Target Trajectory Estimation within a Sensor Network

Adrien Ickowicz
IRISA/CNRS, 35042, Rennes,

J.-Pierre Le Cadre,
IRISA/CNRS,35042, Rennes,France

***Abstract*—This paper deals with the estimation of the trajectory parameters for a target moving within a sensor network. We are especially interested by fusing binary information at the network level. This binary information is related to the local target behavior; i.e. its distance from a given sensor is increasing (-) or decreasing (+). In this domain, seminal contributions include [3]. However, in this rich framework we choose to focus on even simpler observations so as to put in evidence the limits and the difficulties of the decentralized binary framework. More specifically, the binary sequences $\{-,+\}$ can be (locally) summarized by the times of closest point approach (cpa). So, we consider that the available observations, at the network level, are the estimated values of the cpa times. The analysis is also greatly simplified if we assume that the target motion is rectilinear and uniform or a leg-by-leg one. First, we examine the observability requirements for the trajectory parameters. Though the observations do not permit a complete observability, this study allows us to determine the observable part of the state vector. Moreover, we show that observable and unobservable parts are separated. Thus, it is possible to develop simple and efficient methods for estimating the observable parameters. In the case of a single-leg trajectory, we resort to a simple maximum-likelihood estimator, while for the case of multiple-leg trajectories other methods are presented. It is then possible to give confidence intervals for the unobservable components of the state vector. Finally, the constant velocity assumption is relaxed through diffusion process, whether continuous or discrete-time.**

I. Introduction

We consider a sensor network, made with N sensors (e.g. video),with (known) positions. Each sensor can only gives us a binary $\{-,+\}$ information [3], i.e. whether the target-sensor distance is decreasing $(-)$ or increasing $(+)$. Even if many seminal contributions deals with proximity important papers [1], [2], we decide here to focus on the binary $\{-,+\}$ information [3]. From this binary sequence, it is possible to infer the time-period for which the target-sensor distance is locally minimum. For the i-th sensor, we denote t^i_{cpa} this time-period.

Of course, it may be argued that the (relatively) rich binary $\{-,+\}$ information has been considerably reduced. However, it has the great advantage to put in evidence the basic limitations of the treatments, the effects of the process noise, etc. . All the derivations are made within a unique and elementary framework.

This paper is organized as follows. First, the geometric framework is introduced. Though quite elementary, this section will be of constant use subsequently. Then, elementary linear algebra is used to investigate observability requirements when observations are only made of the t^i_{cpa}. Even, if complete observability cannot be achieved by using these observations it is shown that the observable part can be separated from the unobservable one, which is generally one-dimensional.

It is then possible to develop methods for estimating the observable part of the target state vector. First, this is done for a "deterministic" target. If the target trajectory is deterministic, the difficulty we have to face is to estimate the maneuver time-periods T_i. It is shown that when the sensor network is sufficiently dense the error for estimating them can be bounded above.
Then, we relax the hypothesis of deterministic velocity and allow the target to have a diffusive motion. In a first paragraph, we present a continuous-time modeling, and the corresponding estimator of the t_{cpa}. The accuracy of a "continuous-time" estimator is considered, based upon the general framework of stochastic processes. Then, we turn toward a discrete-time modeling, and more specifically a hierarchical Markov chain modeling. We conclude this paper by simulation results, illustrating the accuracy of the estimators and the pertinence of our derivations and modelings.

II. From Cartesian to CPA coordinates (constant velocity)

Consider that the reference sensor is located at the origin O. The target starts from the M_0 point and follows a rectilinear and uniform trajectory ($\mathbf{v}$: velocity vector). Then, the closest-point-approach (cpa) point M_{cpa} is characterized by: the vectors $\overrightarrow{OM}_{\mathsf{cpa}}$ ($\overrightarrow{OM}_0 = \mathbf{x}_0$) and $\overrightarrow{M_0M}_{\mathsf{cpa}}$ are orthogonal, so that we have (see fig. 1):

$$(x_0 + t_{\mathsf{cpa}}\, v_x)\, v_x + (y_0 + t_{\mathsf{cpa}} v_y)\, v_y = 0 \ ,$$
so, that :
$$t_{\mathsf{cpa}} = -\frac{\langle \mathbf{x}_0, \mathbf{v}\rangle}{\|\mathbf{v}\|^2} \text{ ,with: } \mathbf{v} = (v_x, v_y)^T \ , \ \mathbf{x}_0 = (x_0, y_0). \tag{1}$$

Similarly, we obtain the following expression of the closest distance from the sensor, $\mathsf{cpa} \triangleq \|\overrightarrow{OM}_{\mathsf{cpa}}\|$:

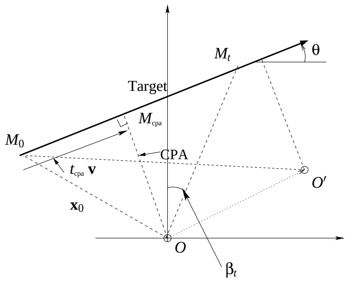

Figure 1. *The CPA geometry*

$$\begin{aligned}
\|\overrightarrow{OM}_{\mathsf{cpa}}\| &= \left[(x_0+t_{\mathsf{cpa}}v_x)^2+(y_0+t_{\mathsf{cpa}}v_y)^2\right]^{1/2} \qquad (2)\\
&= \left[\|\mathbf{x}_0\|^2-\frac{\langle \mathbf{x}_0,\mathbf{v}\rangle^2}{\|\mathbf{v}\|^2}\right]^{1/2},\\
&= \|\mathbf{x}_0\|\,|\sin(\mathbf{x}_0,\mathbf{v})| = \frac{|\det(\mathbf{x}_0,\mathbf{v})|}{v}\ .
\end{aligned}$$

So there is a $1:1$ mapping between Cartesian and cpa coordinates defined as as follows:

$$\begin{pmatrix}\mathbf{x}_0\\ \mathbf{v}\end{pmatrix} \to \begin{pmatrix}|\det(\mathbf{x}_0,\mathbf{v})|\\ \langle \mathbf{x}_0,\mathbf{v}\rangle\\ \theta\\ v=\|\mathbf{v}\|\end{pmatrix} \to \begin{pmatrix}\frac{|\det(\mathbf{x}_0,\mathbf{v})|}{v}=\mathsf{cpa}_{ref}\\ v\\ \theta\\ -\frac{\langle \mathbf{x}_0,\mathbf{v}\rangle}{\|\mathbf{v}\|^2}=t^{ref}_{\mathsf{cpa}}\end{pmatrix}, \tag{3}$$

where θ is the target heading. Considering that O is the reference (ref) position and that $\mathbf{t}_i \stackrel{\Delta}{=} \overrightarrow{OO}_i$ and denoting t^i_{cpa} the cpa time for the i-th sensor, the following relation will be of constant use subsequently:

$$\begin{aligned}
t^i_{\mathsf{cpa}} &= -\frac{\langle \mathbf{x}_0-\mathbf{t}_i,\mathbf{v}\rangle}{v^2}\ , \qquad (4)\\
&= t^{ref}_{\mathsf{cpa}}+\frac{\langle \mathbf{t}_i,\mathbf{v}\rangle}{v^2}\ .
\end{aligned}$$

Let us stress that the vector $\boxed{\mathbf{t}_i}$ is assumed to be known and that eq. 5 is only conditionnally valid. These conditions will not be detailed here, but will be considered in the next sections.

III. OBSERVABILITY ANALYSIS

We consider that the observation is restricted to the t_{cpa}. Various assumptions (about target trajectory) will be considered, but in all the cases the approach we take here is purely deterministic with a binary answer (yes or no). For reasons that will appear clearly through the observability analysis, let us denote $\boxed{\tau_{i,j} \stackrel{\Delta}{=} t^j_{\mathsf{cpa}}-t^i_{\mathsf{cpa}}}$, the difference of cpa times. The aim of this section is to find out whether it is possible to determine the target trajectory parameters.

a) The case of a constant velocity vector: Assume that the target follows a rectilinear trajectory, with a constant velocity vector $\mathbf{v}$, then, from the previous section, we know that:

$$\tau_{i,j} = \frac{\langle \mathbf{t}_j-\mathbf{t}_i,\mathbf{v}\rangle}{\|\mathbf{v}\|^2}\ . \tag{5}$$

Now, the question is: is-it possible that two velocity vectors (say $\mathbf{v}_1$ and $\mathbf{v}_2$) produce the same set of $\{\tau_{i,j}\}$, i.e. $\tau_{i,j}(\mathbf{v}_1)=\tau_{i,j}(\mathbf{v}_2)\ \forall(i,j)$. This would mean that:

$$\langle \mathbf{t}_j-\mathbf{t}_i, \frac{1}{v_1^2}\mathbf{v_1}-\frac{1}{v_2^2}\mathbf{v_2}\rangle = 0\ ,\ \forall\,(i,j)\ . \tag{6}$$

If $\mathrm{span}\,[\{\mathbf{t}_j-\mathbf{t}_i\}]=\mathbb{R}^2$, then we deduce from eq. 6:

$$\frac{1}{v_1^2}\,\mathbf{v_1} = \frac{1}{v_2^2}\,\mathbf{v_2}\ , \tag{7}$$

so that $\mathbf{v_1}=\mathbf{v_2}$. Indeed, considering eq. 7 we deduce from the equality of norms that $v_1=v_2$, the vector equality $\mathbf{v_1}=\mathbf{v_2}$ then follows straightforwardly. The velocity vector $\mathbf{v}$ is observable, but not the position vector $\mathbf{x}_0$. This is quite obvious since t_{cpa} can also be written as

$$t_{\mathsf{cpa}} = -\frac{1}{v^2}\,\langle \mathbf{x}_0+\lambda\,\mathbf{v}^{\perp},\mathbf{v}\rangle\ ,$$

where $\mathbf{v}^{\perp}$ is a vector orthogonal with $\mathbf{v}$. Reciprocally, if the t_{cpa} are equal altogether, then under the same conditions ($\mathrm{span}\,[\{\mathbf{t}_j-\mathbf{t}_i\}]=\mathbb{R}^2$), we have $\langle \mathbf{x}_0,\mathbf{v}\rangle=\langle \mathbf{x}_0^{'},\mathbf{v}\rangle$ which means that $\mathbf{x}_0^{'}=\mathbf{x}_0+\lambda\ \mathbf{v}^{\perp}$. Thus, we see that the dimension of the observable space is equal to 3. The only unobservable parameter is the λ parameter. Of course, unobservability of the λ parameter can be drastically mitigated if proximity sensors are added in the sensor network. However, we shall focus now on multileg target trajectory.

b) The case of a multileg trajectory: Assume now that the target follows a multileg trajectory. For instance, we shall first restrict to a two-leg trajectory and assume that the "maneuvering" time T_1 is known. This means that we have:

$$\begin{cases} 1-\text{st leg}:\ t^i_{\mathsf{cpa},1}=\frac{\langle -\mathbf{x}_0+\mathbf{t}_i,\mathbf{v}_1\rangle}{v_1^2}\ ,\\ 2-\text{nd leg}:\ t^j_{\mathsf{cpa},2}=\frac{\langle -\mathbf{x}_0-T_1\ \mathbf{v}_1+\mathbf{t}_j,\mathbf{v}_2\rangle}{v_2^2}\ ,\end{cases} \tag{8}$$

where i are sensor index for cpa arising on the 1-st leg, and j for the 2-nd leg. It is known (see previous paragraph) that $\mathbf{v}_1$ and $\mathbf{v}_2$ are observable on each leg. Assume now that eq. 8 holds true for two vectors $\mathbf{x}_0$ and $\mathbf{x}_0^{'}$, this would mean that:

$$\begin{cases} 1-\text{st leg}:\ \frac{\langle -\mathbf{x}_0+\mathbf{t}_i,\mathbf{v}_1\rangle}{v_1^2}=\frac{\langle -\mathbf{x}_0^{'}+\mathbf{t}_i,\mathbf{v}_1\rangle}{v_1^2}\ ,\\ 2-\text{nd leg}:\ \frac{\langle -\mathbf{x}_0-T_1\ \mathbf{v}_1+\mathbf{t}_j,\mathbf{v}_2\rangle}{v_2^2}=\frac{\langle -\mathbf{x}_0^{'}-T_1\ \mathbf{v}_1+\mathbf{t}_j,\mathbf{v}_2\rangle}{v_2^2}\end{cases} \tag{9}$$

Subtracting right member from left member on each row of eq. 9, we deduce from eq. 9:

$$\begin{cases} \langle \mathbf{x}_0^{'} - \mathbf{x}_0, \mathbf{v}_1 \rangle = 0 \ , \\ \langle \mathbf{x}_0^{'} - \mathbf{x}_0, \mathbf{v}_2 \rangle = 0 \ . \end{cases} \tag{10}$$

Thus, if $\mathbf{v}_1$ and $\mathbf{v}_2$ are not colinear, then we deduce that $\mathbf{x}_0 = \mathbf{x}_0^{'}$. This reasoning can be easily extended to a general multileg trajectory.

Consider now the observability of *both* T_1 and $\mathbf{x}_0$, from eq. 9 we have:

$$\begin{cases} 1-\text{st leg}: \ \langle \mathbf{x}_0^{'} - \mathbf{x}_0, \mathbf{v}_1 \rangle = 0 \ , \\ 2-\text{nd leg}: \ -\frac{\langle \mathbf{x}_0^{'} - \mathbf{x}_0 + \Delta T_1 \mathbf{v}_1, \mathbf{v}_2 \rangle}{v_2^2} + \Delta T_1 = 0 \ , \\ \text{with: } \ \Delta T_1 = (T_1^{'} - T_1) \ . \end{cases} \tag{11}$$

Practically, this means that we have:

$$\begin{cases} \mathbf{x}_0^{'} = \mathbf{x}_0 + \lambda \, \mathbf{v}_1^{\perp} \ , \\ \langle \lambda \, \mathbf{v}_1^{\perp} + \Delta T_1 \, \mathbf{v}_1, \mathbf{v}_2 \rangle = \Delta T_1 \, v_2^2 \ . \end{cases} \tag{12}$$

This equation has a certain importance since it shows that we have only a single equation for determining the two parameters λ and $(T_1^{'} - T_1)$. Following the same idea with a 3-leg trajectory, we obtain the necessary conditions:

$$\begin{cases} \mathbf{x}_0^{'} = \mathbf{x}_0 + \lambda \, \mathbf{v}_1^{\perp} \ , \\ \langle \lambda \, \mathbf{v}_1^{\perp} + \Delta T_1 \, \mathbf{v}_1, \mathbf{v}_2 \rangle = \Delta T_1 \, v_2^2 \ , \\ \langle \lambda \, \mathbf{v}_1^{\perp} + \Delta T_1 \mathbf{v}_1 + \Delta T_2 \, \mathbf{v}_2, \mathbf{v}_3 \rangle = (\Delta T_1 + \Delta T_2) \, v_3^2 \ , \end{cases} \tag{13}$$

Again, we note that we have two linear equations for determining 3 unknown parameters (λ, $(T_1^{'} - T_1)$, $(T_2^{'} - T_2)$), and the same problem whatever the number of legs. Denoting l the number of legs, the dimension of the target state vector is $(2\,l + 2)$, while under mild conditions the dimension of the observable space is $(2\,l + 1)$. Of course is $\mathbf{x}_0$ is known, then the λ parameter is zero, which means that the target trajectory is completely observable. Moreover, if only $r_0 \triangleq \|\mathbf{x}_0\|$ is available, then the problem becomes fully "observable", which means that the only ambiguity which remains is $\mathbf{x}_0^{'} = \mathbf{x}_0 - 2\frac{\langle \mathbf{x}_0, \mathbf{v}_1^{\perp} \rangle}{\|\mathbf{v}_1\|^2}$, which is the symmetric (w.r.t. $\mathbf{v}_1$) trajectory.

However, even if we cannot have the complete observability of the target state vector, we can infer convenient estimates of T_1, T_2 from the t_{cpa} sequences via the observability of the $\mathbf{v}_i$ vectors.

We shall try now to investigate more precisely the uncertainty in the target trajectory we can infer from eq. 13. Thus for a 2-leg trajectory eq. 13 yields:

$$\lambda \, \langle \mathbf{v}_1^{\perp}, \mathbf{v}_2 \rangle + \Delta T_1 \ \left(\langle \mathbf{v}_1, \mathbf{v}_2 \rangle - v_2^2 \right) = 0 \ . \tag{14}$$

If $v_1 = v_2$, the above equation becomes:

$$\lambda \ \cos(\theta/2) + \Delta T_1 \ \sin(\theta/2) = 0 \ , \ \theta = \measuredangle(\mathbf{v}_1, \mathbf{v}_2) \ .$$

Thus, we see that it is impossible to separate the uncertainty we have in $\mathbf{x}_0$ (λ) in the first hand and in T_1 in the second one. Eq. 15 gives us the equation of the domain -here a segment of a straight line- where the maneuver can occurs. Practically, we have bounds about ΔT_1 (time between 2 t_{cpa}) and/ or λ which alow us to bound this domain.

For a p leg-by-leg trajectory, and denoting $\mathbf{\Delta T} \triangleq (\Delta T_1, \cdots, \Delta T_{p-1})$, we have:

$$\begin{aligned} & \mathcal{A} \, \mathbf{\Delta T} = \lambda \, \mathbf{W} \ , \\ & \text{with:} \\ & \mathcal{A} = \begin{pmatrix} \langle \mathbf{v}_1, \mathbf{v}_2 \rangle - v_2^2 & 0 & \cdots & 0 \\ \langle \mathbf{v}_1, \mathbf{v}_3 \rangle - v_3^2 & \langle \mathbf{v}_2, \mathbf{v}_3 \rangle - v_3^2 & \cdots & 0 \\ \vdots & \vdots & & 0 \\ \langle \mathbf{v}_1, \mathbf{v}_p \rangle - v_p^2 & \langle \mathbf{v}_2, \mathbf{v}_p \rangle - v_p^2 & & \langle \mathbf{v}_{p-1}, \mathbf{v}_p \rangle - v_p^2 \end{pmatrix} \\ & \text{and:} \\ & \mathbf{W} = \left(\langle \mathbf{v}_1^{\perp}, \mathbf{v}_2 \rangle, \langle \mathbf{v}_1^{\perp}, \mathbf{v}_3 \rangle, \cdots, \langle \mathbf{v}_1^{\perp}, \mathbf{v}_p \rangle \right)^T \ . \end{aligned} \tag{15}$$

Having studied the observability, we shall now turn toward the estimation.

IV. Maximizing the likelihood:

Let us denote $\hat{\tau}_i$ the estimated value of t^i_{cpa} on the i-th sensor. We suppose that $\hat{\tau}_i$ is normally distributed, i.e.:

$$\begin{aligned} & \hat{\tau}_i = \tau_i(\mathbf{x}_0, \mathbf{v}) + e_i, \quad e_i \sim \mathcal{N}(0, \sigma^2) \ , \\ & \text{where:} \\ & \tau_i(\mathbf{x}_0, \mathbf{v}) = \frac{\langle -\mathbf{x}_0 + \mathbf{t}_i, \mathbf{v} \rangle}{v^2} \ . \end{aligned} \tag{16}$$

As seen previously, it is not possible to infer *both* $\mathbf{x}_0$ and $\mathbf{v}$ from the $\{\hat{\tau}_i\}$. However, if we are concerned with the estimation of $\mathbf{v}$ only, it is worth considering the differences of the cpa times, i.e. $\hat{\tau}_{i,j}$, with:

$$\begin{aligned} & \hat{\tau}_{i,j} = \tau_{i,j}(\mathbf{v}) + e_i - e_j \ , \\ & \text{with:} \\ & \tau_{i,j}(\mathbf{v}) = \frac{\langle \mathbf{t}_i - \mathbf{t}_j, \mathbf{v} \rangle}{v^2} \ . \end{aligned} \tag{17}$$

It must be emphasized that the vectors $(\mathbf{t}_i - \mathbf{t}_j)$ (relative positions of sensors) are assumed to be known and that the $\tau_{i,j}(\mathbf{v})$ do not involve the $\mathbf{x}_0$ vector. Observation is now made of the vector of $\hat{\tau}_{i,j}$ differences, i.e.:

$$\begin{aligned} & \hat{\boldsymbol{\tau}} = \boldsymbol{\tau}(\mathbf{v}) + \mathbf{e} \ ; , \\ & \boldsymbol{\tau}(\mathbf{v}) = (\cdots, \tau_{i,j}(\mathbf{v}), \cdots) \ , \ \mathbf{e} = (\cdots, e_i - e_j, \cdots) \ . \end{aligned} \tag{18}$$

The noise vector $\mathbf{e}$ is still normally distributed with zero mean, but with a non-diagonal covariance matrix R. The likelihood function $L(\hat{\boldsymbol{\tau}}|\mathbf{v})$, then stands as follows:

$$L\left(\hat{\boldsymbol{\tau}}|\mathbf{v}\right) = \text{cst}(\det R)^{-1/2} \exp\left(-\frac{1}{2} \, \|\hat{\boldsymbol{\tau}} - \boldsymbol{\tau}(\mathbf{v})\|_R^2 \right) \ . \tag{19}$$

The MLE estimation of the $\mathbf{v}$ vector is defined by $\widehat{\mathbf{v}} = \arg\max_{\mathbf{v}} \ L\left(\hat{\boldsymbol{\tau}}|\mathbf{v}\right)$ and can be done by any iterative method, or even an MCMC one if prior and constraints are added. Now, the likelihood functional $L(\mathbf{v}) \triangleq \sum_{i,j} (\hat{\tau}_{i,j} - \tau_{i,j}(\mathbf{v}) \ )^2$ is not convex in general (see the Hessian of $L\left(\hat{\boldsymbol{\tau}}|\mathbf{v}\right)$) . However, assuming that the $\hat{\tau}_{i,j}$ are tightly estimated, which means that $\hat{\tau}_{i,j} = \tau_{i,j}(\mathbf{v}_0)$, where $\mathbf{v}_0$ is the exact velocity vector, then the following property holds true:

Proposition 1: Under the above assumption, the following implication is valid:

$$\nabla L(\mathbf{v}) = \mathbf{0} \Longrightarrow \sum_{i,j} [\tau_{i,j}(\mathbf{v}_0) - \tau_{i,j}(\mathbf{v})\ ]^2 = 0\ ,$$

where $\mathbf{v}_0$ is the true target velocity. Thus, under identifiability conditions, we have $\mathbf{v} = \mathbf{v}_0$.

Proof: Let us recall the expression of $\nabla L(\mathbf{v})$:

$$\begin{aligned} \nabla L(\mathbf{v}) &= -2 \sum_{i,j} (\bar{\tau}_{i,j} - \tau_{i,j}(\mathbf{v})\ ) \\ &\times \left(\frac{1}{v^2} (\mathbf{t}_i - \mathbf{t}_j) - \frac{2}{v^4} \langle \mathbf{t}_i - \mathbf{t}_j, \mathbf{v} \rangle\ \mathbf{v}\ \right) \end{aligned} \quad (20)$$

where, for the sake of brevity, we denote $\bar{\tau}_{i,j}$ for $\tau_{i,j}(\mathbf{v}_0)$. Thus, $\nabla L(\mathbf{v}) = \mathbf{0}$ is equivalent to:

$$\begin{aligned} &\sum_{i,j} (\bar{\tau}_{i,j} - \tau_{i,j}(\mathbf{v})\ ) \frac{1}{v^2} (\mathbf{t}_i - \mathbf{t}_j) = \\ &2 \sum_{i,j} (\bar{\tau}_{i,j} - \tau_{i,j}(\mathbf{v})\ ) \frac{\langle \mathbf{t}_i - \mathbf{t}_j, \mathbf{v} \rangle}{v^4}\ \mathbf{v}\ . \end{aligned} \quad (21)$$

For the rest of the proof, we denote $\bar{\mathbf{v}}$, a vector for which $\nabla L(\bar{\mathbf{v}}) = \mathbf{0}$ The above equality (eq. 21) is a vectorial equality, which implies the following equality via a scalar product with the $\bar{\mathbf{v}}$ vector.

$$\begin{aligned} &\sum_{i,j} (\bar{\tau}_{i,j} - \tau_{i,j}(\bar{\mathbf{v}})\ ) \frac{1}{\bar{v}^2} (\langle \mathbf{t}_i, \bar{\mathbf{v}} \rangle - \langle \mathbf{t}_j, \bar{\mathbf{v}} \rangle) = \\ &2\sum_{i,j} (\bar{\tau}_{i,j} - \tau_{i,j}(\bar{\mathbf{v}})\ ) \frac{1}{\bar{v}^2} (\langle \mathbf{t}_i, \bar{\mathbf{v}} \rangle - \langle \mathbf{t}_j, \bar{\mathbf{v}} \rangle)\ , \\ &\text{so that, we have:} \\ &\sum_{i,j} (\bar{\tau}_{i,j} - \tau_{i,j}(\bar{\mathbf{v}})\ )\ \tau_{i,j}(\bar{\mathbf{v}}) = 0\ . \end{aligned} \quad (22)$$

The same kind of result is obtained if we replace the scalar product with the $\bar{\mathbf{v}}$ vector, by the scalar product with the $\mathbf{v}_0$ vector, yielding:

$$\begin{aligned} &\sum_{i,j} (\bar{\tau}_{i,j} - \tau_{i,j}(\bar{\mathbf{v}})\ ) \frac{1}{\bar{v}^2} (\langle \mathbf{t}_i, \mathbf{v}_0 \rangle - \langle \mathbf{t}_j, \mathbf{v}_0 \rangle) = \\ &2 \sum_{i,j} (\bar{\tau}_{i,j} - \tau_{i,j}(\bar{\mathbf{v}})\ ) \frac{\langle \mathbf{t}_i - \mathbf{t}_j, \bar{\mathbf{v}} \rangle}{\bar{v}^4} \langle \bar{\mathbf{v}}, \mathbf{v}_0 \rangle\ , \\ &\text{or:} \\ &\left(\frac{{v_0}^2}{\bar{v}^2} \right) \sum_{i,j} (\bar{\tau}_{i,j} - \tau_{i,j}(\bar{\mathbf{v}}))\ \bar{\tau}_{i,j} = \\ &2 \frac{\langle \bar{\mathbf{v}}, \mathbf{v}_0 \rangle}{\bar{v}^2} \sum_{i,j} (\bar{\tau}_{i,j} - \tau_{i,j}(\bar{\mathbf{v}})\ )\ \tau_{i,j}(\bar{\mathbf{v}})\ . \end{aligned} \quad (23)$$

Now, we know from eq. 22 that $\sum_{i,j} (\bar{\tau}_{i,j} - \tau_{i,j}(\bar{\mathbf{v}})\ )\ \ \tau_{i,j}(\bar{\mathbf{v}})$ is zero, hence we have also:

$$\sum_{i,j} (\bar{\tau}_{i,j} - \tau_{i,j}(\bar{\mathbf{v}}))\ \bar{\tau}_{i,j} = 0\ . \quad (24)$$

Gathering eqs 22 and 24, we obtain:

$$\begin{aligned} &\sum_{i,j} (\bar{\tau}_{i,j} - \tau_{i,j}(\bar{\mathbf{v}}))\ \ \tau_{i,j}(\bar{\mathbf{v}}) - \sum_{i,j} (\bar{\tau}_{i,j} - \tau_{i,j}(\bar{\mathbf{v}}))\ \ \bar{\tau}_{i,j} = 0\ , \\ &\text{or:} \\ &\sum_{i,j} (\bar{\tau}_{i,j} - \tau_{i,j}(\bar{\mathbf{v}}))^2 = 0\ , \end{aligned} \quad (25)$$

which means that $\bar{\tau}_{i,j} = \tau_{i,j}(\bar{\mathbf{v}})$, whatever the couple (i,j). Finally, we have obtained that the equality $\nabla L(\mathbf{v}) = \mathbf{0}$ implies the equality $\bar{\tau}_{i,j} = \tau_{i,j}(\bar{\mathbf{v}})\ ,\ \forall (i,j)$, and under identifiability condition we have then $\bar{\mathbf{v}} = \mathbf{v}_0$.

The previous reasoning can be easily extended to a multileg scenario: $\{\mathbf{x}_0, \mathbf{v}_1, \cdots, \mathbf{v}_k\}$, with the same conclusion.

V. Estimating the Maneuver Time Periods

We have seen in the previous sections the importance to have a convenient estimation of the maneuver time periods (for a leg-by-leg trajectory). For the sake of simplicity, we shall focus here on a unique velocity change. To that aim, an original method has been developed.

Let us consider the sensor network and more specifically the sensor pairs (see fig. 2). We define an equivalence relation $\mathcal{R}$ on the sensors pairs (i,j), by:

$$(i,j)\ \mathcal{R}\ (k,l) \Longleftrightarrow\ (\mathbf{t}_i - \mathbf{t}_j) = (\mathbf{t}_k - \mathbf{t}_l)\ . \quad (26)$$

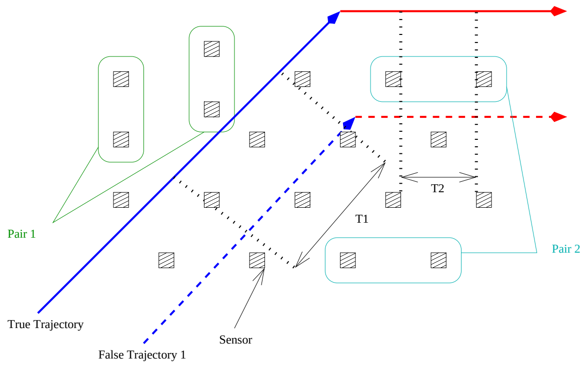

Figure 2. *The target changes its direction*

We then consider each pair of sensors belonging to given class of equivalence. As each pair is oriented in the same way, the difference of the t_{cpa} between the two sensors of each pair should be the same for each pair of the class, for a given velocity vector. Restricting to two classes (1 and 2), the situation can be modelled as follows:

$$\left\{ \begin{array}{ll} \Delta^1_{t_{\text{cpa}}}(\mathbf{v}_1) + \varepsilon^1_1 & \Delta^2_{t_{\text{cpa}}}(\mathbf{v}_1) + \varepsilon^2_1 \\ \Delta^1_{t_{\text{cpa}}}(\mathbf{v}_1) + \varepsilon^1_2 & \Delta^2_{t_{\text{cpa}}}(\mathbf{v}_1) + \varepsilon^2_2 \\ \vdots & \vdots \\ \Delta^1_{t_{\text{cpa}}}(\mathbf{v}_1) + \varepsilon^1_p & \Delta^2_{t_{\text{cpa}}}(\mathbf{v}_1) + \varepsilon^2_p \\ - - - & - - - \\ \Delta^1_{t_{\text{cpa}}}(\mathbf{v}_2) + \varepsilon^1_{p+1} & \Delta^2_{t_{\text{cpa}}}(\mathbf{v}_2) + \varepsilon^2_{p+1} \\ \vdots & \vdots \end{array} \right. \quad (27)$$

along with:

$$\Delta^1_{t_{\mathsf{cpa}}}(\mathbf{v}_1) = \frac{\langle \Delta \mathbf{t}, \mathbf{v}_1 \rangle}{v_1^2} \quad \Delta^1_{t_{\mathsf{cpa}}}(\mathbf{v}_2) = \frac{\langle \Delta \mathbf{t}, \mathbf{v}_2 \rangle}{v_2^2} \ . \quad (28)$$

Notice that these quantities (e.g. $\Delta^1_{t_{\mathsf{cpa}}}(\mathbf{v}_1)$) does not necessarily exist at each time-period. The errors ε_I are normally distributed. Rougly speaking, we have a time series for the first class (left column of eq. 27) and the second class (right column of eq. 27). Thus, if the target maneuver then each time series can be separated in two parts, from which we can infer the target velocity and an estimation of the maneuver time-period.

A usual method we can resort is to model each time series via a mixture of normal densities. There are many methods to estimate the parameters of the mixture density (e.g. EM [7], moments [6], etc.). Here, we choose to focus on the moment method. We thus consider the following modeling of the time series:

$$q(x, \Theta) = \pi \ p(x, \theta_1) + (1 - \pi) \ p(x, \theta_2) \ , \quad (29)$$

and our aim is to estimate the parameter vector $\Theta = (\pi, \theta_1, \theta_2)$. By writing the likelihood of our modeling, we can express the theoretical moments of our sample, and from the data we can determine the values of the empirical moments [6]. Then, by equalizing the empirical and the theoretical moments, we deduce a non-linear system, and ,solving it, an estimation of the Θ vector [5].
More precisely, we use the following equations, where M_i is the i-th (theoretical) moment of $q(x, \Theta)$, m_i is the empirical moment and $x_i = m_i - M_i$:

$$\begin{aligned} 0 &= \pi \, x_1 + (1 - \pi) \, x_2 \ , \quad (30) \\ M_2 &= \pi \, x_1^2 + (1 - \pi) \, x_2^2 + \eta^2 \ , \\ M_3 &= \pi \, x_1^3 + (1 - \pi) \, x_2^3 \\ &\quad + 3\eta^2 (\pi \, x_1 + (1 - \pi) \, x_2) \ , \\ M_4 &= \pi \, x_1^4 + (1 - \pi) \, x_2^4 \ , \\ &\quad + 6\eta^2 (\pi \, x_1^2 + (1 - \pi) \, x_2^2) + 3\eta^4 \ , \end{aligned}$$

where η^2 is the variance of ε. For the uniqueness in the method of moments in the case of mixtures law of two normal distributions, see [4].

VI. Stochastic velocity motion

A. Diffusive continuous-time process

Up to now, it was considered that the target motion was (piecewise)-deterministic, we shall now extend our analysis to a random motion, first modelled by a continuous-time stochastic process. For the sake of brevity, we restrict to a **mono**-dimensional analysis: e.g. a target evolving on a known road network.

$$dx_t \ = \ v \ dt + \sigma \ dw_t \ , \quad (31)$$

where $(x_t)_{t \in [0,T]}$ is the target position, $(w_t)_{t \in [0T]}$ a Brownian motion (also known as a Wiener Process); v is a constant and σ^2 is the variance of the instantaneous velocity. Such an Ito process has a quite simple solution:

$$x_t \ = \ x_0 + t \ v + \sigma \ w_t \ . \quad (32)$$

Let x_c be the position of the closest point (cpa) for the i-th sensor. Since the network topology is assumed to be known, x_c is perfectly known. We would like to have an estimator of the t_{cpa} time-period, for the stochastic process described in eq. 32. For any sensor, a natural estimator $\hat{t}_0$ is defined by:

$$\hat{t}_0 = \inf \{ t | \ x_t \geq x_c \} \ . \quad (33)$$

The object of this paragraph is to investigate the statistical properties of $\hat{t}_0$ (convergence, bias, variance, etc.). To that aim, we first express the likelihood of $\hat{t}_0$. For a continuous-time process, it is simply:

$$\mathcal{L}(\hat{t}_0) = \frac{\partial}{\partial t} \ P(\hat{t}_0 \leq t) \ . \quad (34)$$

We then have to deal with [1]:

$$\begin{aligned} P(\hat{t}_0 \leq t) &= P(\hat{t}_0 \leq t, x_t \geq x_c) + P(\hat{t}_0 \leq t, x_t \leq x_c) \\ &= P(\hat{t}_0 \leq t, x_t \geq x_c) \quad \text{see the footnote} \\ &= P(x_t \geq x_c) \\ &\quad \text{because } \{\hat{t}_0 \leq t\} \subset \{x_t \geq x_c\} \ . \quad (35) \end{aligned}$$

So, to determine exactly the likelihood of our modeling, we must calculate $P(x_t \geq x_c)$. The right calculation is the following one:

$$\begin{aligned} P(x_t \geq x_c) &= P(x_0 + t \ v + \sigma \ w_t \geq x_c) \ , \\ &= P\left(w_t \geq \frac{(x_c - x_0 - t \ v)}{\sigma} \right) \ . \ (36) \end{aligned}$$

Now, when t is fixed, w_t follows a centered Gaussian law, with a standard deviation equal to $\sqrt{t}$. So, we have:

$$\begin{aligned} P(x_t \geq x_c) &= \int_{b_t}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-u^2/(2t)} du \ , \quad (37) \\ \text{with :} & \\ b_t &= \frac{(x_c - x_0 - t \ v)}{\sigma} = \frac{v}{\sigma} \left(t_{\mathsf{cpa}} - t \right) \ . \end{aligned}$$

To continue the calculation, we need to perform a differentiation of an (improper) parametrized integral whose both integrand and lower bound depend on the parameter (t). So, let us remind the Leibniz formula: assume $g(t) = \int_{\alpha(t)}^{\beta(t)} f(t, u) du$, then under appropriate conditions:

$$\begin{aligned} g'(t) &= \int_{\alpha(t)}^{\beta(t)} \frac{\partial f}{\partial t} f(u, t) du + \beta'(t) f(\beta(t), t) \\ &\quad - \alpha'(t) f(\alpha(t), t) \ . \quad (38) \end{aligned}$$

Applied to eq. 38, Leibniz rule yields:

$$\frac{\partial}{\partial t} P\left(x_t \geq x_c \right) = \begin{cases} \frac{1}{2t\sqrt{2\pi t}} \int_{b_t}^{\infty} \left(\frac{u^2}{t} - 1 \right) e^{-\frac{u^2}{2t}} du \ , \\ + \frac{v}{\sigma \sqrt{2\pi t}} e^{-\frac{v^2 (t_{\mathsf{cpa}} - t)^2}{2t\sigma^2}} \ . \end{cases}$$

[1] Assuming $\frac{\sigma}{v}$ is very small, then the target has a very negligible probability to come back under x_c, and then $P(\hat{t}_0 \leq t, x_t \leq x_c)$ becomes negligible

To simplify that expression, we first split the above integral, yielding:

$$\begin{aligned} I_t &= \frac{1}{2t\sqrt{2\pi t}} \int_{b_t}^{\infty} \Big(\frac{u^2}{t} - 1\Big) e^{-\frac{u^2}{2t}} du , \qquad (39) \\ &= \frac{1}{2t\sqrt{2\pi t}} \int_{0}^{\infty} \Big(\frac{u^2}{t} - 1\Big) e^{-\frac{u^2}{2t}} du , \\ & -\frac{1}{2t\sqrt{2\pi t}} \int_{0}^{b_t} \Big(\frac{u^2}{t} - 1\Big) e^{-\frac{u^2}{2t}} du . \end{aligned}$$

The first integral is zero. Thus, it is sufficient to consider the second one:

$$\begin{aligned} I_t &= \frac{-1}{2t\sqrt{2\pi t}} \int_{0}^{b_t} \Big(\frac{u^2}{t} - 1\Big) e^{-\frac{u^2}{2t}} du , \qquad (40) \\ &= \frac{-1}{2t\sqrt{2\pi t}} \int_{0}^{b_t} \frac{u^2}{t} e^{-\frac{u^2}{2t}} du , \\ & +\frac{1}{2t\sqrt{2\pi t}} \int_{0}^{b_t} e^{-\frac{u^2}{2t}} du . \end{aligned}$$

Then, performing an integration by part, we obtain:

$$I_t = \frac{b_t}{2t\sqrt{2\pi t}} e^{-\frac{{b_t}^2}{2t}} du , \qquad (41)$$

so that, we finally obtain for $\mathcal{L}_t$ an expression as simple as:

$$\mathcal{L}_t = \frac{(t_{\mathsf{cpa}} + t)\ v}{2\sigma\ t\sqrt{2\pi t}} \exp\left(-\frac{b_t^2}{2t}\right) . \qquad (42)$$

To ensure our estimator is a good one require additional calculations. Using eq. 42, it is easily shown that the likelihood is maximum for a value of t (say $t_{\max}$ given by [2]:

$$t_{\max} = t_{\mathsf{cpa}} + \left(1 - \frac{3}{4}\frac{1}{t_{\mathsf{cpa}}}\right) \frac{\sigma^2}{v^2} + o(\frac{\sigma^2}{v^2}) . \qquad (43)$$

The proof of eq. 43, is sketched below:

- Differentiate (w.r.t. t) the density $\mathcal{L}_t$,
- The derivative is zero for values of t which are roots of a certain 3-rd order polynomial,
- Obtain the exact expression of the real positive root via the Cardan method, eq. 43 is deduced from the above step via an expansion around t_{cpa}.

Even if the density of $\hat{t}_0$ does not seem to be symmetric (see eq. 42), we see that $t_{\max}$ is close to t_{cpa} as soon as the ratio σ^2/v^2 is sufficiently small. A last step is to calculate the expectation of $\hat{t}_0$. Under the same assumption and using eq. 42, we have:

$$\mathbb{E}\left[\hat{t}_0\right] = t_{\mathsf{cpa}} - \frac{\alpha}{2} + \frac{3\alpha}{8\ t_{\mathsf{cpa}}} + o(\frac{\sigma^2}{v^2}) , \qquad (44)$$

where:

$$\alpha = -2\sigma + (8\pi + 16\sigma - 20)\frac{\sigma^2}{v^2}$$

[2] τ_{cpa} is deterministic.

Noticing that α is usually quite smaller than t_{cpa}, we conclude that the bias of the estimator of t_{cpa} is equal to σ.

The expression of the variance of our estimator can be calculated by the same way, which we do not provide here. However, simulation results provide interesting results about the effects of σ on $\mathsf{var}(\hat{t}_0)$.

B. *Discrete-Time Markov modeling*

We can also consider a discrete-time modeling of the time-varying target velocity. Instead of considering a continuous Ito process, we turn now toward a hierarchical Markov chain model. Let us define a N-state velocity Markov chain. Assuming the velocity is one dimensional, we have a N-by-N transition matrix for the velocity chain. Then, the position chain is implicitly defined as in figure 3. Then, the first part of our approach deals with the choice of

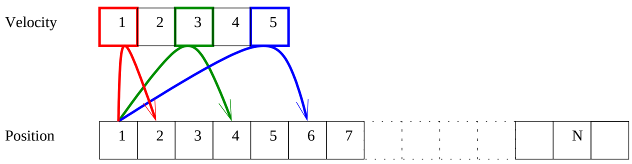

Figure 3. *Link between the position Markov chain and the velocity Markov chain*

the probabilities of transition for the velocity. They should be chosen so that the stationary law would be as close as possible to a normal law, whose mean is the right value of the target velocity (say i_0). More precisely, we choose:

$$\begin{aligned} &\text{if} \quad i = i_0 \\ &\quad P(v_{n+1} = j | v_n = i) = \alpha_k \quad \text{if } | i - j | = k , \\ &\text{if} \quad i < i_0 \\ &\quad P(v_{n+1} = j | v_n = i) = \beta_k \quad \text{if } | j - (i+1) | = k , \\ &\text{if} \quad i > i_0 \\ &\quad P(v_{n+1} = j | v_n = i) = \gamma_k \quad \text{if } | j - (i-1) | = k , \end{aligned}$$

where we have the following assumptions:

$$\sum \alpha_i = \sum \beta_i = \sum \gamma_i = 1 , \qquad (45)$$

and,

$$\forall (i,j) \in [0,k]^2, i > j \Rightarrow \begin{cases} \alpha_i & < & \alpha_j \\ \beta_i & < & \beta_j \\ \gamma_i & < & \gamma_j \end{cases}$$

The position chain is then straightforwardly deduced :

$$\forall k \in \mathbb{N}^* \begin{cases} P(s_{n+1} = i + k | s_n = i) & = & P(v_n = k) , \\ P(s_{n+1} = i - k | s_n = i) & = & 0 . \end{cases}$$

Similarly to eq. 33, we define the discrete-time t_{cpa} estimator by:

$$\hat{k}_0 = \inf\{k | s_k > s_{\mathsf{cpa}}\} , \qquad (46)$$

and the speed estimator :

$$\hat{v}_0 = \frac{\Delta \mathsf{cpa}_{i,j}}{\hat{k}_0} \qquad (47)$$

We refer to the "Simulations" section for experimental results of that method.

VII. SIMULATION RESULTS

We shall now investigate the previous developments via simulations. The simple maximum likelihood estimation will be presented in a first part, in which we will deal with a comparison between two sensor networks. Then we will present few results about a diffusive target.

A. Basic MLE

To verify the effect of the sensor distribution, we test two scenarios. In the first one, the sensors are uniformly distributed over the surveillance domain. In the second one, the sensors are put at random over the same domain. Then, considering a constant speed target motion, the effects of the changes in direction are examined. On fig.

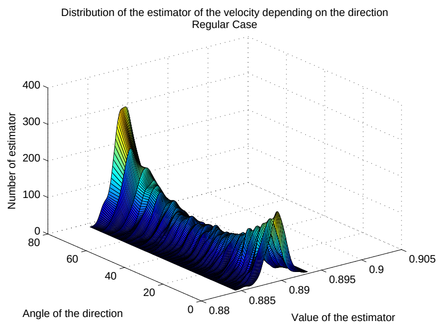

Figure 4. *Uniformly distributed network*

4 the distribution of the velocity estimator in the case of a deterministic regular distribution of the sensors is presented. In fig. 4, we can notice that it is for a middle heading angle (around 45 deg.) that the distribution seems to be the less peaky (though unbiased). This result can be compared with fig. 5 which presents the results for a randomly distributed sensor network. This problem no longer holds. This is also illustrated by fig. 6, where the

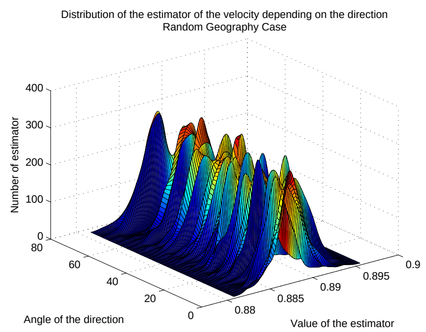

Figure 5. *Randomly distributed network*

target heading is varying from 0 to 80 deg. . Not surprisingly, looking at the variance of the estimators (see fig. 7), we notice that we have a 400% increase of the variance between the 0 degrees direction and the 40 degrees one for a regular network, while this phenomenon does not appear if sensors are put at random. Thus, the statistical

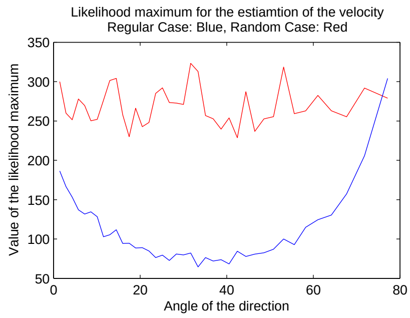

Figure 6. *Maximum of Likelihood*

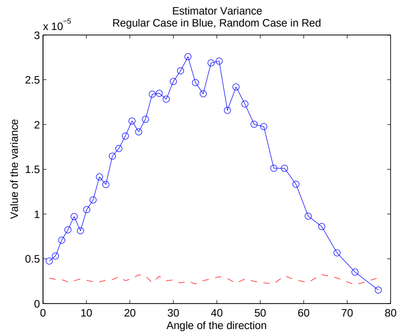

Figure 7. *Variance of the estimators*

behavior of the maximum likelihood estimator is tightly dependent of the sensor distribution. However, this result should be seriously mitigated for a a multi-leg trajectory, since it has been shown that a regular repartition has definite advantages for estimating both velocity, position and maneuvers. More generally, optimizing the topology of the sensor network seems to be an important direction of future research in this context.

B. Continuous-time diffusive target

Assuming that the trajectory follows now a diffusive process, our aim is to examine the statistical behavior of the t_{cpa} estimator we defined, and whose density has been theoretically derived as a function of the process noise. In fig. 8, we present the empirical histogram of the t_{cpa} estimator obtained via simulation and compared with the exact density (see eq. 42). As we could expect, the two distributions present a good agreement. Two important facts have to be underlined. The first one is that even if the mean of the distribution is not the actual value of t_{cpa}, the maximum of the likelihood corresponds to the real value. The second one is that the distribution of the speed estimator can be straightforwardly inferred from the $\hat{t}_{\mathsf{cpa}}$ distribution, since both are deterministically related.

As in the deterministic section, we would like to know the influence of the variance of the stochastic process on the variance of the t_{cpa} estimator (see fig. 9), and of the velocity estimator (see fig. 10). Looking at both figures, the effect of the σ parameter is quite visible. However, while $\mathsf{var}\left[\hat{t}_{\mathsf{cpa}}(\sigma)\right]$ depends (almost) linearly of σ; the

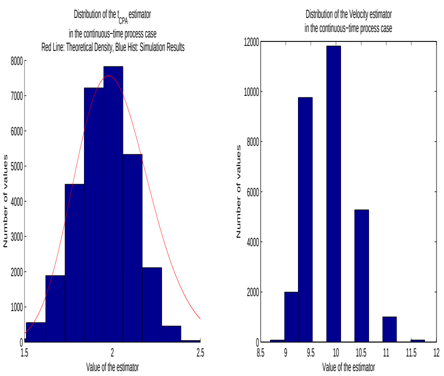

Figure 8. *Left:Distribution of the estimator ($\mathbf{v} = 10\ m/s$, $\sigma = 1, x_c = 20\,m, x_0 = 0\,m$). Right: Distribution of the velocity estimator (same parameters)*

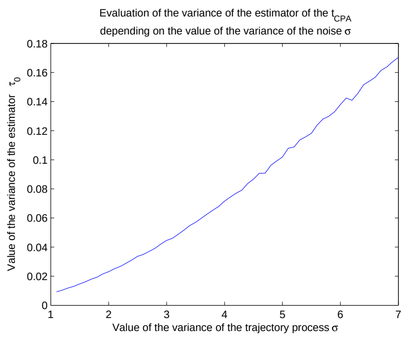

Figure 9. *Variance of $\hat{t}_{\mathsf{cpa}}$ as a function of σ ($v = 10m/s$, $x_c = 20\,m$, $x_0 = 0\,m$)*

effect of σ on $\mathsf{var}(\hat{v}(\sigma))$ is quite non linear. So, a great attention should be paid on the estimation of the variance σ^2 of the t_{cpa} modeling.

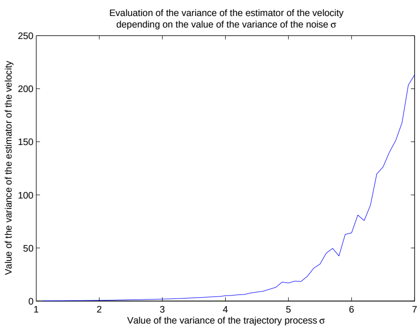

Figure 10. *Variance of the velocity estimator*

C. Diffusive Discrete-time Process

Finally, let us consider a discrete-time modeling of the target velocity. A single result is proposed, but which underline the accuracy of our discrete-time estimator. This time, unlike the previous results, even the mean of the estimator seems to fit the correct value. Once again, the maximum of the likelihood corresponds to the correct values of both the t_{cpa}, and the speed.

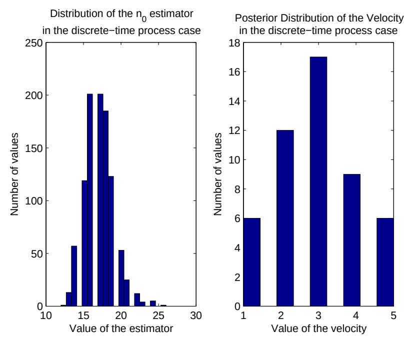

Figure 11. *Left: Distribution of the estimator of the time n_0 (discrete-time t_{cpa}). Right: Posterior distribution of the velocity. Modeling parameters: $v = 3$, $N_v = 5$, $N_c = 50$*

VIII. Conclusion

In this paper, we chose to focus on the use of the t_{cpa} estimates at the level of information processing for a sensor network. Though this information is rather poor, it has been shown that it can provide clear insights about the limitations of the trajectory estimation via binary informations. Three different models of the target trajectory have been considered; ranging from the deterministic one to the continuous-time and discrete time random models. The limitations due to the rough nature of the measurements have been carefully considered. The fundamental uncertainty about target position can be seriously reduced via proximity sensors. Another way is to consider a ”sufficiently” dense network and using the information given by the target maneuvers.
A large part of this paper is devoted to what happens when the target departs from a deterministic model. For the continuous-time model, it has been possible to perform a theoretical analysis. The pertinence of the estimators has been shown and original methods have been developed. Perspectives for future work should be centered around the use of the binary informations [3] for multitarget tracking.

References

[1] X. Wang and B. Moran , Multitarget Tracking Using Virtual Measurements of Binary Sensor Networks. *Proc. of the 9-th Int. Conf. on Information Fusion*, Jul. 2006.

[2] L. Lazos, R. Poovendran and J.A. Ritcey , Probabilistic Detection of Mobile Targets in Heteregeeous Sensor Networks. *Proc. of the 6-th IPSN*, Apr. 2007.

[3] J. Aslam, Z. Butler, F. Constantin, V. Crespi, G. Cybenko, D. Rus, Tracking a moving object with a binary sensor network. *Proc. of the 1st international Conference on Embedded Networked Sensor Systems*, Nov 2005, pp. 150–161

[4] E. Monfrini, Uniqueness in the method of moments for mixtures of two normal distributions. *Compte Rendu Academie des Sciences, Paris*, Ser I, vol. 336 (2003), pp. 89–94

[5] C.-T. Bhattacharya , A simple method of resolution of a distribution into Gaussian components. *Biometrics*, vol. 23, (1967), pp. 115–137.

[6] B.-G. Lindsay and P. Basak, Multivariate Normal Mixtures: A Fast Consistent Method of Moments. *JASA*, vol. 88, no. 422, (1993), pp. 468–476.

[7] A.P. Dempster, N. Laird and D.B. Rubin, Maximum likelihhod from incomplete dat via the EM algorithm. *Journal of the Royal Statistical Society*, B-39, (1977), pp. 1–38.