The Cross-Entropy method for solving a variety of hierarchical search problems

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***Abstract*— This paper introduces a common method, based on the Cross-Entropy method, in oder to solve a variety of search problems when search resources are scarce compared to the size of the space of search. In particular, we solve: detection and information search problems, a detection search game, and a two-targets detection search problem. Our approach is built of two steps: first, decompose a problem in a hierarchical manner (two optimization levels) and then, solve the global level using the Cross-Entropy method. At local level, different solutions are conceivable, depending of the kind of the problem. Problems of interest are in the field of combinatorial optimization and are considered to be hard to solve: we find optimal solution in most cases with a reasonable computation time.**

Keywords: Sensor management, Cross-Entropy, information games, multitarget.

I. Introduction

Search theory is a major discipline within the field of operations research. During the second world war, through his work on antisubmarine warfare, B.O. Koopman [4] laid the principles of search theory: the aim is to find the best manner of searching for an object, given the probabilities on the location of the searched object and the characteristics of the searchers. One can extract two classes of search theory problems:

- the one-sided class: in such search problems, decisions are only made by the searcher
- the two-sided class: in this case, decisions are made both by searchers and searched (search games . . .)

Furthermore, the search problem can be static or dynamic (time dependent). Actually, there is substantial literature on the topic. However, few papers deal with the optimization of hierarchical search problems, despite hierarchical approaches could be very useful. That is especially the case when the space of search is large and the search resources are scarce or when probabilities on the location of the searched object are imprecise. Here we will present algorithms based on the Cross-Entropy method, which was developed by R. Rubinstein [8] in order to optimize some static hierarchical search problems.

This paper is organized as follows: in section II, the framework of our work is presented; followed by the study of four static hierarchical search problems:

- the first one deals with the optimization of the repartition of the search resources, in order to maximize the probability of detecting a unique stationary target,
- the second one deals with the optimization of the repartition of the search , in order to maximize the information gain on the location of a unique stationary target,
- the third one deals with the optimization of a search game, in order to maximize the probability of detecting a unique stationary target,
- the last one deals with the two-targets static detection problem (there are two objects to detect).

II. Framework

In real life, one may want to search a target in a large space of search. Furthermore, available search resources (search resources will be called sensors in the following) might be scarce, compared to the size of the space of search. Thus we need to optimize the use of these resources in order to maximize the probability of detecting the target(s). One way to do that is to partition the space of search into search zones, to which we will then have to choose to allot (or not) one (or more) sensor(s). So, the optimization of the search policy for a sensor will be restricted to a unique zone. Then we can define two interconnected optimization levels:

- a global level: which will manage the allotment of sensors to search zones,
- a local level: which will manage the optimization of a search policy for each sensor, when an allotment has been defined.

Such kind of problems are often very hard to solve because of their complexity. In order to overcome this difficulty we may use simulation methods, as the Cross-Entropy method. Here we especially use the Cross-Entropy method in the optimization of global level.

In this section, we will first introduce the main notations, common to the four hierarchical problems; and then present the Cross-Entropy method.

A. Definitions and notations

1) The space of search: We denote E the space of search. We consider that it has been partitioned into $|Z|$ smaller search zones in order that a sensor can explore efficiently a whole zone (time constraints, autonomy, velocity. . .). A zone is denoted $\mathcal{E}_z$.

Furthermore, characteristics of the land (due to vegetation...) within E are variable: we discretize E into small homogeneous areas. We call them units and denote them E_u. Thus, a search zone is a set of units.

2) The target(s): Sensors are searching for a set $\mathcal{T}$ of target, an elementary target called t. Each of these targets is hidden into one of the units of the space of search. The location of a target t is characterized by a prior probability $\alpha_{t,u}$, over the units of E. Thus:

$$\forall t \in \mathcal{T} \ , \sum_u \alpha_{t,u} = 1 \tag{1}$$

This prior may be relatively informative and results (in general) from operational considerations, previous searches, etc.
Remark: In the following, if $|\mathcal{T}| = 1$, the prior probability of the target will be denoted α_u.

3) Means of search: We will consider that the search is carried out by $|S|$ sensors, an elementary one called s. These sensors can be men, radars, etc. A sensor can only be alloted to a unique search zone (but none, one, or more than one sensors can be alloted to the same search zone). We consider that an allotment of the sensors to the search zones is represented by a mapping $m \ : \ s \ \longmapsto \ z$. Moreover, each sensor disposes of a limited capacity of search Φ_s. For example, it could be the exploration time to share between units of a zone. Investigation of a unit E_u, if the explorer is s, requires a quantity of resource $\varphi_s(E_u)$ (we consider that the resources of each sensor are continuous). As a consequence, the investigation carried out by a sensor is constrained by its capacity of search:

$$\sum_{u | E_u \in \mathcal{E}_z} \varphi_s(E_u) \leq \Phi_s, \tag{2}$$

where $\mathcal{E}_z$ is the zone to which the sensor s is alloted. Of course, if a sensor s is alloted to a zone $\mathcal{E}_z$, $\forall u | \ E_u \notin \mathcal{E}_z$, $\varphi_s(E_u) = 0$. The aim is then to optimize sensors' use.

4) Visibility coefficients: We must qualify the quality of detection of a sensor for a target, over a unit. This is done by means of visibility coefficients, $w^t_{s,u} \in \ ]0;1[$, which are characteristic of:

- the sensor s
- the target t
- the unit E_u.

A visibility coefficient $w^t_{s,u} = 0$ means that the sensor s has no visibility on the target t, if it is hidden into the unit E_u. That is to say: whatever the resource quantity of sensor s alloted for the search in E_u, the target will remain undetected.
As for the prior probabilities of the targets, when $|\mathcal{T}| = 1$, $w^t_{s,u}$ will be denoted $w_{s,u}$.

5) Probabilities of non detection: The detection of a target t hidden in a unit E_u by the sensor s, both depends on the quantity of resource $\varphi_s(E_u)$ of the sensor invested in the search in unit E_u and on the visibility coefficient $w^t_{s,u}$ (see [4]). Thus, the probability of not detecting the target t if it is hidden into the unit E_u, and if it is searched by sensor s is:

$$\exp(-w^t_{s,u} \ \varphi_s(E_u)) \tag{3}$$

Furthermore, detections made by different sensors are considered to be independent. As a finite set of sensors is alloted to a search zone, and because a unit belongs to only one search zone, the non detection probability if the target t is hidden into E_u is:

$$\prod_{s | s \in m^{-1}(z)} \exp(-w^t_{s,u} \varphi_s(E_u)) \tag{4}$$

At last the non detection probability associated to a search zone $\mathcal{E}_z$, when there is only one target to search is:

$$\sum_{u | E_u \in \mathcal{E}_z} \alpha_u \prod_{s | s \in m^{-1}(z)} \exp(-w_{s,u} \varphi_s(E_u)) \tag{5}$$

B. The CE method

1) Brief description: The Cross-Entropy method is a simulation method which has been developed by Rubinstein [8] in order to evaluate rare events probabilities (e.g. failure probabilities, etc.). However, this method can also be employed to solve combinatorial optimization problems (see [8]): first, the problem must be translated into a stochastic problem. The set of feasible solutions of the original problem is then regarded as a set of events subjected to an importance density. Thus, finding the optimal solution to the original problem is considered as a rare event. We present next the Cross-Entropy algorithm.

2) General algorithm: Let B be an evaluation function for the draws.

1) Choose a family of probability laws (P_ν) representative of the problem;
2) Initialize the law : $\nu = \nu_0$;
3) Perform until convergence:
   - Draw N samples $m_1, m_2, \ldots, m_N$ according to P_ν;
   - Select the $R = \rho \times N$, $0 < \ \rho \ < 1$, best draws, according to $B(m_j)$;
     Call them $\tilde{m}_1, \tilde{m}_2, \ldots, \tilde{m}_R$;
   - Find ν' minimizing the Kullback distance, i.e. maximizing: $\displaystyle\sum_{r=1}^{R} \ln(P_\nu \ (\tilde{m}_r))$;
   - Set $\nu := \nu'$;

III. Detection and Information problems

A. Common Settings

The aim of this section is to compare optimization of the search problem described in section II, when the optimization functional is a detection function, and when it is an information function. In both cases, sensors are searching for a motionless target. A prior on the location of the target is also given. Then, the optimization problem is of the following form:

$$\min_{\varphi, \ m} \ F(m, \varphi_s) \tag{6}$$

$$\text{s.t.:} \begin{cases} \forall z, \ \forall s \in m^{-1}(z), \ \displaystyle\sum_{u | E_u \in \mathcal{E}_z} \varphi_s(E_u) \ \leq \ \Phi_s \\ m \ : \ 1...|S| \ \longmapsto \ 1...|Z| \\ \forall u \, , \ \forall s \, , \ \varphi_s \ (E_u) \geq \ 0 \end{cases}$$

where F is separable in z. This problem is hierarchical: thus we can slip it into two interconnected levels:

- a global level: find the best allotment of sensors to search zones
- a local level: when an allotment is given, find the best sensors' resource sharing

Thus, a modification in the sensors allotment implies a need to compute again the local settings. Conversely, the solutions of the local level determine the quality of the global allotments.

B. Detection problem

We are interested in the optimization of a detection function . Thus, the optimization functional F define in (6) is:

$$F(m,\varphi)=\sum_z \sum_{u|E_u\in\mathcal{E}_z} \alpha_u \prod_{s\in m^{-1}(z)} \exp(-w_{s,u}\varphi_s(E_u)). \tag{7}$$

We will first describe the optimization of the local level.

1) Resource sharing: In the particular case where at most one sensor is allotted to a search zone, say sensor s alloted to zone $\mathcal{E}_z$, the following discrete problem must be solved:

$$\min_{\varphi_s} \sum_{u|E_u\in\mathcal{E}_z} \alpha_u \exp(-w_u^s\ \varphi_s(E_u)) \tag{8}$$

$$\text{s.t.:}\begin{cases} \sum_{u|E_u\in\mathcal{E}_z} \varphi_s\ (E_u)-\Phi_s\le 0 \\ \forall\ E_u\in\mathcal{E}_z\ \ \varphi_s\ (E_u)\ge 0 \end{cases}$$

This is a convex problem. According to the Karush, Kuhn and Tucker (KKT) conditions, it could be optimized by solving:

$$\sum_{u|E_u\in\mathcal{E}_z} \varphi_s(E_u)\ -\ \Phi_s\ =\ 0, \tag{9}$$

with:

$$\forall\ E_u\ \in\mathcal{E}_z,\ \ \varphi_s\ (E_u)=\max\left[\ 0\ ,\ -\frac{\ln\left(\frac{\lambda}{w_u^s\ \ \alpha(E_u)}\right)}{w_u^s}\ \right] \tag{10}$$

We must find the value of the multiplier λ, associated to the resource consumption constraint of sensor s (see (2)). Due to the convexity of the objective functional, λ is unique and can easily be found by means of a dichotomy search.

However, more than one sensor can be allotted to the same search zone. When a subset $m^{-1}(z)$ of sensors is allotted to the zone $\mathcal{E}_z$, the following problem has to be solved:

$$\min_{\varphi} \sum_{u|E_u\in\mathcal{E}_z} \alpha(E_u) \prod_{s\in m^{-1}(z)} \exp(-w_u^s\ \varphi_s(E_u)) \tag{11}$$

$$\text{s.t.:}\begin{cases} \forall s\in m^{-1}(z),\ \sum_{u|E_u\in\mathcal{E}_z} \varphi_s(\ E_u\ )-\Phi_s\le 0 \\ \forall s\in m^{-1}(z),\ \forall\ E_u\in\mathcal{E}_z\ \ \varphi_s(\ E_u\ )\ge 0 \end{cases}$$

This problem is also a convex problem. Thus, a simple method for optimizing resource sharing is to optimize successively each variable φ_s, according to the following scheme:

1) Initialize $\varphi_s(E_u)=0$ for each sensor $s\ \in m^{-1}(z)$,
2) Compute the non-detection probability for the search zone $\mathcal{E}_z$, call it $Prob_1^z$,
3) For each sensor $s\ \in m^{-1}(z)$, do:
   - Compute the best resource sharing for the current sensor, according to a de Guenin optimization (for all other sensors, the variables $\varphi_s(E_u)$ are fixed to their last values),
   - $\forall u$, memorize $\varphi_s(E_u)$,
4) Compute the new non-detection probability, call it $Prob_2^z$,
5) If $Prob_1^z\approx Prob_2^z$ then stop; else set $Prob_1^z:=Prob_2^z$ and go to 3.

We now need to present the global optimization algorithm.

2) Sensors-to-zones Allotments: In this section, we present two approaches in order to solve the sensors-to-zones allotment (global level): linear programming (LP) and the Cross-Entropy method (CE). We will see that when the allotment is not injective, LP can not be used to solve the hierarchical optimization problem.

a) Linear Programming: In the case where the mapping m is injective, the hierarchical problem is quite simple. Then, it could be optimized optimally with a method coupling LP (global level) and the algorithm of de Guenin (local level). In the case where LP is used, the LP program has $Z\times S$ continuous variables (one per couple sensor-zone). These variables are denoted x_z^s. They are continuous and they take values in $[0;1]$. Costs C_z^s of these variables are defined by means of the de Guenin algorithm, and are the optimal solutions of:

$$C_z^s\ =\min_{\varphi_s} \sum_{u|E_u\in\mathcal{E}_z} \alpha(E_u)\ \exp\left(-w_u^s\ \varphi_s(E_u)\right) \tag{12}$$

Then we solve the following problem:

$$\min \sum_z \sum_{s|s\in m^{-1}(z)} C_z^s x_z^s \tag{13}$$

$$\text{s.t.:}\begin{cases} \forall s\sum_z\ x_z^s\ =1 \\ \forall z\sum_s\ x_z^s\ =1 \\ \forall s,\forall z:\ \ x_z^s\ \ge\ 0 \end{cases}$$

However, in the general case (injective assumption is removed), it is no more possible to model the allocation of a sensor to a zone by means of a continuous variable. Indeed, the optimization problem at global level becomes an integer programming problem (IP). Thus, in large dimensions, classical methods will fail to obtain an optimal solution in low time consumption. An alternative can be to use the CE, which is adaptive, and is particularly indicated for solving such kind of problems (see [8]).

3) Cross-Entropy method: The de Guenin [2] optimization at local level can be coupled with the use of CE at global level. Here, the principle of the CE algorithm is to draw particular allotments of sensors to search zones that will be evaluate and then selected, in order to obtain a drawing law which gives the optimal allotment. First, we must choose a family of probability laws, (P_ν), describing a probabilistic choice of m. The aim is to find: $\nu^* \in \arg\min_\nu \sum_m P_\nu(m)B(m)$, with $B(m) = \min_\varphi F(m, \varphi)$. We choose to associate to each sensor a discrete probability law $P(\mathcal{E}_z|s)$, which represents the probability of allotting the sensor s to the zone $\mathcal{E}_z$. Thus:

$$\forall s, \ \sum_z P(\mathcal{E}_z|s) = 1. \tag{14}$$

These probability laws will be described by a matrix $\mathbf{M} =$

$$\begin{pmatrix} P(\mathcal{E}_z = \mathcal{E}_1|s=1)^M & \dots & P(\mathcal{E}_z = \mathcal{E}_{|Z|}|s=1)^M \\ P(\mathcal{E}_z = \mathcal{E}_1|s=2)^M & \dots & P(\mathcal{E}_z = \mathcal{E}_{|Z|}|s=2)^M \\ \vdots & \ddots & \vdots \\ P(\mathcal{E}_z = \mathcal{E}_1|s=|S|)^M & \dots & P(\mathcal{E}_z = \mathcal{E}_{|Z|}|s=|S|)^M \end{pmatrix} \tag{15}$$

where $P(\mathcal{E}_z|s)^M$ is the probability of allotting sensor s to $\mathcal{E}_z$, according to $\mathbf{M}$. As explained in the second section, the CE iteration works as follows:

- Generate $|N|$ draws of the sensors-to-zones mapping, denoted m^k, with $1 \leq k \leq |N|$. These draws are sampled according to the probability laws described by $\mathbf{M}$
- Select the $|R| = \rho \times |N|$ best samples according to F and denote them $\tilde{m}^1 ... \tilde{m}^{|R|}$.

The update $\mathbf{M}'$ of $\mathbf{M}$ is given by the minimization of the Kullback-Leiber distance. This results in the following updating formula:

$$P(\mathcal{E}_z|s)^{M'} = \frac{|k, \ 1 \leq k \leq |R|, \ \tilde{m}^k(s) = \mathcal{E}_z|}{|R|} \tag{16}$$

This algorithm describe the case where m can be any mapping (injective or not). However, in the case where m is an injective mapping, some adaptations must be operated. In fact, the probability law must return only injective allotments. One way to do that is to use the same laws ($\mathbf{M}$) than in the general case, but with a penalization or a reject on the samples where the allotment is not injective.

C. Information Problem

In the previous section, we have shown how CE method can be employed for the optimization of combinatorial problems. However, the optimization functional was simple enough so that we can solve the low-level part (resource sharing) with classical means. Here, we use the Cross-entropy method in order to solve problems with more complicated optimization functionals. We are particularly interested in the optimization of information problems, as described by Nakai [7].

1) The information fonctionnal: We want to allot the sensors to the zones, and then the resources of the sensors to the units of the zones, in order to maximize the gain of information concerning the location of the target. The optimization problem is the same as for the detection problem (see (6)), except of the functional. The expected amount of information concerning the location of the target under the search policy (m, φ) is given by:

$$I_{m,\varphi} = \sum_z I^z_{m,\varphi}, \text{ with } I^z_{m,\varphi} = H^z(\alpha) - G^z_{m,\varphi} H^z(\alpha'). \tag{17}$$

where:

$$G^z_{m,\varphi} = \left(\sum_{u|E_u \in \mathcal{E}_z} \alpha_u \prod_{s|s \in m^{-1}(z)} \exp(-w^s_u \varphi_s(E_u)) \right) \tag{18}$$

$H^z(\alpha)$ is the well-known entropy function used by Shannon in communication network:

$$H^z(\alpha) = - \sum_{u|E_u \in \mathcal{E}_z} \alpha_u \ln(\alpha_u); \tag{19}$$

and α' is the posterior distribution of the position of the target given that the prior distribution is α and that it is not detected by the policy (m, φ).

$$\forall E_u \in \mathcal{E}_z, \ \alpha'_u = \frac{\alpha_u \prod_{s|s \in m^{-1}(z)} \exp(-w^s_u \varphi_s(E_u))}{\sum_{u|E_u \in \mathcal{E}_z} \alpha_u \prod_{s|s \in m^{-1}(z)} \exp(-w^s_u \varphi_s(E_u))} \tag{20}$$

Thus, we have to solve the problem (see (6)) with the following optimization fonctional:

$$F_{m,\varphi} = - \sum_z \sum_{u|E_u \in \mathcal{E}_z} \alpha_u \prod_{s|s \in m^{-1}(z)} \left[\exp(-w^s_u \varphi^s_u) \ \ln(J^z_{m,\varphi}) \right] \tag{21}$$

where:

$$J^z_{m,\varphi} = \frac{\sum_{u'|E_{u'} \in \mathcal{E}_z} \alpha_{u'} \prod_{s|s \in m^{-1}(z)} \exp(-w^s_{u'} \varphi^s_{u'})}{\alpha_u \prod_{s|s \in m^{-1}(z)} \exp(-w^s_u \varphi^s_u)} \tag{22}$$

As for the detection search problem, the two levels of hierarchy are:

- global level: find the best allotment of sensors to search zones;
- local level: find the best resource sharing among the units of the zones.

However, this time we can not find by a direct computation the optimal resource sharing at local level for a given allotment at high level. Thus we will use the Cross-Entropy method for optimization at both levels.

2) Solving the information search problem: We solve the information search problem by using the CE method at both high and low level: we define two CE algorithms, one for each level. For the global level, we employ the CE method described in section III-B.3 (denoted CEg). For the local level, we will use another CE method, denoted CEl, based on the optimization of multi-normal distributions. These two methods are interconnected as follows:

- for each allotment drawn by CEg
  - for each zone

- * find the best resource sharing for all the sensors alloted to the zone, according to the information functional, using CEl
  - value the contribution of the zone to the optimization functional
- value the allotment

In the following, we describe the CE method for the optimization of local level.

a) Algorithm CEl: Let $\mathcal{E}_z$ be a zone to which sensors $\{\sigma_1, \sigma_2, ...\}$ are allotted by a draw of CEg. Let $|U_z|$ be the number of units such that: $E_u \in \mathcal{E}_z$. For each sensor σ_i allotted to the zone, we define a $|U_z|-1$ dimensional multi-normal distribution. We call them $\mathcal{N}_{\sigma_i}(\Sigma_{\sigma_i}, \mu_{\sigma_i})$. The aim is to optimize $\mathcal{N}_{\sigma_i}$ for each sensor, according to the information functional. Thus, we introduce the following algorithm:

- repeat until convergence of the $\mathcal{N}_{\sigma_i}$
  - for n in 1..N
    - * **A**: draw a resource sharing for each sensor;
    - * **B**: denote $v^{\sigma_i}_{n,j}$ the value for the draw n, for the sensor σ_i and for the dimension j of $\mathcal{N}_{\sigma_i}$;
    - * **C**: value the information functional restricted to the zone.
  - select the R best draws, according to the information functional;
  - for each σ_i, update $\mathcal{N}_{\sigma_i}$:
    - * $\mu^{\sigma_i}_j = \dfrac{1}{R}\displaystyle\sum_{r=1}^{r=R} v^{\sigma_i}_{r,j}$;
    - * $\Sigma^{\sigma_i}_{j,k} = \dfrac{1}{R}\displaystyle\sum_{r=1}^{r=R}(v^{\sigma_i}_{r,j}-\mu^{\sigma_i}_j)(v^{\sigma_i}_{r,k}-\mu^{\sigma_i}_k)$.

In order to explain the steps **A**,**B** and **C**, we have to clarify the built of the laws $\mathcal{N}_{\sigma_i}$. Thus, we describe the algorithm employed in order to draw the resource sharing for a given sensor.

b) Drawing in $\mathcal{N}_s$: A random resource sharing for the sensor s, alloted to the zone $\mathcal{E}_z$, according to $\mathcal{N}_s(\Sigma_s, \mu_s)$, is obtained by the following algorithm:

- draw $y^s_1, ..., y^s_{|U_z|-1}$ in $]-1;1[$, such that $\forall k$ odd, $(y^s_k)^2 + (y^s_{k+1})^2 \leq 1$;
- $\forall k$ odd, compute $z^s_k = \dfrac{\sqrt{-2\ln((y^s_k)^2+(y^s_{k+1})^2)}}{(y^s_k)^2+(y^s_{k+1})^2}$;
  Set: $Y^s_k = z^s_k\ y^s_k$ and $Y^s_{k+1} = z^s_k\ y^s_{k+1}$.
  At that point, we have: $\forall k$, $Y^s_k \in \mathcal{N}(0,1)$.
- set $(x^s_1, ..., x^s_{|U_z|-1}) = H(Y^s_1, ..., Y^s_{|U_z|-1}) + \mu$, where H is the Cholesky decomposition of Σ_s.
  Hence: $\forall k$, $x^s_k \in \mathcal{N}(\Sigma_s, \mu_s)$.
- $\forall k$, set: $\theta^s_k = \exp(x^s_k)$.
  We obtain a vector $\theta^s = (\theta^s_1, ..., \theta^s_{|U_z|-1})$ where each θ^s_k corresponds to a line, such that: $\varphi^s_{k+1} = \theta^s_k\ \varphi^s_k$.
- The value φ^s_u is given by:

$$\varphi^s_u = \frac{\Phi_s}{1 + \left(\displaystyle\sum_{j=1}^{j=u-1}\prod_{k=j}^{k=u-1}\frac{1}{\theta_k}\right) + \left(\displaystyle\sum_{j=u}^{j=|U|-1}\prod_{k=u}^{k=j}\theta_k\right)} \tag{23}$$

IV. Detection Game

In the previous section, we described a search problem where a detailed prior probabilistic density on the location of the target was given; However, in the real world this prior is usually unknow. Instead, we often have a global idea on the position of the target, that can be modelled by a prior over the zones of the space of search. In this section we describe a detection problem where a prior on the location of the target over the zones is given. We solve a hierarchical detection problem where global level is the optimization of allotments of sensors to search zones (as in the preceding sections) and local level is the optimization of a min-max problem between the target and the searchers.

A. The min-max problem

We want to solve the problem described in (6). However, we do not know the prior α_u but only a prior on the location of the target over the zones: α_z, such that:

$$\sum_z \alpha_z = 1. \tag{24}$$

We denote $\alpha_{u|z}$ the probability that the target may be hidden into the unit E_u if it is known that the target is hidden into the zone $\mathcal{E}_z$. We have:

$$\forall \mathcal{E}_z,\ \sum_{u|E_u\in\mathcal{E}_z}\alpha_{u|z} = 1. \tag{25}$$

As a unit belongs to a unique search zone:

$$\forall E_u \in \mathcal{E}_z,\ \alpha_u = \alpha_z\alpha_{u|z} \tag{26}$$

Thus, we solve the following problem:

$$\min_m \sum_z \alpha_z \left(\min_{\varphi_s}\max_{\alpha}\sum_{u|E_u\in\mathcal{E}_z}\alpha_{u|z}\prod_{s|s\in m^{-1}(z)}\exp(-w^s_u\varphi_s(E_u))\right) \tag{27}$$

$$\text{s.t.:}\begin{cases}\mathcal{E}_z,\ \displaystyle\sum_{u|E_u\in\mathcal{E}_z}\varphi_s\ (E_u) - \Phi_s \leq 0\\ \forall\mathcal{E}_z,\ \forall\ E_u\in\mathcal{E}_z\ \ \varphi_s\ (E_u)\geq 0\end{cases}$$

Here again, we can split the problem into two interconnected levels:

- a global level: optimization of allotment of sensors to search zones
- a local level: optimization of both prior of the target $\alpha_{u|z}$ and of the repartition of resources of sensors alloted to a search zone.

We do not describe the optimization of global level, as we use the CE algorithm (CEg) described in section III-B.3. The evaluation of allotments for each zone is given by the result of the local optimization presented in the following section.

B. The local detection game

We consider the zone $\mathcal{E}_z$ where a set of sensors has been alloted by a draw of m. We introduce a min-max problem with two players. The first one is the target, which wants to hide into one of the units of the zone of search. The second one is the set of sensors alloted to the zone. The target wants to maximize the probability of non detection into the zone, while the sensors want to minimize the probability of non detection into the zone. We have the following optimization problem:

$$\max_{\alpha_{u|z}} \min_{\varphi_s} D(\varphi_s, \alpha_{u|z}) \tag{28}$$

where: $D(\varphi_s, \alpha_{u|z}) = \sum_{u|E_u \in \mathcal{E}_z} \alpha_{u|z} \prod_{s \in m^{-1}(z)} \exp(-w_u^s \varphi_s(E_u))$ (29)

$$\text{s.t.:} \begin{cases} \sum_{u|E_u \in \mathcal{E}_z} \varphi_s(E_u) = \Phi \\ \sum_{u|E_u \in \mathcal{E}_z} \alpha_{u|z} = 1 \\ \forall E_u \in \mathcal{E}_z, \ \alpha_{u|z} \geq 0 \\ \forall E_u \in \mathcal{E}_z, \ \varphi_s(E_u) \geq 0 \end{cases}$$

Thus, the optimization functional are :

- for the target: $\left[\max_{\alpha_{u|z}} \ \min_{\varphi_s} \ D(\varphi_s, \alpha_{u|z})\right]$;
- for the sensors: $\left[\min_{\varphi_s} \max_{\alpha_{u|z}} \ D(\varphi_s, \alpha_{u|z})\right]$.

In a two-person zero-sum game, if there is a couple $(\varphi_s^*, \alpha_{u|z}^*)$ such that:

$$D(\varphi_s^*, \alpha_{u|z}) \leq D(\varphi_s^*, \alpha_{u|z}^*) \leq D(\varphi_s, \alpha_{u|z}^*), \tag{30}$$

then:

$$\max_{\alpha_{u|z}} \min_{\varphi_s} D(\varphi_s, \alpha_{u|z}) = \min_{\varphi_s} \max_{\alpha_{u|z}} D(\varphi_s, \alpha_{u|z}) = D(\varphi_s^*, \alpha_{u|z}^*). \tag{31}$$

In the case where only one sensor is allotted to the search zone, the strategies of the target and of the sensors are given by the result of Nakai [7]:

$$\varphi_s^*(E_u) = \frac{\Phi_s}{w_u^s} \frac{1}{\sum_{u|E_u \in \mathcal{E}_z} \frac{1}{w_u^s}} \tag{32}$$

$$\alpha_{u|z}^* = \frac{1}{w_u^s} \frac{1}{\sum_{u|E_u \in \mathcal{E}_z} \frac{1}{w_u^s}} \tag{33}$$

The following deals with the optimization of the search problem when more than one sensors are allotted to the same search zone.

1) Optimization of the target strategy: We restrict the study to the units that belong to the zone $\mathcal{E}_z$. We consider that the optimal search effort, φ_s^* is given. The optimization problem for the target is then:

$$\min_{\alpha_{u|z}^*} -D(\varphi_s^*, \alpha_{u|z}) \tag{34}$$

$$\text{s.t.:} \begin{cases} \sum_u \alpha_{u|z} = 1 \\ \forall u, \ \alpha_{u|z} \geq 0 \end{cases}$$

A multiplier μ is associated to first constraint and a multiplier $\theta_u \geq 0$ is associated to each positivity constraint.The KKT optimality conditions give, $\forall u$:

$$-\prod_s \exp(-w_s^u \varphi_s(E_u)) + \mu - \theta_u = 0$$

$$\mu \left(\sum_u \alpha_{u|z} - 1\right) = 0$$

$$\theta_u \left(-\alpha_{u|z}\right) = 0$$

Two cases are identified:

- if $\alpha_{u|z}^* > 0$: $\prod_s \exp(-w_u^s \varphi_s^*(E_u)) = \mu$,
- if $\alpha_{u|z}^* = 0$: $-\prod_s \exp(-w_u^s \varphi_s^*(E_u)) < \mu$.

2) Optimization of the sensor strategy.: We consider that the optimal probability of the location of the target into the zone $\mathcal{E}_z$, $\alpha_{u|z}^*$, is given. The optimization problem for the target is then:

$$\min_{\varphi_s} D(\varphi_s, \alpha_{u|z}^*) \tag{35}$$

$$\text{s.t.:} \begin{cases} \forall s, \ \sum_u \varphi_s(E_u) = \Phi_s \\ \forall s, \ \forall u, \ \varphi_s(E_u) \geq 0 \end{cases}$$

A multiplier ν_s is associated to each constraint of the first bloc. For each positivity constraint, we associate a multiplier $\tau_u^s \geq 0$.The KKT optimality conditions give:

$$\forall s, \ \forall u, -w_s^u \alpha_{u|z}^* \exp(-w_u^s \varphi_s^*(E_u)) \prod_{s' \neq s} \exp(-w_u^{s'} \varphi_{s'}^*(E_u)) + \nu_s - \tau_u^s = 0;$$

$$\forall s, \ \nu_s \left(\varphi_s^*(E_u) - \Phi_s\right) = 0;$$

$$\forall s, \ \forall u, \ \tau_u^s \left(-\varphi_s^*(E_u)\right) = 0.$$

Two cases arise:

- if $\varphi_s^*(E_u) > 0$:

$$-w_s^u \alpha_{u|z}^* \exp(-w_u^s \varphi_s^*(E_u)) \prod_{s' \neq s} \exp(-w_u^{s'} \varphi_{s'}^*(E_u)) = \nu_s,$$

- if $\varphi_s^*(E_u) = 0$:

$$-w_s^u \alpha_{u|z}^* \exp(-w_u^s \varphi_s^*(E_u)) \prod_{s' \neq s} \exp(-w_u^{s'} \varphi_{s'}^*(E_u)) < \nu_s.$$

For a given sensor s, the dual functional is:

$$L(\varphi_s^*, \alpha_{.|z}) = \sum_{u|\varphi_u^* > 0} -\frac{\nu_s}{w_u^s} + \sum_{u|\varphi_u^* = 0} -\frac{\nu_s + \tau_u}{w_u^s} + \nu_s(\Phi_s - \sum_u \varphi_s^*(E_u)) - \sum_u \tau_u^s \varphi_s^*(E_u).$$

The maximization of L considering the τ_u^s implies that $\forall u, \ \tau_u^s = 0$.

So, at the optimum we have:

$\forall s, \ -w_s^u \alpha_{u|z}^* \prod_s \exp(-w_s^u \varphi_s^*(E_u)) + \nu_s = 0$ and $\nu_s > 0$.

Thus, if we consider the problem (34): $\forall u, \ \alpha_{u|z}^* > 0$.

3) Global conditions.: By considering both optimality conditions of the target strategy and of the sensors strategy, we determine a set of equivalent solutions such that:

$$\alpha^*_{u|z} = \frac{1}{\sum_{u'} \frac{1}{\sum_s w^s_u \prod_{s'} \exp(-w^{s'}_{u'} \varphi^*_{s'}(E_{u'})}} \tag{36}$$

and $\varphi^*_s(E_u) =$

$$\frac{1}{w^s_u} \left[\frac{\Phi_s - \sum_{u'} \left(\frac{\sum_{s' \neq s} -w^{s'}_{u'} \varphi_{s'}(E_{u'})}{w^s_{u'}} \right)}{\sum_{u'} \frac{1}{w^s_{u'}}} + \sum_{s' \neq s} -w^{s'}_u \varphi_{s'}(E_u) \right] \tag{37}$$

The optimization of the global min-max problem requires to find one solution in this set. We want to choose the solution that minimize the difference between the quantities of resource alloted to two units (see problem (38)):

$$\min_{\varphi_s} \sum_s \left(\sum_{u|E_u \in \mathcal{E}_z} \sum_{u' > u|E_{u'} \in \mathcal{E}_z} (\varphi_s(E_u) - \varphi_s(E_{u'}))^2 \right) \tag{38}$$

$$\text{s.t.:} \begin{cases} \forall s, \ \forall u, \ (37) \\ \forall s, \ \forall u, \ \varphi_s(E_u) \geq 0 \end{cases}$$

This problem is solved by the algorithm of Uzawa.

V. Two-target detection problem

In many real world problems, the sensors want to detect a set of target. Here we describe a detection problem when there are two targets to detect, say $t1$ an $t2$. We want to solve the following problem:

$$\min_{m, \varphi_s} \sum_z \sum_{u|E_u \in \mathcal{E}_z} \max_t \left[\alpha_{u,t} \prod_{s \in m^{-1}(z)} \exp(-w^s_{u,t} \varphi_s(E_u)) \right] \tag{39}$$

$$\text{s.t.:} \begin{cases} \forall z, \ \sum_{u|E_u \in \mathcal{E}_z} \varphi_s(E_u) = \Phi \\ \forall z, \ \forall E_u \in \mathcal{E}_z, \ \varphi_s(E_u) \geq 0 \end{cases}$$

As usual, we have a hierarchical problem with two levels. In order to solve it, we use the algorithm CEg an CEl introduced in section III-B.3 and III-C.2.a for the information search problem:

- for each allotment drawn by CEg
  - for each zone
    - find the best resource sharing for all the sensors alloted to the zone, according to the detection functional, using CEl
  - value the contribution of the zone to the optimization functional
- value the allotment

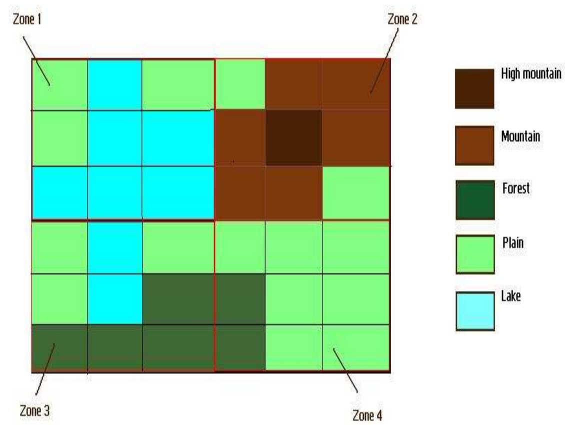
The only difference is that, for a zone $\mathcal{E}_z$, the draws of CEl are selected according to: $\sum_{u|E_u \in \mathcal{E}_z} \max_t \left[\alpha_{u,t} \prod_{s \in m^{-1}(z)} \exp(-w^s_{u,t} \varphi_s(E_u)) \right]$. In fact, for each unit of the zone we select the worst case, according to the detection of the two targets, that is to say: $\max \{ \alpha_{u,t1} \prod_{s \in m^{-1}(z)} \exp(-w^s_{u,t1} \varphi_s(E_u)), \ \alpha_{u,t2} \prod_{s \in m^{-1}(z)} \exp(-w^s_{u,t2} \varphi_s(E_u)) \}$.

VI. Results

In this section, we will present main results on the same example. In fact, this aims to point out differences of approaches.

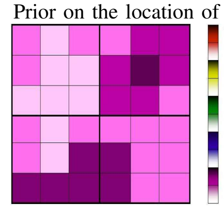
a) Data: We describe a space of search, divided into four search zones (see Fig. 1), each made of nine units. These units can be of four kind (forest, lake, plain, mountain and high mountain).

Fig. 1. The space of search.



The target is hidden in the space of search according to a prior. In this example, we consider that the targets prefers to hide in mountain, high mountain and forest (see Fig. 2).

Fig. 2. Prior on the location of the target.



This target is searched by two sensors. The first one has a good visibility over lake, mountain, and plain (see Fig. 3); while the second one has a good visibility over plain, forest and mountain (see Fig. 4).

b) Results for the detection and information search problems: In the case where the optimization functional is a detection functional (see section III-B), the optimal solution (see Fig. 5) is obtained in 30 iterations, in among 2 seconds. We can see that the sensors prefer to search in units where

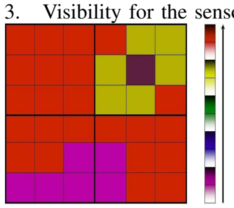


Fig. 3. Visibility for the sensor 1.

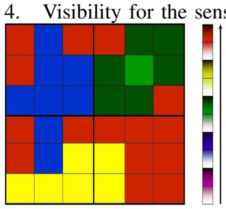


Fig. 4. Visibility for the sensor 2.

the target has most chances to be.

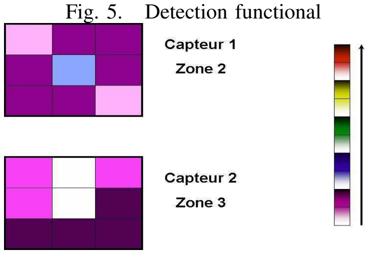


Fig. 5. Detection functional

In the case where an information functional (see section III-C) is employed, the optimal solution (see Fig. 6) is obtained in 30 iterations for an execution time of among 1 minute. This delay is due to the two levels of CE. The results of the information optimization are different from the results of the detection optimization. In fact the sensors will search a little more in the units where the target has few chances to be. That is especially the case for sensor 2 in zone 3: with the detection optimization, no resource was allocated for the search in the lake units.

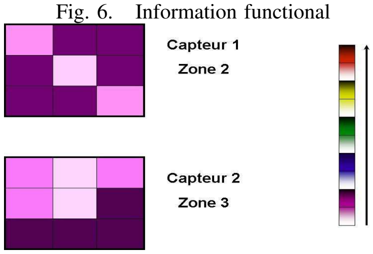


Fig. 6. Information functional

c) Result for the detection search game: For this example, we consider a detection search game (section IV) such that the target has equal chances to be in each zone. The optimal solution (see Fig. 7) is obtain in 162 iterations, for 2 seconds of execution time. We can see that the prefers to hide where sensors have the lower visibility.

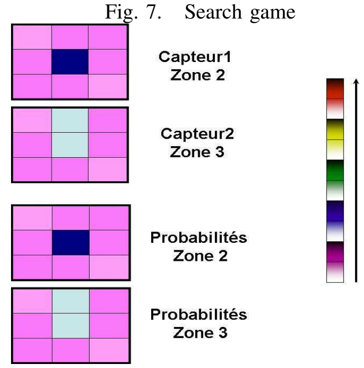


Fig. 7. Search game

d) Results for the two-targets detection problem: We consider the two-target detection problem (see section V) when the prior of both targets is setted to the same prior (see Fig. 2), and visibility of the sensors are the same, whatever the target (see Fig. 3 and Fig. 4). Of course, we obtain the same results that for the detection problem (see Fig. 5). The algorithm takes 26 iterations and among 10 minutes to find the optimal solution (due to the two levels of CE).

Tests have been made with success for more complicated instances of these search problems, with up to 20 searchs zones and 10 sensors. However, due to graphical constraints, we can not expose them here.

VII. Conclusion

We have introduced a federative method in order to optimize a variety of search problems. This approach is based on the use of the CE algorithm and on the hierarchical splitting of the problems studied. We are able to find optimal or even good solutions in reasonnable time consumption for big instances of search problems. However, the use of two levels of CE may imply high convergence time of algorithms. That is especially the case for the two-targets optimization. Our future work will aim to investigate further multi-targets search problems.

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