Lemma 5

Case $(S_1 \,;\, S_2, \sigma) \rightarrow^{k_0+1} \sigma''$

There exists k_1 and k_2 such that

$$(S_1, \sigma) \rightarrow^{k_1} \sigma' \quad \text{and} \quad (S_2, \sigma') \rightarrow^{k_2} \sigma'' \quad \text{with} \quad k_1 + k_2 = k_0 + 1$$

By induction hypothesis,

$$(S_1, \sigma) \Downarrow \sigma' \quad \text{and} \quad (S_2, \sigma') \Downarrow \sigma''$$

By the NS rule SEQ, we conclude that $(S_1 \,;\, S_2, \sigma) \Downarrow \sigma''$.

Case $(\,\texttt{while}\; b\; \texttt{do}\; S, \sigma)$
$\rightarrow (\,\texttt{if}\; b\; \texttt{then}\; (S \,;\, \texttt{while}\; b\; \texttt{do}\; S)\; \texttt{else skip}\,, \sigma) \;\rightarrow^{k_0}\; \sigma''$

From the induction hypothesis, we get

$$(\,\texttt{if}\; b\; \texttt{then}\; (S \,;\, \texttt{while}\; b\; \texttt{do}\; S)\; \texttt{else skip}\,, \sigma) \Downarrow \sigma''$$

In Exercise 4.5, we proved this command semantically equivalent to $\texttt{while}\; b\; \texttt{do}\; S$, hence $(\,\texttt{while}\; b\; \texttt{do}\; S, \sigma) \Downarrow \sigma''$.

Other cases same technique

Outline

Extension of *While* (1) : termination operator

We extend the *While* language with the command `abort` .

Informal description: *the command halts execution of the program.*

One way of modeling `abort` : semantic rules stay unchanged and the configurations of form (`abort` , σ) are blocking.

- SOS: `abort` is different from `skip` and from `while true do skip` .
- NS: `abort` is different from `skip` but equivalent to `while true do skip` .

One solution that allows the NS to distinguish between termination by `abort` and non-termination: introduce special state σ_{abort} and

$$(\texttt{abort}, \sigma) \Downarrow \sigma_{abort}$$

But: must modify all other rules to take σ_{abort} into account!

Extensions to *While* (2) : non-deterministic choice

Extend *While* with the non-deterministic choice operator $c_1 \,\square\, c_2$.

Informal description: *choose non-deterministically to execute one of* c_1 *and* c_2.

The language now becomes non-deterministic.

- Formalisation as an SOS

$$\text{CH1}\ \frac{}{(c_1 \,\square\, c_2, \sigma) \to (c_1, \sigma)}$$

$$\text{CH2}\ \frac{}{(c_1 \,\square\, c_2, \sigma) \to (c_2, \sigma)}$$

- Formalisation as a NS

$$\text{CH1}\ \frac{(c_1, \sigma) \Downarrow \sigma'}{(c_1 \,\square\, c_2, \sigma) \Downarrow \sigma'}$$

$$\text{CH2}\ \frac{(c_2, \sigma) \Downarrow \sigma'}{(c_1 \,\square\, c_2, \sigma) \Downarrow \sigma'}$$

SOS can choose an expression that loops, while NS will always choose to eliminate non-termination.

Ex : `(x := 1)` $\square$ `( while true do skip )`

Extension of *While* (3) : concurrency

Add a parallel composition to commands:

$$c ::= \ldots \mid (c_1 \parallel c_2).$$

How can we extend the SOS semantics? The natural semantics?