Available online at www.sciencedirect.com
ScienceDirect
Aerospace Science and Technology 11 (2007) 279–288
Aerospace Science and Technology
www.elsevier.com/locate/aescte

A branch-and-bound algorithm applied to optimal radar search pattern

Pierre Dodin [a,*], Pierre Minvielle [a], Jean-Pierre Le Cadre [b]

[a] *CEA CESTA, BP 2 33114 Le Barp, France*
[b] *IRISA/CNRS, Campus de Beaulieu, Rennes, France*

Received 31 May 2006; received in revised form 6 December 2006; accepted 8 December 2006

Available online 23 February 2007

Abstract

In the article, the radar acquisition problem, e.g. the determination of a directional energy allocation sequence, is studied. The radar search pattern goal is the detection of a moving target whose initial location is approximately known. We have turned towards the general search theory where the observer allocates indivisible search efforts while the target presence probability spreads due to its dynamics. A few years ago, a Branch and Bound algorithm was proposed to determine the optimal sequence for a conditionally deterministic target. This operational research algorithm supposes a negative exponential detection function and a one over N detection logic, meaning that the target is declared detected if it has been detected once over a horizon of N looks. We have applied it to a narrow-beam tracking radar attempting to acquire a ballistic target. Non-trivial search patterns, such as expanding-contracting spirals, are obtained.

Keywords: Optimization; Search theory; Radar search pattern; Beam scheduling

1. Introduction

Target acquisition is a common problem for narrow-beam tracking radars [1]. During the target acquisition stage, the radar must operate in a search mode over a limited volume of space. This limited volume corresponds to the prior uncertainty on the target location. For instance, it can be provided by a hand-over coming from other sensors or by an early-warning system for an ATBM (Anti-Tactical Ballistic Missile) defence [6]. Typically, a cued electronic beam scanning radar must seek the target in a 3-dimensional growing error basket. Therefore, the radar needs to determine a sequence of pulses or looks in successive appropriate directions. This sequence, determined over a fixed temporal horizon, should optimize the chances to detect the moving target, once or more times. There are classic acquisition search patterns for agile beam radars, such as rectangular raster scans [1], fence or ellipsoidal search patterns [6], which can be dedicated to various operational configurations. However, these semi-empirical patterns do not necessarily provide the best search. Other patterns could offer a higher probability of detection of the target or could require less resource or energy. In this article, we investigate this scheduling problem within the *search theory* framework.

Search theory [7,8,10] came into being during Wold War II with the work of B.O. Koopman and his colleagues [7] in the ASWORG (Antisubmarine Warfare Operations Research Group). Now a major discipline within the field of operations research, it treats the following problem: how best to search an object when the amount of searching efforts is limited and when only probabilities of the possible position of the object are given. Roughly speaking, the aim is to find a spatio-temporal repartition of search effort in order to optimize the probability of detection. In the initial framework of B.O. Koopman, the goal was to compute a continuous spatial repartition of the search efforts, the sum of efforts being bounded by a constant. Later, multi-scan search strategies have been developed for moving targets in the aim to maximize the probability of detecting them within a fixed amount of time. It is quite usual to model the target trajectory via a Markovian diffusion. No-

* Corresponding author.
E-mail addresses: pdodin@arthrovision.biz (P. Dodin), pierre.minvielle@cea.fr (P. Minvielle), lecadre@irisa.fr (J.-P. Le Cadre).

1270-9638/$ – see front matter
doi:10.1016/j.ast.2006.12.007

ticeably, this problem reveals to be formidably difficult as far as an iterative algorithm, called Forward And Backward (FAB) [2] provide a feasible way to solve it.

And yet, due to the continuous repartition hypothesis, a direct application of the FAB algorithm for the radar acquisition issue cannot lead to a search pattern determination. The only way to determine a search pattern is to consider that search efforts are indivisible. Practically, we have also to adapt the search cells to the radar beamwidth. Then, the search consists of successive cell search moves which depend on the target probability of presence. As a matter of fact, this problem has already been studied in the past. An integer search allocation starting from the FAB algorithm has been suggested [12]. Yet the most powerful technique [3,5] is a Branch and Bound ($B\&B$) approach. Generally, $B\&B$ methods are well-known exact optimization methods that consist in enumerating cleverly the solution space. Also called implicit enumeration methods, their aim is to divide the solution space in smaller and smaller subsets, most of them being eliminated by bounding. In Hohzaki and Iida work [5], the $B\&B$ approach was developed above all in the conditionally deterministic target dynamic case, i.e. when the target dynamic is conditioned by a set of (random) parameters. Practically, it is the good way [5] to represent the target location uncertainty by a beam of possible trajectories and the conditioning parameters may be the ballistic coefficient, the initial parameters of the reentry phase, etc.

The article purpose is the application of the Hohzaki B&B approach to the radar acquisition search pattern issue. It is illustrated in the following application, the acquisition of a ballistic target by a narrow-beam radar. The main assumption of [5] is effectively checked: the target dynamic is conditionally deterministic. We show how we adapt the B&B approach to the radar acquisition issue. Besides, we especially describe an efficient strategy search, i.e. an appropriate way to explore fastly the solution space tree; which is a determining factor in the approach efficiency.

The article is organized as follows. In Section 2, we present essential and appropriate elements of the search theory framework developed by Hohzaki and Iida [5]. Among the main results, we especially stress the upper bound existence induced by a relaxation method. In Section 3, we describe the intuitive heuristic tree search that is part of the B&B algorithm. It offers an explicit way to explore the tree made by the solution space. Finally, in Section 4, we describe the application to the determination of a radar search pattern for ballistic target acquisition. Various results emphasize the interest of this approach.

2. Search pattern constrained to a beam of trajectories

At first, let us say a few words about the acquisition problem with the intention of making this current section clearer and more concrete. Notice that it will be further detailed in Section 4. Assume that the search space can be divided into a certain number of cells, whose sizes are linked to the radar main beamwidth. Notice that each cell of the angular 2D grid may not necessarily be provided with the same elementary detection. Indeed, each cell may be dedicated at a given time to a different area in range of the 3D real space and the radar detection capability is classically a function decreasing with range. In straightforward terms, the radar goal is to perform a sequence of scans or looks in order to maximize an acquisition probability, which is a global detection function. This is undoubtedly an optimization issue. On the other hand, the dynamic of a non-maneuvering ballistic target is quasi-Keplerian before the re-entry. Hence, the conditionally deterministic assumption is checked. Moreover, for a cued radar using a hand-over information, the initial position and the initial speed are known with a bounded precision.

Actually, the optimization problem of the radar search pattern perfectly fits with the formalism of Hohzaki and Iida [5]. Their formalism, based on a path formulation, was first suggested by [3]. In this approach, each path is a possible trajectory for the target, with an assigned probability. The paths go through a certain number of cells in the observation space. Concerning the searcher, a decision variable indicates which cell is observed at each step of time. The optimization goal is to maximize the probability of detecting the target at least once. To solve the combinatorial problem, Hohzaki and Iida develop a B&B search method. Note that this kind of method was beforehand suggested in [11,12]. In [5], the authors establish all the required elements and especially develop a duality-based method able to solve efficiently a relaxed version of the problem. It is subsequently massively used in their application of the B&B technique.

Next, we introduce the search theory framework, developed by Hohzaki and Iida. The interest of a B&B algorithm is tightly related to the accuracy of the bound in use. This bound is calculated via a relaxation of integrity constraints. This part is rather intricate since it involves multiperiod optimization. So, it will be carefully considered in the next section. We refer to [5] for the full theoretical developments.

2.1. Search theory framework

We introduce now the fundamentals of the search theory formalism given by Hohzaki and Iida [5]. Notice that for clarity we have chosen to simplify what is not relevant to our context. Thus, for the sake of completeness, simplified proofs of the main steps are provided. So, notations of the reference paper (see [5]) will be adopted.

Assuming that the target moves among a finite number of cells $\mathbf{K} = 1, \ldots, K$ in discrete time $\mathbf{t} = 1, \ldots, T$, let us introduce the following points:

- A path ω is represented by a sequence of cells $\{\omega(t)\}_{t=1}^{T}$, while $\omega(t)$ is the cell in which the target is located at time t.
- The target moves on a path ω chosen with probability $\pi(\omega)$ among the finite set of possible paths Ω.
- A searcher knows the probability law of the target paths in advance and moves along cells $\mathbf{K}$ looking for the target. From a time period τ to period $\tau + 1$, the searcher is al-

lowed to move from a cell i to one of the adjacent cells $I(i)$ and to examine only the cell where he is.[1]

- The conditional probability that the searcher detects the target by looking at cell i at time τ, given that the target is there, is $p(\varphi(i,\tau)) = 1 - \exp(-\alpha_i \varphi(i,\tau))$, where α_i is a visibility parameter in cell i and $\varphi(i,t)$ is the search effort in cell i at time t. Notice that this exponential detection probability was firstly introduced in Koopman seminal work. Here, the decision variable $\varphi(i,t)$ is equal to one if the searcher looks at time t into cell i, otherwise it is equal to zero ($\{0,1\}$ search effort).

The searcher wants to find a search pattern, i.e. a sequence of binary decision variables (or search efforts), which maximizes the probability of detecting the target at least one time. At first, let us formulate in equation (1) the probability of detecting the target *at least one time* during period $[1,t]$:

$$P_1^t = 1 - \sum_{\omega \in \Omega} \pi(\omega) \exp\left[-\sum_{\tau=1}^{t} \alpha_{\omega(\tau)} \varphi\big(\omega(\tau),\tau\big)\right]. \tag{1}$$

The global criterion (2), proposed by Hohzaki and Iida, is:

$$R_1^T = P_1^T(\varphi,\pi). \tag{2}$$

Consequently, the problem P_0 can be formulated in the following way:[2]

$$\mathsf{P}_0 \quad \left| \begin{array}{l} \max\limits_{\varphi} R_1^T(\varphi) \quad \text{subject to,} \\ \varphi(s,0) = 1; \displaystyle\sum_{i=1}^{K} \varphi(i,t) = 1, \quad t \in \mathbf{T}, \\ \varphi(i,t) \leqslant \displaystyle\sum_{j \in I(i)} \varphi(j,t+1), \quad i \in \mathbf{K},\ t \in [0, T-1], \\ \varphi(i,t) \in \{0,1\}, \quad i \in \mathbf{K},\ t \in \mathbf{T}. \end{array} \right. \tag{3}$$

It is an integer optimization problem, with a concave and separable criterion. Unfortunately, such an integer number problem is known to be tough to solve. Hohzaki and Iida [5] chose to apply the Branch and Bound method. This optimization technique is well-known in the integer linear programming field. It enumerates cleverly the solution space, using separation and evaluation steps. In order to bring the method into play efficiently, it is necessary here to use an optimal value of a relaxed problem generated from the original problem $P0$ as the upper bound of the objective function. In the following subsections, we present concisely the B&B method and the associated issues.

2.2. A Branch and Bound algorithm

Branch and Bound methods are exact optimization methods that consist in enumerating cleverly the solution space. Also called implicit enumeration methods, they aim is to divide the solution space in smaller and smaller subsets, most of them being eliminated by bound calculus before being constructed explicitly. B&B methods can be applied to NP-hard problems of common size where they are more effective than exhaustive enumeration. Even if B&B methods can be quite different, they all resort to the three following components: a branching rule which separates and partitions the solutions, an evaluation or bounding function which is the key factor to avoid the aforementioned exhaustive enumeration and eventually the search strategy which defines the next node to separate and the related separation decisions.

Let us briefly describe the B&B method applied to the search pattern program $P0$. While the tree of all possibilities is progressively being established, the method advances in the tree diagram by forcing an integer constraint as it comes across each tree level. The choice of the integer constraint is suggested by a relaxation program, i.e. a derived program for which various integer constraints have been put out. For each integer constraint to force, the relaxation is performed over a time horizon, from the next arborescence level to the end of the tree. Once the end is reached with this iterative procedure, a bound is obtained (to be reset each time a better solution is found). At this step, a heuristic search is launched, which goes back and forth between alternative paths. At that time, it is important to stress that the method makes use of the fact that it is not worth going forward if the relaxation is lower that the bound. Indeed, each path obtained with the same prior integer constraint will have in this case a lower value than the bound itself.

The B&B method is indeed a powerful technique, but it requires several conditions. First, one should know how to split the criterion, the first part of it being integer and the second being real. Moreover, one has to make sure that the criterion, split in this way, is congruent with the evolution of probabilities. Indeed, since new *a posteriori* probabilities (knowing the first observations) are to appear, the criterion need to be consistent with them. Finally, one must ensure that the relaxed criterion is greater than the global integer criterion, otherwise there is no justification of the Branch and Bound procedure.

2.3. Criterion splitting and relaxation

In order to use the B&B method, it is necessary to define relaxed problems for fathoming branches. Of course, such relaxation must lead to convenient bounds and (far) easier optimization problems. More specifically, relaxing integer constraints will allow us to consider optimization of differentiable functionals. Even with this simplification, the problem remains difficult since it is a multiperiod one. Again, it will be the FAB algorithm which will be the workhorse.

First, it is necessary to define the *a posteriori* probability $\Lambda_t \pi(\omega)$ that the target takes path ω and remains *undetected* up to the period t, i.e.:

$$\Lambda_t \pi(\omega) = \frac{\pi(\omega) \exp\big(-\sum_{\xi=1}^{t} \alpha_{\omega(\xi)} \varphi(\omega(\xi),\xi)\big)}{1 - P_1^t(\varphi,\pi)}. \tag{4}$$

Note that $\exp(-\sum_{\xi=1}^{t} \alpha_{\omega(\xi)} \varphi(\omega(\xi),\xi))$ is simply the probability that a target following a path ω remains undetected up to

[1] The “adjacent move” constraint is optional and will not be used in the acquisition application.

[2] $I(i)$: adjacent cells.

the period t; while the denominator is simply a normalization factor so as to ensure the condition $\sum_{\omega\in\Omega} \Lambda_t\pi(\omega) = 1$.

As said previously, the B&B approach needs to know how to split the criterion. The first part consists of integer decision variables forced by the beginning of the chosen path, while the second part consists of decision variables for which integer constraints are relaxed. Furthermore, if the splitting decision happens at time $t+1$, we need a formulation of the search problem on the interval $[t+1, T]$. A simple adaptation of the detection probability on $[t+1, T]$ is:

$$P_{t+1}^T = 1 - \sum_{\omega\in\Omega} \pi(\omega) \exp\left(-\sum_{\xi=t+1}^{T} \alpha_{\omega(\xi)}\varphi\big(\omega(\xi), \xi\big)\right). \tag{5}$$

Therefore, the global reward on the interval $[t+1, T]$ is:

$$R_{t+1}^T = P_{t+1}^T(\varphi, \pi). \tag{6}$$

Let $\widetilde{R}_{t+1}^T(\varphi, \Lambda_t\pi)$ be a solution of the P_1 problem, which is a relaxation of P_0 (where integrity constraints are relaxed) on $[t+1, T]$. The formulation of $P1$ is (with dual variables in brackets):

$$\mathsf{P}_1 \quad \left|\begin{array}{l} \max\limits_{\varphi} P_{t+1}^T(\varphi,\pi) \quad \text{subject to:} \\ \varphi(s,0) = 1, \\ \sum\limits_{i=1}^{K} \varphi(i,\tau) = 1, \\ \quad \forall\tau \in [t+1, T] \text{ (dual variables: } \lambda(\tau)), \\ \varphi(i,t) \geqslant 0, \\ \quad \forall i \in \mathbf{K},\ \forall t \in [t+1, T] \text{ (dual variables: } \nu(i,\tau)). \end{array}\right. \tag{7}$$

The following lemmas are fundamentals (see [5] for proof) for partitioning.

Lemma 1. *Partitioning the expected reward*

$$R_1^T(\varphi,\pi) = R_1^t(\varphi,\pi) + \big(1 - P_1^t(\varphi,\pi)\big) R_{t+1}^T(\varphi, \Lambda_t\pi). \tag{8}$$

Proof. First, using Eq. (4) and the exponential properties, we obtain:

$$R_{t+1}^T(\varphi, \Lambda_t\pi) = 1 - \frac{1}{1 - P_1^t(\varphi,\pi)} \sum_{\omega\in\Omega} \pi(\omega) \times \underbrace{\exp\left[-\sum_{\tau=1}^{t} \alpha_{\omega(\tau)}\varphi\big(\omega(\tau)\big)\right] \exp\left[-\sum_{\tau=t+1}^{T} \alpha_{\omega(\tau)}\varphi\big(\omega(\tau)\big)\right]}_{\exp[-\sum_{\tau=1}^{T}\alpha_{\omega(\tau)}\varphi(\omega(\tau))]}. \tag{9}$$

Thus we have:

$$\begin{aligned} &P_1^t(\varphi,\pi) + \big(1 - P_1^t(\varphi,\pi)\big) R_{t+1}^T(\varphi, \Lambda_t\pi) \\ &\quad = P_1^t(\varphi,\pi) + \left(\frac{1 - P_1^t(\varphi,\pi)}{1 - P_1^t(\varphi,\pi)}\right)\left[1 - P_1^t(\varphi,\pi) - \sum_{\omega\in\Omega}\pi(\omega)\exp\left(-\sum_{\tau=1}^{T}\alpha_{\omega(\tau)}\varphi\big(\omega(\tau)\big)\right)\right] \\ &\quad = R_1^T(\varphi,\pi). \qquad \square \end{aligned} \tag{10}$$

Lemma 2 *(The partitioned bound). The following inequality holds true*:

$$R_1^T(\varphi,\pi) \leqslant R_1^t(\varphi,\pi) + \big(1 - P_1^t(\varphi,\pi)\big) \widetilde{R}_{t+1}^T(\varphi, \Lambda_t\pi). \tag{11}$$

For a proof, it is sufficient to note that $R_{t+1}^T(\varphi, \Lambda_t\pi) \leqslant \widetilde{R}_{t+1}^T(\varphi, \Lambda_t\pi)$, which is evident. Indeed, the interest of these lemmas is to show that the criterion has a compatible structure with the *a posteriori* probabilities, which results in an upper bound when the variables of the second part are relaxed. This bound will be used intensively for the *B&B* algorithm. Let us now consider its calculation. The optimization problem we have now to deal with is:

$$\mathsf{P}_1' \quad \left|\begin{array}{l} \max\limits_{\phi} P_{t+1}^T(\varphi,\pi), \quad \text{subject to:} \\ \sum\limits_{i=1}^{K} \varphi(i,\tau) = 1, \quad \tau = t+1,\dots,T, \\ \varphi(i,\tau) \geqslant 0, \quad i = 1,\dots,K,\ \tau = t+1,\dots,T. \end{array}\right. \tag{12}$$

This is now a classical optimization problem, for a differentiable concave functional on a convex domain. A natural way to solve it is to use duality. The Lagrangian (denoted $\mathsf{L}(\varphi)$) of the above problem is:

$$\begin{aligned} \mathsf{L}(\varphi) = {} & \widetilde{R}_{t+1}^T(\varphi, \Lambda_t\pi) \\ & + \sum_{\tau=t+1}^{T} \lambda(\tau)\left(1 - \sum_{j=1}^{K}\varphi(j,\tau)\right) \\ & + \sum_{j=1}^{K}\sum_{\tau=t+1}^{T} \nu(j,\tau)\varphi(j,\tau). \end{aligned} \tag{13}$$

Let us denote $\{\varphi^*(i,\tau)\}_{(i,\tau)}$, the solution of the primal problem P_1' (see Eq. (12)). Then, Karush–Kuhn–Tucker (KKT) conditions yield:

$$\begin{cases} \dfrac{\partial\mathsf{L}(\varphi)}{\partial\varphi(i,\tau)}\big(\varphi^*(i,\tau)\big) = 0, \quad \forall i,\ \forall\tau, \\ \nu(i,\tau)\varphi^*(i,\tau) = 0, \quad \nu(i,\tau) \geqslant 0 \quad \forall i,\ \forall\tau. \end{cases} \tag{14}$$

Since the Lagrangian is a differentiable functional, we have:

$$\frac{\partial\mathsf{L}(\varphi)}{\partial\varphi(i,\tau)} = B_{i\tau}\exp\big[-\alpha_i\varphi(i,\tau)\big] - \lambda(\tau) + \nu(i,\tau),$$

with:

$$B_{(i,\tau)} = \alpha_i \sum_{\omega\in\Omega} \Lambda_t\pi(\omega)\delta_{i\omega(\tau)} \times \exp\left[-\sum_{\xi=t+1,\xi\neq\tau}^{T} \alpha_{\omega(\xi)}\varphi\big(\omega(\xi),\xi\big)\right]. \tag{15}$$

Notice that the Kronecker symbol $\delta_{i\omega(\tau)}$ (see Eq. (15)) there indicates that the calculation of the partial derivative is restricted to the target paths passing through cell i at step τ. Thus, at the optimum, KKT conditions yield for each time period τ:

$$\left|\begin{array}{l} \varphi^*(i,\tau) > 0 \Rightarrow \lambda(\tau) = B_{(i,\tau)}\exp\big[-\alpha_i\varphi^*(i,\tau)\big], \\ \varphi^*(i,\tau) = 0 \\ \quad \Rightarrow B_{(i,\tau)}\exp\big[-\alpha_i\varphi^*(i,\tau)\big] = \lambda(\tau) - \underbrace{\nu(i,\tau)}_{\geqslant 0} \leqslant \lambda(\tau). \end{array}\right. \tag{16}$$

Considering the strictly decreasing property of the $\exp(-\alpha_i x)$ function, we have:

$$\varphi^*(i,\tau) > 0 \Rightarrow \varphi^*(i,\tau) = \frac{1}{\alpha_i}\ln\left(\frac{B_{(i,\tau)}}{\lambda(\tau)}\right) \triangleq \rho^{-1}_{(i,\tau)}\big(\lambda(\tau)\big),$$
$$\varphi^*(i,\tau) = 0 \Rightarrow \varphi^*(i,\tau) \leqslant \rho^{-1}_{(i,\tau)}\big(\lambda(\tau)\big), \tag{17}$$

where:

$$\rho_{(i,\tau)}(x) = B_{(i,\tau)}\exp(-\alpha_i x).$$

The above condition makes sense since the objective functional is separable which implies that the search efforts at period τ (i.e. $\varphi_{(i,\tau)}$) have no effect on the value of $B_{(i,\tau)}$. Thus, it has been proved that the optimal distribution of search effort at period τ is defined by:

$$\varphi^*(i,\tau) = \big[\rho^{-1}_{(i,\tau)}\big(\lambda(\tau)\big)\big]_+, \tag{18}$$

with:

$$[x]_+ = x \quad \text{if } x \geqslant 0, \qquad [x]_+ = 0 \quad \text{if } x < 0.$$

It remains to determine the convenient values of the dual parameters $\lambda(\tau)$ $(t+1 \leqslant \tau \leqslant T)$. To that aim, the following property is instrumental.

Proposition 3. *For every period $\tau \in \{t+1, T\}$, there exists a finite Lagrange multiplier $\lambda^*(\tau)$ such that*:

$$\sum_{i=1}^{K}\big[\rho^{-1}_{(i,\tau)}\big(\lambda^*(\tau)\big)\big]_+ = 1 \quad \textit{and}$$
$$\varphi^*(i,\tau) = \big[\rho^{-1}_{(i,\tau)}\big(\lambda^*(\tau)\big)\big]_+.$$

Proof. This means that all the constraints are satisfied for this value of the dual parameter $(\lambda^*(\tau))$. Taking implicitly[3] into account the positivity of the search efforts $\varphi_{(i,\tau)}$ leads to consider the following simplified Lagrangian, i.e.:

$$\left|\begin{aligned} &\mathsf{L}(\lambda(\tau),\varphi) = \widetilde{R}^T_{t+1}(\varphi, \Lambda_t\pi) \\ &\qquad + \sum_{\tau=t+1}^{T}\lambda(\tau)\left(1 - \sum_{j=1}^{K}\varphi(j,\tau)\right), \\ &\varphi(i,\tau) \geqslant 0 \quad \forall i, \forall \tau. \end{aligned}\right. \tag{19}$$

Thanks to the previous results, the dual functional $\psi(\lambda(\tau)) \triangleq \max_\varphi \mathsf{L}(\lambda(\tau),\varphi)$ is:

$$\begin{cases} \psi\big(\lambda(\tau)\big) = \widetilde{R}^T_{t+1}\big(\varphi^*\big(\lambda(\tau)\big), \Lambda_t\pi\big) \\ \qquad + \displaystyle\sum_{\tau=t+1}^{T}\lambda(\tau)\left(1 - \sum_{j=1}^{K}\varphi^*\big(j,\lambda(\tau)\big)\right), \\ \varphi^*\big(i,\lambda(\tau)\big) = \big[\rho^{-1}_{(i,\tau)}\big(\lambda(\tau)\big)\big]_+. \end{cases} \tag{20}$$

This is the dual functional and its great advantage is that our problem is reduced to the minimization of a *monodimensional* functional. Furthermore, classical optimization results assert that it is convex. From Eq. (17), we note that $\lambda(\tau)$ is necessarily positive. When $\lambda(\tau)$ tends toward zero, then $\varphi^*(i,\tau)$ tends toward infinity (see Eq. (16)). But it is easily shown that $\lambda(\tau)\sum_{i=1}^{K}\varphi^*(i,\tau)$ tends toward zero (see Eq. (16) and recall that $\lim_{x\to\infty}\ln x/x = 0$), hence:

$$\lim_{\lambda(\tau)\to 0}\psi\big(\lambda(\tau)\big) = 1 - \sum_{\omega\in\Omega}\Lambda_t\pi(\omega)\delta_{i\omega(\tau)} \times \exp\left[-\sum_{\xi=t+1,\xi\neq\tau}^{T}\alpha_{\omega(\xi)}\varphi\big(\omega(\xi),\xi\big)\right]. \tag{21}$$

So, $\lim_{\lambda(\tau)\to 0}\psi(\lambda(\tau)) > 0$, while

$$\lim_{\lambda(\tau)\to 0}\sum_{j=1}^{K}\varphi^*\big(j,\lambda(\tau)\big) = \infty.$$

On the same way, there is a maximal value for $\lambda(\tau)$ (say $\bar{\lambda}_\tau$). Examining Eq. (16), we see that $\bar{\lambda}_\tau$ is bounded above by $\max_j B_{(j,\tau)}$. Furthermore, we have (see Eq. (16)):

$$\sum_{j\in K}\varphi^*(j,\bar{\lambda}_\tau) = 0. \tag{22}$$

The function $\rho^{-1}_{(i,\tau)}$ being continuous, it is inferred that there is a unique value of the parameter $\lambda(\tau)$ (say $\lambda^*(\tau)$) such that:

$$\sum_{i=1}^{K}\big[\rho^{-1}_{(i,\tau)}\big(\lambda^*(\tau)\big)\big]_+ = 1, \tag{23}$$

which ends the proof. □

Practically, $\lambda^*(\tau)$ is found by a dichotomy search among the interval $[0, \bar{\lambda}(\tau)]$ and is the solution of the dual optimization problem:

$$\mathsf{D}_\tau \quad \begin{cases} \lambda^*(\tau) = \arg\min_{\lambda(\tau)}\psi\big(\lambda(\tau)\big), \\ \psi\big(\lambda(\tau)\big) \text{ given by Eq. (20).} \end{cases} \tag{24}$$

From classical duality results, we know that for the value of $\lambda(\tau)$ minimizing $\psi(\lambda(\tau))$ the equality constraints (Eq. (12)) are also satisfied.

Now, we must stress that the primal problem P_1' (see Eq. (12)) corresponds to a multiperiod optimization. The only feasible way for solving is to use a Forward And Backward optimization procedure [2,4], described in Fig. 1 and summarized below:

- Forward: for each $\tau = t+1, \to, T$, solve the dual problem $\mathsf{D}_\tau \to \varphi^*(i,\tau)$,
- Forward: update the $B_{i,\tau+1}$, solve the dual problem $\mathsf{D}_{\tau+1}$,
- Forward: up to $\tau = T$.
- Backward: go back to $t+1$ and reiterate the Forward process, up to convergence.

Consequently, the first iteration of the FAB algorithm consists in starting from a random feasible allocation φ, solving the dual problem (see Eq. (23)) at step $\tau = t+1$ (the other periods being unaltered), taking account of the result, and then again

[3] This means that the Lagrangian is defined on the (convex) domain of constraints $\varphi_{i,\tau} \geqslant 0, \forall i$.

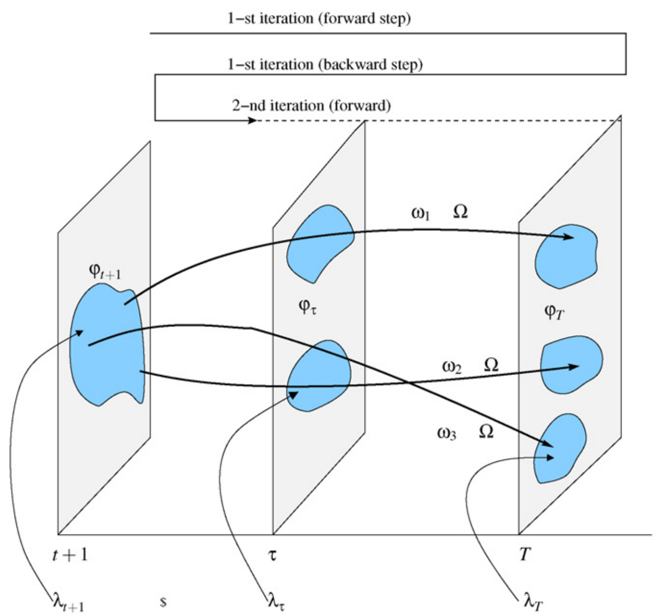

Fig. 1. Principle of the FAB procedure.

solving in the same way at step $\tau = t + 2$, etc. This leads to a closed loop iterative process: once step T is reached, we start over again backward until improvement remains sufficiently great. For convergence analysis of the procedure, we refer to [2,4].

3. The solution tree exploration

This section is dedicated to the search strategy, meaning the way the algorithm goes round the solution tree, defines the next node to separate and the related separation decisions. In B&B methods, the search strategy is known to be decisive in the global efficiency. We present here a heuristic search strategy specifically designed for the former section problem. This explicit search makes choices in order to obtain quickly an inferior integer bound as tight as possible. In other words, the heuristic distinguishes itself from an exhaustive search and attempts to explore as less as possible the solution tree, taking advantage of the bound information. It is important to emphasize that, although the search strategy is a heuristic, the B&B method provides in the end the optimal solution.

We choose here to show the search strategy principles by illustrating its behavior in a basic example. After a first presentation without using the bound, we show how the heuristic search manages to use the bound to efficiently explore the tree of all possibilities.

3.1. A depth-first approach

On the left part of Fig. 2, the tree of all the possibilities is represented in case of a search among 3 steps of time and 3 possibilities (i.e. 3 search cells). On the right part, we show the algorithm initialization. For each step of time, a sort is performed among all integer decisions for the current period, by taking into account the past decisions and by relaxing the future decisions. In this example, this leads to 3 sorts. The sort is represented in the decreasing direction (according to the criterion), at each step of time, by the numbers 1, 2 and 3. The first solution is obtained by making the best choice at each step of time. Next, in this depth-first approach, the tree is going to be explored by successive backtracks.

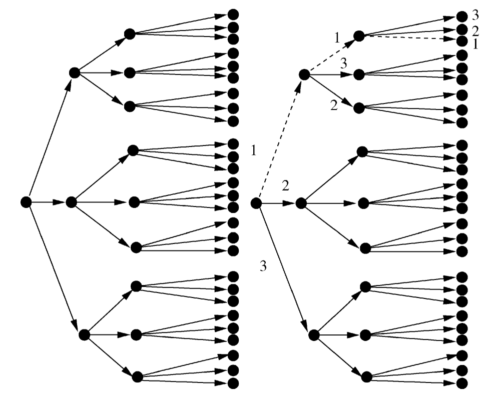

Fig. 2. Step 0 and 1 of the exploration algorithm.

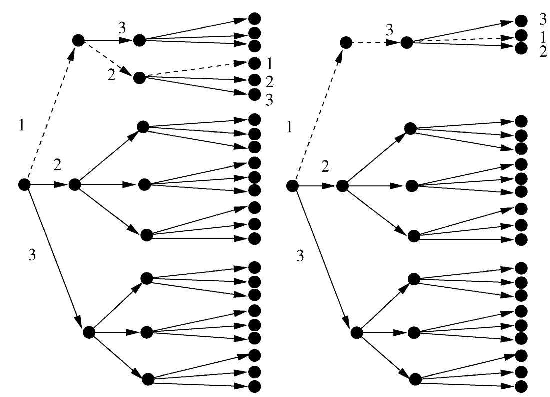

Fig. 3. Step 2 and 3 of the exploration algorithm.

On Fig. 3, we show what happens next. On the left tree, there is no alternative at time 3. The strategy is to go back to time 2 and to take the second alternative. Then, the heuristic search uses again the depth-first procedure previously described for the initialization. It finds the best solution that takes these new constraints into account. On the right tree, it carries on with the third alternative. On Fig. 4, since there is no possible improvement at time 2 and 3, the search strategy goes back further until time 1. There, it takes the second best alternative and starts over the exploration again from this step of time, using the new constraints, etc.

3.2. The use of the bound

Let us start again the basic example to show how the search strategy can take advantage of the bound. At time 1, the first integer solution is obtained. It provides the first bound. Each integral solution of bigger value is going to replace the bound. At time 2, on the left part of Fig. 3, the best alternative is still chosen, but this time, before going further, the algorithm initiates a test to know if the relaxation problem gives a higher result than the current bound. If it does so, the search can be resumed. Otherwise, it is not worth carrying on forward. Indeed,

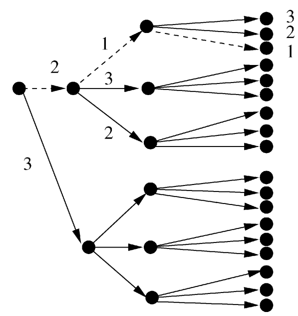

Fig. 4. Step 4 of the exploration algorithm.

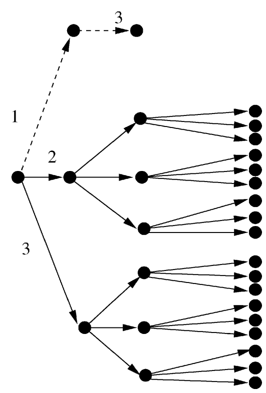

Fig. 5. Step 3 of the exploration algorithm using the bound.

if the obtained relaxation gives a result lower than the bound, there is no possibility to get a better integer solution than the solution whose value is the bound. Here, the obtained relaxation has a larger value than the bound and consequently a forward exploration is launched.

Fig. 5 represents the step 3: during the computation of the third alternative, the relaxation has given a result lower than the bound, therefore no search is launched. On Fig. 6, during the second alternative computation at time 1, once again the relaxation is lower than the bound and no exploration is launched.

4. Application to ballistic target acquisition

In this section, we describe the application of the B&B method to the determination of the optimal search pattern for the radar acquisition of a ballistic target. The assessment scenario is described in Fig. 7. Similarly to [9], a cued narrow-beam radar is located near the objective. It attempts to acquire an Inter Continental Ballistic Missile (ICBM) in the aim to track it during its re-entry. At one point, the radar gets a hand-over coming from upper sensors, such as the early warning system [6]. Typically, the hand-over consists of an estimation of the target position and velocity. The radar may need to update the information and predict the target trajectory at the time of acquisition. In Fig. 7, the 3-dimensional error basket represents the (3σ) error ellipsoid envelop. Basically, the incertitude on the target position is growing, due to the incertitude on the speed. To acquire the target, the radar beam scheduling process must lead to a sequence of looks or scans in appropriate directions. In a very close context, [6] briefly mention the "ellipsoidal scan",

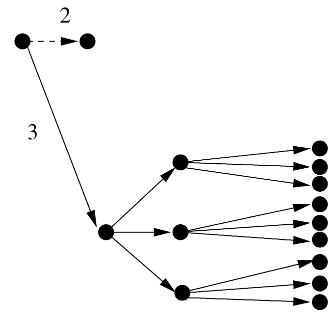

Fig. 6. Step 4 of the exploration algorithm using the bound.

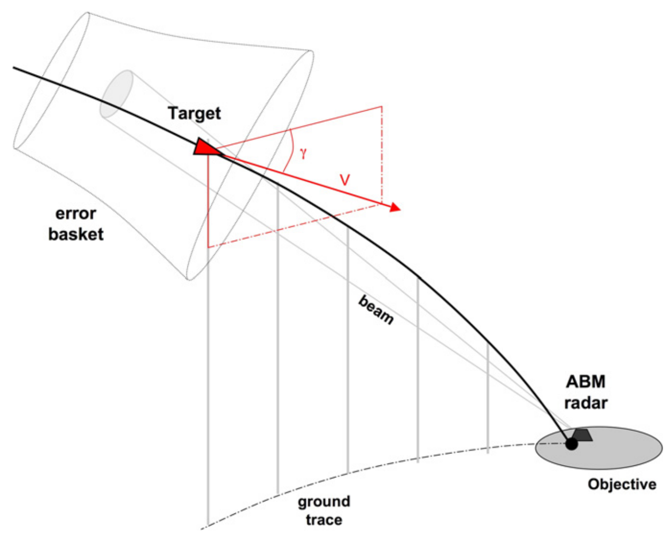

Fig. 7. The acquisition scenario.

where the successive overlapped beams are uniformly arranged to cover the error ellipsoid projected onto the radar angular coordinate system.

After the enumeration of the main scenario assumptions, we present the required modeling adaptations to fit to the developed search pattern optimization approach. Finally, we present the obtained results.

4.1. The acquisition scenario

In this subsection, we enumerate the main simple simulation assumptions about the target and the radar. One must notice that the following values are arbitrary, most of them coming from [9].

4.1.1. The ICBM target

- Long-range trajectory with [$\gamma = 25^\circ - V = 6800\ \mathrm{m\,s^{-1}}$] at 120 km altitude (see Fig. 7).
- Simulated dynamics (ballistic with drag [9]): gravitation (spherical Earth), atmospheric density (exponential).
- Re-entry deceleration: constant ballistic coefficient ($\beta = 5.10^{-4}\ \mathrm{m^2\,kg^{-1}}$).
- Target RCS (Radar Cross Section): $\sigma = 1.5\ \mathrm{m^2}$ (in the radar band).

4.1.2. The cued narrow-beam radar

- Situated at the impact point (see Fig. 7).
- Target acquisition below 120 km altitude (time 0) with a 1 Hz scan rate.
- Main lobe beamwidth: 0.6° in both azimuth and elevation.
- Detection range: compatible with the target acquisition at 120 km altitude.
- Acquisition requirement: at least one detection over the scans.

4.1.3. The hand-over features

- Position incertitude (when the target is at 120 km altitude): 3000 m (1σ).
- Speed incertitude (when the target is at 120 km altitude): 3000 m (1σ).

4.2. The modeling adaptation

4.2.1. The beam of ballistic trajectories

As described in Section 2, the trajectory incertitude is modeled by a set of possible trajectories, each of them being compatible with the hand-over information. Practically, it consists in generating random initial conditions at time 0, according to the position and speed ellipsoidal incertitude. Then, the set of possible trajectories is created by using an ODE solver (cf. ballistic propagator [9]).

4.2.2. The detection function

Let us recall the elementary detection criterion: $p_{\omega(\tau)} = 1 - \exp(-\alpha_{\omega(\tau)})$, if the searcher looks inside the cell $\omega(\tau)$ at the current time τ. We assume here that the visibility parameter is straightforwardly linked to the Signal to Noise Ratio (*SNR*), i.e. $\alpha_{\omega(\tau)} = SNR$. This simplification, compared to more complex expressions [1], is reasonable in the current context since the *SNR* does not vary much.

For a monostatic radar, the link between *SNR* and the range R, as well as other parameters (target RCS σ, radar power, pulse duration, etc.), can be expressed in simple terms by the so-called radar equation (Ref. [6]):

$$SNR = \frac{\sigma \Omega_0}{R^4} \tag{25}$$

where R is the target range to the radar and Ω_0 is a synthetic term taking into account all the other terms. As told previously about the detection range, Ω_0 is beforehand tuned in the scenario simulation so that the detection probability is good enough at acquisition time.

4.2.3. Target and search spaces

In Fig. 8, we present the space cutting in (azimuth, elevation, range) along a spherical window. The beam is represented in blue and is characterized by its direction, i.e. azimuth and elevation values.[4] To simplify, we assume here that the beam is squared with no overlap.

[4] For colours in figures see the web version of this article.

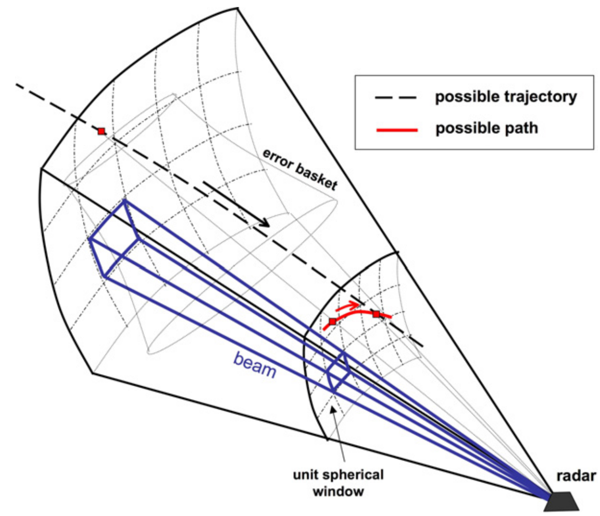

Fig. 8. The radar search cells.

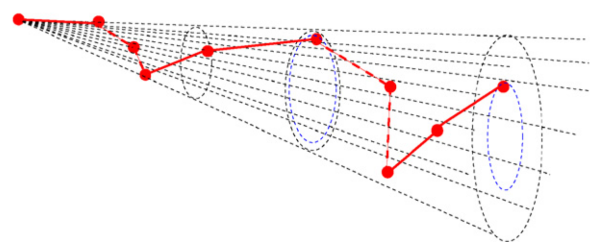

Fig. 9. The optimal search pattern.

The target moves in the 3D space. The visibility function can be computed for each path in function of the range, azimuth and elevation. The search space is 2D, in azimuth and elevation. The grid is defined according to the beamwidth. Furthermore, the search space bound is obtained by considering the beam of possible trajectories. That defines the number of required cells of the search space.

4.3. Radar search pattern results

Let us first mention that the search space is in this example a checkerboard of 4×4 cells. The radar allows itself 10 scans or looks to acquire the target, meaning here to detect it at least once. On the other hand, the beam is made of 80 random trajectories representing the radar incertitude on the target.

We show on Fig. 9 the optimal search pattern. In order to correctly visualize the error ellipsoid expansion, we add artificially the time as a new dimension to the space search dimensions. In this specific case, the optimal search pattern turns out to be an expanding-contracting spiral. At the beginning, the search visits the extremities of the diffusion cone. Next, after having reduced drastically the posterior probability of presence there, the search goes towards the cone center. The *a posteriori* uncertainty cone section is represented by a blue dot circle.

In Figs. 10–12, we show the solution evolution during the B&B algorithm, starting from the first solution to the final optimal (0–1) solution. Notice that this first solution is not the myopic search, but the first integer solution, i.e. the depth-first

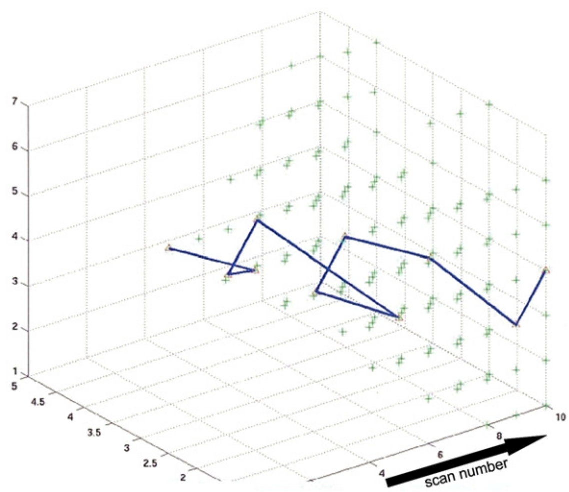

Fig. 10. The initial solution.

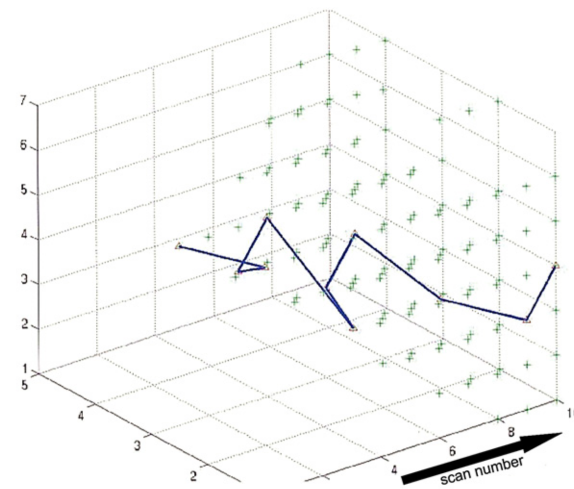

Fig. 12. The final optimal solution.

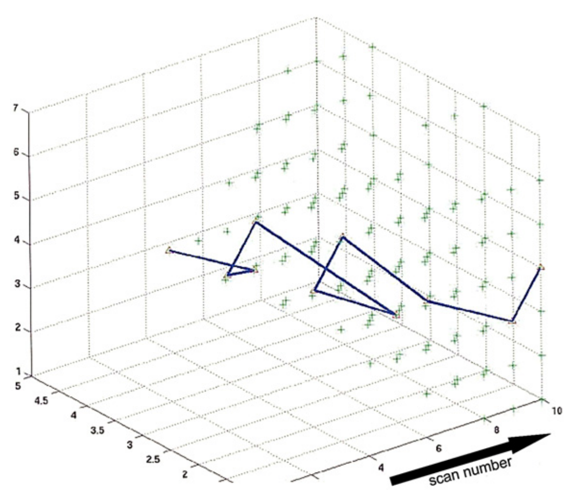

Fig. 11. An intermediate solution.

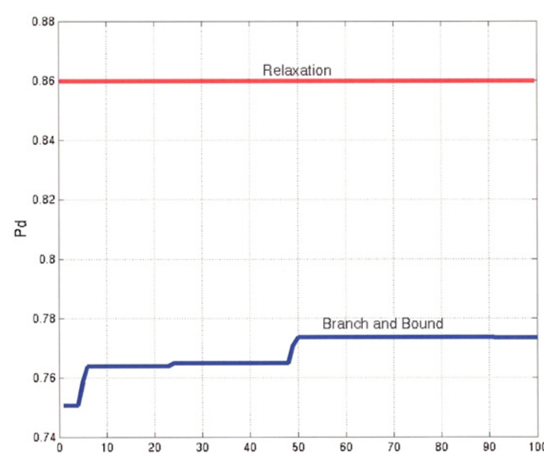

Fig. 13. The evolution of the detection probability.

algorithm initialization. In Fig. 13, the evolution of the detection probability is represented. The optimum is found after a few tens of tentative progress. We have compared the optimal search to a heuristic search, called myopic search. The myopic strategy consists in choosing, at each step of time, the cell with the highest *a posteriori* probability of presence. The detection probability for the myopic method and the B&B method are respectively 77% and 74%. The difference is here low, around 4%. It may be higher in other situations. Notice that the total (unfeasible) relaxation is more than 86%.

Concerning the computation load, it depends on the problem complexity, i.e. essentially on the number of possible search cells and the number of scans. As the search pattern can be computed off-line, there is no severe requirement. Yet, we have observed that the B&B method runs up against difficulties when the scan number is higher than a few tens. Indeed, the method is not able to overcome the combinatorial explosion. It is there possible to set up a sub-optimal adaptation of the B&B algorithm, as in [10].

5. Conclusion

An efficient Branch and Bound method has been applied to the optimization of the sequence of looks. The method consists of an adaptation of search theory and above all of Hohzaki and Iida works [5]. It has been tested in the situation where a narrow-beam radar attempts to acquire a long-range ballistic missile from an upper sensor hand-over. The powerful dedicated algorithm is able to provide the optimal search pattern solution in a limited amount of time. In certain conditions, the search pattern proves to be an expanding-contracting spiral which first visits the extremities of the diffusion cone and later concentrates gradually on the center.

Numerous "simple" extensions could be proposed to the described Branch and Bound method. A sub-optimal adaptation could be carried out to deal with tough combinatorial situations. Besides, the method could benefit from the introduction of cost functions, the generalization to Markovian targets, etc. A markedly harder extension would be to take the confirmation matter into account. Similarly to [8], it could consist in substituting the one detection at least rule for a "p over n" detection rule.

References

[1] D.K. Barton, Modern Radar System Analysis, Artech House, 1988.
[2] S.S. Brown, Optimal search for a moving target in discrete time and space, Operations Research 28 (6) (1980) 1275–1289.
[3] J.N. Eagle, J.R. Yee, An optimal branch-and-bound procedure for the constrained path moving target search problem, Operations Research 38 (1) (1990) 110–114.
[4] K. Iida, R. Hohzaki, The optimal search plan for a moving target minimizing the expected risk, Journal of the Operations Research Society of Japan 31 (2) (1988) 294–320.
[5] R. Hohzaki, K. Iida, Optimal strategy of route and look for the path constrained search problem with reward criterion, European Journal of Operational Research 100 (1) (1999) 167–179.
[6] K.H. Keil, Generic case study: Evaluation of early warning satellites cueing radars against TBM, in: Proceedings of the SPIE Conference on Signal and Data Processing of Small Targets, vol. 3809, 1999, pp. 297–307.
[7] B.O. Koopman, Search and Screening: General Principles with Historical Applications, MORS Heritage Series, Alexandria, Virginia.
[8] J.-P. Le Cadre, G. Souris, Searching tracks, IEEE Trans. on Aerospace Electron. Syst. 36 (4) (2000) 1149–1166.
[9] P. Minvielle, Tracking a ballistic re-entry vehicle with a sequential Monte-Carlo filter, in: Proceedings of the IEEE Aerospace Conference, vol. 4, 2002, pp. 1773–1787.
[10] J.-P. Le Cadre, G. Souris, Un panorama des méthodes d'optimisation de l'effort de recherche en détection, Traitement du Signal 16 (6) (1999) 403–424.
[11] T.J. Stewart, Search for a moving target when searcher motion is restricted, Computer and Operation Research 6 (1979) 129–140.
[12] T.J. Stewart, Experience with a branch-and-bound method for constrained searcher motion, in: B. Haley, L. Stone (Eds.), Search Theory and Applications, Plenum, New York, 1980, pp. 247–253.