Robust Detection of Target Maneuvers

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Abstract - ***This paper is concerned with the development of information processing for robust detection of target maneuvers. Indeed, the price to pay for having good performance for target maneuver detection is to accept that the event that a target maneuver will be falsely detected will be not so infrequent. Classical remedies exist and perform satisfactorily like the test based on consecutive exceeds. It is proved here that this test is robust and emphasis is put on performance analysis via Markov chain analysis. It is then possible to derive general method for optimizing the test so as to satisfy operational requirements.***

Keywords: detection, target maneuvers, Markov chains

1 Introduction

The detection of target maneuvers is far to be a new subject, and has been widely investigated [1], [2]. Not surprisingly, "elementary" detection of target maneuvers will rely on classical methods. By "elementary", we mean here a test directly based on filter outputs (i.e. innovations). However, operational requirements lead to consider a specific problem. The probability that a target maneuver be falsely detected, throughout a certain duration, must be bounded above.
This has a fundamental importance, especially for surveillance applications where the observer (e.g. a plane) has to monitor a large geographic area, including a considerable number of targets in its field of view. Only a few of them are maneuvering, so it is necessary to develop a system ensuring both a good maneuver detection and a low level of false target maneuver detection *at the system level.*

A comparison of performances of four maneuver tests show us the good performance of a simple test based on consecutive threshold exceeds. This test is clearly defined at an information processing level, since inputs are the outputs of a test, itself designed at a signal processing level. This comparative study shows that this test is robust and reliable. Robustness is essentially inherited from the level of elementary P_{fa} required at the signal processing level. Thus, it is no longer necessary to use dubious assumptions about the tail distribution, a definite advantage for our problem. The success of this architecture is based on exploitation of elementary detection trends (see [3]).
In order to analyze the performance of this test, a Discrete Time Markov Modeling (DTMC) is definitely the convenient framework [3]. It is then possible to take benefit of classical DTMC results for investigating the temporal behavior of this test (e.g. occupancy time-distribution, inter-visit times) and, thus, to adjust (signal processing) parameters (P_{fa}) so as operational requirements be satisfied. Not surprisingly, this is a very general setting which can lead to numerous extensions, briefly described here.

This paper is organized as follows. The state/observation modeling is presented in section 2, followed by a section devoted to hypotheses testing. The aim of section 4 is to analyze the behavior of multiple exceeds tests via DTMC behavior. Some extensions are presented in section 5.

2 State/observation space modelings and filtering

Throughout this paper, the following state-observation modeling will be considered:

$$\left\{ \begin{array}{l} \mathbf{x}_{k+1} = F_k\,\mathbf{x}_k + \mathbf{u}_k + v_k \\ z_k = H_k\,\mathbf{x}_k + w_k\ . \end{array} \right. \tag{1}$$

Thus $\mathbf{x}_k$ is the system state "at time-period k" $\mathbf{x}_k = (rx_k, ry_k, vx_k, vy_k)^T$, which represents the target relative (Cartesian) position and velocity. The vector $\mathbf{u}_k$ represents the target controls, while v_k is a **small** diffusion noise ($\mathsf{cov}(v_k) = Q_k$). The transition matrix is simply:

$$F = \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix} \otimes \mathrm{Id}_2\ . \tag{2}$$

The matrix H_k links the state $\mathbf{x}_k$ with the observation z_k, while w_k is the measurement noise (white Gaussian noise, $\mathsf{cov}(w_k) = R_k$). Here, we have:

$$H_k\, \mathbf{x}_k = (rx_k, ry_k)^T \quad , (\, H_k = H\,), \tag{3}$$

which means that the target position is completely observed. Let us also briefly recall the Kalman filter equations:

$$\begin{array}{l}
\text{STATE EXTRAPOLATION} \\
\mathbf{x}_{k+1|k} = F\, \mathbf{x}_{k|k} + \mathbf{u}_k\;, \\
P_{k+1|k} = F\, P_{k|k}\, F^T + Q_k\;, \\
\text{MEASUREMENT PREDICTION} \\
z_{k+1|k} = H\, \mathbf{x}_{k+1|k};\; R_{k+1|k} = H\, P_{k+1|k}\, H^T\;, \\
\text{INNOVATION} \\
I_{k+1} \stackrel{\Delta}{=} z_{k+1} - z_{k+1|k}\;, \\
\mathsf{cov}(I_{k+1}) = \Sigma_{k+1|k} = H\, P_{k+1|k}\, H^T + R_{k+1}\;, \\
\text{KALMAN GAIN}: \; K_{k+1} = P_{k+1|k}\, H\, (\mathsf{cov}(I_{k+1})\,)^{-1}\;, \\
\text{PREDICTED STATE AND ASSOC. COV.} \\
\mathbf{x}_{k+1|k+1} = \mathbf{x}_{k+1|k} + K_{k+1}\, I_{k+1}\;, \\
P_{k+1|k+1} = P_{k+1|k} - K_{k+1}\, \mathsf{cov}(I_{k+1})\, K_{k+1}^T\;.
\end{array} \tag{4}$$

This is the filtering framework we shall use throughout this paper. However, this is not a strong limitation since our framework for robust maneuver detection can be extended to more complex state/observation, including non-linear measurement. This would simply result in changes in the calculation of the transition probabilities.

3 Testing hypotheses

We restrict here to a test between two hypotheses (say H_0 and H_1). The target is **not-maneuvering** (H_0), or maneuvering (H_1). Let us emphasize that our principal concern is the evaluation of the probability of falsely detect a target maneuver, so H_0 will be our primary subject of interest. The tests we shall consider are presented in the table below.

without alternative	**with** alternative
Windowed χ^2 test	
Forgetting factor test	Test with alternative
Test on consecutive exceeds	

3.1 Testing without alternative

Let us define the reduced innovation residue ε_k, by:

$$\varepsilon_k \stackrel{\Delta}{=} I_k^T\, (\mathsf{cov}(I_k)\,)^{-1}\, I_k\;. \tag{5}$$

Under the assumption that it is the right state space model which is used in the Kalman filter, then the reduced innovation residue ε_k follows a (central) χ^2 law with 2 degrees of freedom (the observation dimension). Temporal variability of this test can be attenuated via a forgetting factor, i.e.:

$$\rho_k \stackrel{\Delta}{=} \sum_{i=1}^{k} \alpha^{k-i}\, \varepsilon_i\;, \tag{6}$$

where α is a constant ($0 < \alpha < 1$), which represents the forgetting factor. Alternatively, it is possible to consider the (sliding) window test, i.e. test for:

$$\lambda_k \stackrel{\Delta}{=} \sum_{i=k-L}^{k} \varepsilon(k) > t\;. \tag{7}$$

From previous considerations, we know that λ_k is Chi-square distributed, with $2L$ degrees of freedom. Typical values of L are $\{4, 6, 10\}$.

Another test is to consider an associated test with two consecutive threshold exceeds, i.e. a maneuver is said detected if the following condition is fulfilled $\{\lambda_k > t, \lambda_{k+1} > t\}$, with $L = 1$ (or another value). This test is important from an engineering viewpoint. In this setting, temporal trends of elementary maneuver "detections" are considered [3] and we shall see that "despite" its simplicity, it has very interesting features.

3.2 Testing with alternative

Assume now that the target is maneuvering at the time period m, then an additional (deterministic) input (say $\mathbf{u}$) is added in the state equation (see eq.1). Let us consider two filtering hypotheses. For, the first one (H_0) there is no target maneuver (state noted $\bar{\mathbf{x}}$), while for the second (H_1) a maneuver has occurred (notation: $\mathbf{x}^\nu$). For the two hypotheses, the Kalman filter makes the assumption that the target does not maneuver. This is justified since our aim is target detection, not input estimation [2].

Considering the difference of innovations (noted $\Delta\, I$), associated with these two *target* models, classical calculations yield:

$$\begin{array}{lcl}
\Delta\, I(m+1) & \stackrel{\Delta}{=} & \bar{I}(m+1) - I^\nu(m+1)\;, \\
 & = & H(m+1)\, \mathbf{u}(m)\;,
\end{array}$$

and more generally:

$$\Delta\, I(m+n) = H \sum_{j=1}^{n} \left[\prod_{l=1}^{n-j} F\, (\mathsf{Id} - K(m+l)H))\; \mathbf{u}(m+j-1) \right]\;. \tag{8}$$

Considering this alternative, we see that it is necessary to detect the target maneuver as soon as possible. So we have to manage two conflicting objectives: delay the decision for maximizing the probability of detection, reduce the delay for avoiding divergence [10]. The compromise between these antagonist objectives will now be analyzed via elementary statistics.
First, consider a sliding window of size L, the sum of the associated innovations is:

$$D_L(k) = \sum_{j=k-L+1}^{k} I(j)\;. \tag{9}$$

Under H_0, $D_L(k)$ is normally distributed, i.e. $\mathcal{N}(0, \Sigma_L(k))$, with $\Sigma_L(k) = \sum_{j=k-L+1}^{k} \Sigma(j|j-1)$. Under H_1, $D_L(k)$ has the same covariance, but its mean is different:

$$\mathbb{E}\left[D_L(k)|H_1\right] \triangleq \nu_L(k) = \sum_{j=k-L+1}^{k} \Delta I(j) , \qquad (10)$$

where $\Delta I(j)$ is given by eq. 8. Elementary calculations yields the general form of the alternative test:

$$2\, \nu_L^T(k)\, \Sigma_L^{-1}(k)\, D_L(k) - \nu_L^T(k)\, \Sigma_L^{-1}(k)\, \nu_L(k) \underset{H_0}{\overset{H_1}{\gtrless}} t , \qquad (11)$$

where t is a threshold. Furthermore, the probabilities of false alarm (Pfa) and detection (Pd) are easily calculated, i.e. :

$$\begin{array}{l} \text{Pd} = \mathsf{erf}\left(\frac{\delta_k - t}{2\sqrt{\delta_k}}\right) , \\ \text{with: } \left\{ \begin{array}{l} t = -\delta_k - 2\sqrt{\delta_k}\, \mathsf{erf}^{-1}(\text{Pfa}) , \\ \delta_k = \nu_L^T(k)\, \Sigma_L^{-1}(k)\, \nu_L(k) . \end{array} \right. \end{array} \qquad (12)$$

The above formulas allow us to compute ROC curves. In general, inputs are unknown though they are used in the alternative test (see eq. 8). Inputs are deduced from a constant load turn modeling. The unknown parameter is the radius of the turn, itself depending of the type of ship (400 to 1000 m). So, the turn model can be adapted or not. However, simulation results prove that this does not affect greatly the alternative test.

3.3 Simulation results

We are now using the previous calculations for investigating the behavior of the maneuver tests. Since our principal concern is test robustness, we shall clearly focus on this aspect. Actually, the Gaussianity of the innovation I_k is questionable. So is our assumption about the χ^2 distribution of the residue ε_n (see eq.5). This is not too important as far we are not considering thresholds on the tail, but it may be a fundamental concern for "traditional" tests. So, heavy tails seem a realistic *measurement* modeling. To that aim we consider the following (tail) density modeling:

$$f_\gamma(x) = (1-\gamma)\, \mathcal{N}(m, \sigma) + \gamma\, \frac{(x - m')}{\sigma'^2} \exp\left(-\frac{x - m'}{\sigma'}\right) . \qquad (13)$$

In the above equation, the γ parameter represents a contamination factor (e.g. 0.1 , 0.5), and for a given value of γ the parameters m' and σ' are adjusted so that $f_\gamma(x)$ remains a density with prescribed mean (m) and standard deviation (σ). Sampling from $f_\gamma(x)$ is done via a classical method, i.e. :

$$\begin{array}{l} 1) - \text{sample from } U[0,1] \rightarrow \{y(\kappa)\} , \\ 2) - \text{if } y(\kappa) \leq 1 - \gamma \text{ draw with: } \mathcal{N}(m, \sigma) , \\ 3) - \text{if } y(\kappa) \geq 1 - \gamma \text{ draw with: } \frac{(x-m'}{\sigma'^2} \exp\left(-\frac{x-m'}{\sigma'}\right) . \end{array} \qquad (14)$$

So, measurements are simulated via this algorithm, for various values of the γ parameter.

The probability to falsely detect a maneuver

We have considered various scenarios (see fig. 1), differing only by target parameters. The target is **not** maneuvering.

	speed	range
large ships	18 knots	70 – 50 n.m.
medium ships	18 knots	40 – 25 n.m.
fast ships	35 knots	30 – 20 n.m.

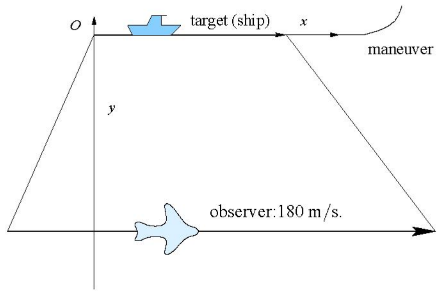

Figure 1: The general target-observer scenario

The value of the state noise variance is $5\ 10^{-4}$ m/s^2. Our aim is to investigate the threshold values in order to have an empirical probability to falsely detect a maneuver (epfd): $0.009 \leq \text{epfd} \leq 0.011$, for 1000 independent trials. So, in the following table, it is the value of the elementary P_{fa} which is presented[1].

	$\gamma = 0$	$\gamma = 0.1$
large ships		
Alternative test $L = 6$	$1.\ 10^{-4}$	$7.5\ 10^{-5}$
χ^2 $L = 6$	$7.5\ 10^{-5}$	**X**
Two consec. exceeds	$5\ 10^{-3}$	$3.7\ 10^{-3}$
fast ships		
Alternative test $L = 6$	$1.8\,10^{-4}$	$1.5\ 10^{-4}$
χ^2 $L = 6$	$3.2\ 10^{-4}$	$1.\ 10^{-5}$
Two consec. exceeds	$11.\ 10^{-3}$	$6.\ 10^{-3}$

Values of the elementary P_{fa}

The results of the forgetting factor test are not presented in this table, since it is impossible to find an acceptable value of P_{fa}, whatever the forgetting factor α. We note also the lack of robustness of the χ^2 test, opposite to the alternative test. Finally,

[1] **X** means that it is impossible to find an acceptable value of P_{fa}

the two consec. exceeds test is robust and have the **definite** advantage to require only large value of the elementaryP_{fa} for assessing the required *global* performance level. This will motivate incoming developments.

Under the same conditions, the detection performance is analyzed via the delay for detecting the target maneuver. Results are summarized in the following table. We note that the performance of the two consec. exceeds test can compete with the best one (the *adapted* alternative test).

	$\gamma = 0$	$\gamma = 0.1$
large ships		
Alternative test $L = 6$	29.05 s.	29.5 s. s
Two consec. exceeds	34.5 s.	37.3 s.
fast ships		
Alternative test $L = 6$	21.0 s.	20.5 s.
Two consec. exceeds	21.5 s.	22.6 s.

Delay for target maneuver detection

4 Modeling the detection-confirmation process

In this section, we shall consider the detection/confirmation process i.e. maneuver detection based on multiple threshold exceeds. First let us briefly recall principal definitions of Markov chains.

4.1 Discrete time Markov models (DTMC)

We consider a random process, indexed by time $\{X_n,\ n \geq 0\}$, defined on a finite space S. This process has the Markov property if it fulfills the following (Markov) property:

$$\forall i, j \in S^2 ,\quad P\left(\mathsf{X}_{n+1} = j | \mathsf{X}_n = i, \mathsf{X}_{n-1}, \cdots, \mathsf{X}_0\right) = P\left(\mathsf{X}_{n+1} = j | \mathsf{X}_n = i\right). \tag{15}$$

Such process is called a Discrete Time Markov Chain (DTMC). An homogeneous DTMC satisfies furthermore:

$$P\left(\mathsf{X}_{n+1} = j | \mathsf{X}_n = i\right) = P\left(\mathsf{X}_1 = j | \mathsf{X}_0 = i\right) \quad ; \forall n \geq 0 . \tag{16}$$

For an homogeneous DTMC, we simply define $p_{i,j} \stackrel{\Delta}{=} P\left(X_{n+1} = j | X_n = i\right)$ $(\forall n \geq 0)$ as the probability of transition from state i to state j. These transition probabilities are represented by a square matrix $\mathbf{P}$, called the transition matrix:

$$\mathsf{P} = \begin{pmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,N} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,N} \\ \vdots & \vdots & \vdots & \vdots \\ p_{N,1} & p_{N,2} & \cdots & p_{N,N} \end{pmatrix} \tag{17}$$

For the problem we have in mind, we first restrict to a 4-dimensional state ($N = 4$) and adopt the following notations:

$$\begin{array}{l} p_t :\ p_t \stackrel{\Delta}{=} P(\varepsilon_n > t) :\ \text{probability of threshold exceeding,} \\ [0]\ \text{event: } \{\varepsilon_n < t\}\ \text{, under threshold ,} \\ [1]\ \text{event: } \{\varepsilon_n \geq t\}\ \text{, over threshold,} \end{array} \tag{18}$$

where n is a time index. The random variable X can take 4 states, according to a certain threshold t:

$$\left| \begin{array}{l} \text{state: } (1) : [0,0] \text{ event: } \{\varepsilon_n < t, \varepsilon_{n+1} < t\} , \\ \text{two consecutive non-detections,} \\ \text{state: } (2) : [0,1] \text{ event: } \{\varepsilon_n < t, \varepsilon_{n+1} > t\} , \\ \text{a non-detection followed by a detection,} \\ \text{state: } (3) : [1,0] \text{ event: } \{\varepsilon_n > t, \varepsilon_{n+1} < t\} , \\ \text{a detection followed by a non-detection,} \\ \text{state: } (4) : [1,1] \text{ event: } \{\varepsilon_n > t, \varepsilon_{n+1} > t\} , \\ \text{two consecutive detections.} \end{array} \right. \tag{19}$$

This detection/confirmation process is illustrated on fig. 2. In the above definition the terms detection

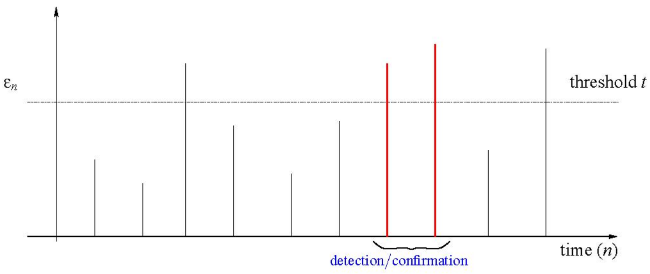

Figure 2: The one-level threshold exceeding detection/confirmation process

is quite "freely", used instead of threshold exceeding (non-detection instead of under threshold). Note that, under the H_0 hypothesis and thanks to the Kalman filer properties, the sequence $\{\varepsilon_n\}$ is i.i.d. [8]. The transition diagram of this DTMC is illustrated by fig. 3, while the associated 4×4 transition matrix P_2 stands as:

$$\mathsf{P}_2 = \begin{pmatrix} 1 - p_t & p_t & 0 & 0 \\ 0 & 0 & 1 - p_t & p_t \\ 1 - p_t & p_t & 0 & 0 \\ 0 & 0 & 1 - p_t & p_t \end{pmatrix} \tag{20}$$

Considering fig. 3 and the transition matrix P_2, we see that this Markov chain is aperiodic and irreducible, ensuring existence of a stationary distribution [5]. State 4 is especially relevant for our analysis. It corresponds to **two** consecutive detections (i.e. event: $\{\varepsilon_n > t, \varepsilon_{n+1} > t\}$). As the elementary probability of detection p_t is directly depending on the threshold t, our main concern will be to adjust t so that have an acceptable rate of false detection. A sim-

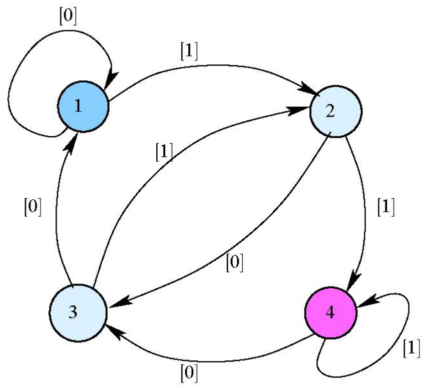

Figure 3: The DTMC diagram

ple calculation shows that:

$$
\mathsf{P_2}^2 = \begin{pmatrix} (1-p_t)^2 & p_t\,(1-p_t) & p_t\,(1-p_t) & p_t^2 \\ (1-p_t)^2 & p_t\,(1-p_t) & p_t\,(1-p_t) & p_t^2 \\ (1-p_t)^2 & p_t\,(1-p_t) & p_t\,(1-p_t) & p_t^2 \\ (1-p_t)^2 & p_t\,(1-p_t) & p_t\,(1-p_t) & p_t^2 \end{pmatrix} \tag{21}
$$

The structure of the matrix $\mathsf{P_2}^2$ is quite particular and is a characteristic feature. Indeed, we see (from eq. 21), that:

$$
\begin{aligned}
&\mathsf{P_2}^2 = \left[(1-p_t)^2\,\mathbb{1}, p_t(1-p_t)\,\mathbb{1}, p_t(1-p_t)\,\mathbb{1}, p_t^2\,\mathbb{1}\right] \ , \\
&\text{where: } \mathbb{1} = (1,1,1,1)^T \ .
\end{aligned} \tag{22}
$$

Clearly, $\mathsf{P_2}^n$ is a rank one matrix for $n \geq 2$. Thus, $\mathsf{P_2}^2$ admits the following factorization:

$$
\begin{aligned}
&\mathsf{P_2}^2 = \mathsf{V}\,\mathsf{W}^T \ , \\
&\text{where:} \\
&\mathsf{V} = (1-p_t)\ \mathbb{1}, \\
&\mathsf{W}^T = \left(1-p_t,\ p_t,\ p_t,\ p_t^2/(1-p_t\,)\right) \ .
\end{aligned} \tag{23}
$$

Furthermore, it is easily shown that $\mathsf{W}^T\,\mathsf{P}_2 = \mathsf{W}^T$. Thus, we have:

$$
\begin{aligned}
\mathsf{P_2}^3 &= \left(\mathsf{V}\,\mathsf{W}^T\right)\ \mathsf{P}_2 \ , \\
&= \mathsf{V}\ \left(\mathsf{W}^T\,\mathsf{P}_2\right) \ , \\
&= \left(\mathsf{V}\,\mathsf{W}^T\right) = \mathsf{P_2}^2 \ .
\end{aligned} \tag{24}
$$

And more generally, whatever $n \geq 4$ we have $\mathsf{P_2}^n = \mathsf{P_2}^2\,\mathsf{P_2}^{n-2} = \mathsf{P_2}^4 = \mathsf{P_2}^2$, yielding the following result:

Proposition 1 *Whatever $n \geq 2$, the following equality holds true:*
$\mathsf{P_2}^n = \mathsf{P_2}^2$.

So, whatever the initial distribution X_0, described by the **row** vector $\mathsf{X}_0 = (x_1, x_2, x_3, x_4)$, we have ($\forall n \geq 2$):

$$
\begin{aligned}
\mathsf{X_0}^{(n)} &= \mathsf{X}_0\,\mathsf{P_2}^n = \mathsf{X}_0\,\mathsf{P_2}^2 \ , \\
&= (\mathsf{X}_0\,\mathsf{V})\ \mathsf{W}^T \ , \\
&= (1-p_t)\ \underbrace{(\mathsf{X}_0\,\mathbb{1})}_{=1}\ \mathsf{W}^T = (1-p_t)\ \mathsf{W}^T \ , \\
&= \left(\,(1-p_t)^2\ , p_t\,(1-p_t)\ , p_t\,(1-p_t)\ , p_t^2\right) \ .
\end{aligned} \tag{25}
$$

Similarly, let us consider the (asymptotic) stationary distribution $\boldsymbol{\pi}$, then $\boldsymbol{\pi}$ is a solution of the balance equation:

$$
\boldsymbol{\pi} = \boldsymbol{\pi}\ \mathsf{P}_2 \ . \tag{26}
$$

Not surprisingly, it is easily shown that:

$$
\boldsymbol{\pi} = \left((1-p_t)^2\ , p_t\,(1-p_t)\ , p_t\,(1-p_t)\ , p_t^2\right) \ . \tag{27}
$$

We are now in position for studying the behavior of this detection/confirmation setup. To that aim, we shall recall and use general results about renewal theory for DTMC [5], [7].

4.2 Behavior of the detection/ confirmation process

We are now investigating the behavior of the detection/ confirmation process via renewal analysis. First, let us define the occupancy time. Let $N_j(n)$ be the number of times the DTMC visits state j over the time span $\{0, 1, \cdots, n\}$ and let:

$$
\begin{aligned}
&\mu_{i,j}(n) = \mathbb{E}\ \left(N_j(n)|X_0 = i\right) \ , \\
&\mathsf{M}(n) \triangleq (\ \mu_{i,j}(n)\ )_{1\leq i\leq 4,\ 1\leq j\leq 4} \ .
\end{aligned} \tag{28}
$$

Then, the following classic property is insightful.

Proposition 2 *(Occupancy Times [5],[7]) Let a time-homogeneous DTMC, with transition probability* P. *Then, the mean occupancy times matrix is given by:*

$$
\mathsf{M}(n) = \sum_{r=0}^{n} \mathsf{P}^r \ . \tag{29}
$$

This result is upsettingly simple, as is its proof that we recall here only for the sake of completeness. Fix i and j, and define the r.v. Z_n as an indicator function; i.e. $Z_n = 1$ if $X_n = j$ and $Z_n = 0$ if $X_n \neq j$. Then $N_j(n) = Z_0 + Z_1 + \cdots + Z_n$, hence we have:

$$
\begin{aligned}
\mu_{i,j}(n) &= \sum_{r=0}^{n} \mathbb{E}\ (Z_r|X_0 = i) \ , && (30) \\
&= \sum_{r=0}^{n} P\ (X_r|X_0 = i) \ , && (31) \\
&= \sum_{r=0}^{n} p_{i,j}^{(r)} \ .
\end{aligned}
$$

Thus, for our application, we have simply [2]:

$$
\begin{aligned}
&\mathsf{M}(n) = \mathsf{Id} + \mathsf{P}_2 + (n-1)\ \mathsf{P_2}^2 \ , \\
&\text{so, that we have:} \\
&\left|\begin{array}{l}
\mu_{1,1}(n) = 2 - p_t + (n-1)(1-p_t)^2 \ , \\
\mu_{4,4}(n) = 1 + p_t + (n-1)p_t^2 \ , \\
\mu_{1,2}(n) = p_t + (n-1)p_t(1-p_t) \ , \\
\mu_{1,4}(n) = (n-1)\ (1-p_t)^2, \text{ etc.}
\end{array}\right.
\end{aligned} \tag{32}
$$

[2] Id: identity matrix

From eq. 32, we see that the $\mu_{i,1}$ $(i = 1, \cdots, 4)$ are the predominant terms, while $\mu_{4,4}$ is quite smaller, as far as p_t is relatively small. To complete this analysis, let us consider now the stationary distribution of $N_j(n)$. To that aim, we define the *occupancy* of state j, by:

$$\hat{\pi}_j \triangleq \lim_{n\to\infty} \frac{\mathbb{E}\ (N_j(n))}{n+1}\ . \qquad (33)$$

Then the following property is especially relevant:

Proposition 3 Occupancy distribution *If the occupancy distribution $\hat{\pi}$ exists, it satisfies :*

$$\hat{\pi}_j = \sum_i \hat{\pi}_i\, p_{i,j} \ \ \textit{and:} \sum_j \hat{\pi}_j\ . \qquad (34)$$

So, for an irreducible DTMC, occupancy distribution and stationary distribution are identical. For instance, we have here $\hat{\pi}_4 = p_t^2$.

Another important concern is the study of the *mean first passage time*, which is simply defined by [5]:

$$\begin{array}{l} T = \min\ \{n \geq 0\ :\ X_n = N\ \}\ , \\ m_i = \mathbb{E}\ (T|X_0 = i)\ , \end{array} \qquad (35)$$

where the DTMC states are indexed from 1 to N. Then, the following result holds true.

Proposition 4 *(Expected First Passage Times [5]) The mean first passages times m_i , $1 \leq i \leq (N-1)$ satisfy the following set of equations:*

$$m_i = 1 + \sum_{j=1}^{N-1} p_{i,j}\, m_j \ \ 1 \leq i \leq N-1\ . \qquad (36)$$

Solving eq. 36, for $N = 4$, yields:

$$m_1 = m_3 = \frac{1}{p_t^2}\ ,\ m_2 = \frac{(1+p_t)}{p_t^2}\ . \qquad (37)$$

Thus, we note that m_1, m_2 and m_3 are similar and rather important for small values of p_t. This analysis has the real advantage to be quite simple, while providing insightful result. However, it cannot be adapted to the inter-visit time (m_N) which is is clearly zero, from its definition. So, we have to turn to another classical result.

Proposition 5 *(Mean Inter-Visit Times [5], [4]) Assume the DTMC is irreducible and let $\boldsymbol{\pi}$ its stationary distribution, then:*

$$m_{j,j} = \frac{1}{\pi_j}\ ,\ 1 \leq j \leq N\ . \qquad (38)$$

Thus, we have here:

$$m_{4,4} = \frac{1}{\pi_4} = \frac{1}{p_t^2}\ . \qquad (39)$$

Let us emphasize the consequences of these elementary calculations. For instance, assume that we can tolerate a false detection rate of say 15 min. (900 sec.), then $p_t = (900)^{-1/2} \simeq 0.0033$. Then, under the non-maneuvering assumption we have seen that the normalized innovation I_n follows a central Chi-square distribution with two degrees of freedom. An adapted threshold value is easily found by inverting the cumulative density function of this Chi-square distribution, yielding $t = 6.8$. We see that the adapted threshold is not situated on the tail of the Chi-square distribution, which means that the two consecutive detection test is quite robust.

Consider now a slight modification of the DTMC. If a maneuver is detected, then the DTMC *remains* on (the absorbing) state 4. This is illustrated on fig. 4. The associated transition matrix $\tilde{\mathsf{P}}_2$ reads:

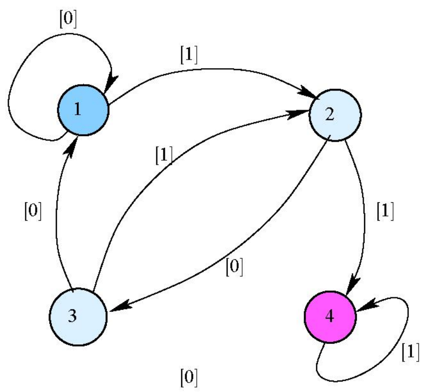

Figure 4: The DTMC diagram: state 4 absorbing

$$\tilde{\mathsf{P}}_2 = \begin{pmatrix} 1-p_t & p_t & 0 & 0 \\ 0 & 0 & 1-p_t & p_t \\ 1-p_t & p_t & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (40)$$

The aim of this modeling is to investigate the probability that the system be **at least** one time in state 4. However, this chain is obviously reducible, so that a stationary distribution does not exist. State 4 is obviously an absorbing state. Admitting an initial state distribution, say X_0, we have to calculate $\mathsf{X}_n \triangleq \mathsf{X}_0\, \tilde{\mathsf{P}}_2^n$. In order to obtain an explicit expression, the following block-factorization is especially useful:

$$\tilde{\mathsf{P}}_2 = \begin{pmatrix} \mathsf{Q} & \mathbf{v}_1 \\ \mathbf{0}^T & 1 \end{pmatrix}\ , \qquad (41)$$

where Q is a 3×3 matrix, and yields:

$$\tilde{\mathsf{P}}_2^n = \begin{pmatrix} \mathsf{Q}^n & \mathbf{v}_n \\ \mathbf{0}^T & 1 \end{pmatrix}\ . \qquad (42)$$

If we are able to provide an explicit expression of Q^n, there is no need to calculate the vector $\mathbf{v}_n$ since the matrix $\tilde{P}_2^n$ is stochastic. The eigensystem of the Q matrix is quite simple, i.e. :

$$\left| \begin{array}{ll} \text{eigenvalues} & \text{eigenvectors} \\ \lambda_1 = 0 & \mathbf{u}_1^T = \left(\frac{-p_t}{1-p_t}, 1, 0\right) \\ \lambda_2 = \frac{1}{2}\left(1 - p_t - \sqrt{\delta}\right) & \mathbf{u}_2^T = \left(1, \frac{(1-p_t)}{\lambda_2}, 1\right) \\ \lambda_3 = \frac{1}{2}\left(1 - p_t + \sqrt{\delta}\right) & \mathbf{u}_3^T = \left(1, \frac{(1-p_t)}{\lambda_3}, 1\right) \\ \delta = \left(1 + 2\,p_t - 3\,p_t^2\right) & . \end{array} \right. \quad (43)$$

From which the following equality is deduced[3]:

$$Q^n = \lambda_2^n\ \left(\mathbf{u}_2\mathbf{u}_2^T\right) + \lambda_3^n\ \left(\mathbf{u}_3\mathbf{u}_3^T\right)\ . \quad (44)$$

Therefore, the coefficient $\tilde{P}_2^n\ (1,4)$ is simply:

$$\begin{aligned} \tilde{P}_2^n\ (1,4) &= 1 - \lambda_2^n\ \left(\mathbf{e}_1^T\mathbf{u}_2\right)\ \left(\mathbf{u}_2^T 1\right)\ , \quad (45) \\ &\quad -\lambda_3^n\ \left(\mathbf{e}_1^T\mathbf{u}_3\right)\ \left(\mathbf{u}_3^T 1\right)\ , \\ &\quad \mathbf{e}_1 = (1,0,0)\ . \end{aligned}$$

Consequently, admitting an initial distribution $\mathsf{X} = (1,0,0,0)$ of the system state, the probability that the state 4 has been attained *at least* at one time within the temporal interval $[0,n]$ is:

$$\begin{aligned} \tilde{P}_2^n\ (1,4) &= 1 - \lambda_2^{n+1}\left(\frac{2\,\lambda_2 + 1 - p_t}{2\,\lambda_2^2 + (1-p_t)^2}\right), \quad (46) \\ &\quad +\lambda_3^{n+1}\left(\frac{2\,\lambda_3 + 1 - p_t}{2\,\lambda_3^2 + (1-p_t)^2}\right)\ . \end{aligned}$$

A second order expansion give us $\tilde{P}_2^n\ (1,4) \simeq (n+1)p_t^2 + \frac{p_t}{3}$. Coming back to the previous results, we take $p_t = 0.011$. Then, we obtain $\tilde{P}_2^{100}\ (1,4) \simeq 0.012$, a value which corresponds to the second order expansion and our previous simulations.

Furthermore, let us denote N_a the number of visits to the transient states, before visiting the absorbing state (state 4 here), then we have:

$$P(N_a = n) = \mathsf{X}_0\ \mathsf{Q}^{n-1}\ (\mathsf{Id} - \mathsf{Q})\ 1\ ,\ \ n \geq 1. \quad (47)$$

Hence, the *expected* number of visits to the absorbing state is simply:

$$\begin{aligned} &\mathbb{E}(N_a) = \sum\nolimits_{n\geq 1} nP(N_a = n) = \mathsf{X}_0\ (\mathsf{Id} - \mathsf{Q})^{-1}\ 1\ , \\ &\text{with:} \\ &(\mathsf{Id} - \mathsf{Q})^{-1}\ 1 = \left(\frac{1+p_t}{p_t^2}, \frac{1}{p_t^2}, \frac{1+p_t}{p_t^2}\right)\ . \end{aligned} \quad (48)$$

As p_t is rather small for our application, we thus have $\mathbb{E}(N_a) \simeq \frac{1}{p_t^2}$, whatever the initial distribution of the transient states.

[3]after normalization of the $\mathbf{u}_2$ and $\mathbf{u}_3$ vectors

5 Extensions

We shall consider now some extensions of the detection/confirmation process. The first one is to consider that detection/confirmation is based on multiple *consecutive* threshold exceeds. In the second one, we shall consider a multi-level detection/confirmation process.

5.1 Detection/confirmation based on multiple consecutive threshold exceeds

Let us now consider a test based on *three consecutive threshold exceeds.* The probability p_t is again the probability of exceeding the threshold t. The events [0] and [1] are similarly defined. The random variable X can take 8 states, according to a certain threshold t:

$$\begin{array}{ll} \text{state: } (1) : [0,0,0] & \text{state: } (5) : [1,0,0] \\ \text{state: } (2) : [0,0,1] & \text{state: } (6) : [1,0,1] \\ \text{state: } (3) : [0,1,0] & \text{state: } (7) : [1,1,0] \\ \text{state: } (4) : [0,1,1] & \text{state: } (8) : [1,1,1] \end{array} \quad (49)$$

while the associated 8×8 transition matrix P_3 stands as (block-matrix):

$$\begin{aligned} &\mathsf{P}_3 = \begin{pmatrix} \mathsf{Q}' & 0 \\ 0 & \mathsf{Q}' \\ \mathsf{Q}' & 0 \\ 0 & \mathsf{Q}' \end{pmatrix} \\ &\text{with:} \\ &\mathsf{Q}' = \begin{pmatrix} 1-p_t & p_t & 0 & 0 \\ 0 & 0 & 1-p_t & p_t \end{pmatrix} \end{aligned} \quad (50)$$

and $\mathsf{0}$ is a 2×4 matrix composed of 0. By the same way, it is proved that:

$$\begin{aligned} {\mathsf{P}_3}^3 &= [\ (1-p_t)^3 1, (1-p_t)^2\, p_t 1, (1-p_t)^2\, p_t 1, \\ &\quad (1-p_t)p_t^2 1, (1-p_t)^2\, p_t 1, (1-p_t)p_t^2 1, \\ &\quad (1-p_t)p_t^2 1, p_t^3 1\ ]\ . \end{aligned} \quad (51)$$

Thus, it is again a rank 1 matrix. So, the reasoning we have used for analyzing ${\mathsf{P}_2}^n$ can be straightforwardly extended for investigating the properties of ${\mathsf{P}_3}^n$.
Not surprisingly, we obtain that ${\mathsf{P}_3}^n = \varepsilon_{n+2} > t$, whatever the value of $n \geq 3$. More generally, and within the same detection/confirmation process[4], we have.

Proposition 6 *Whatever the value of the integer q (q consecutive detections), the following property holds true:*

$$\forall\, n \geq q : \quad {\mathsf{P}_q}^n = {\mathsf{P}_q}^q\ , \quad (52)$$

[4]Which means that a target maneuver is said detected/confirmed if q consecutive detections have occurred

and the matrix $\mathsf{P}_q{}^q$ is a rank one matrix whose stationary distribution is given by its first row; i.e. :

$$\boldsymbol{\pi} = \left[(1-p_t)^q,\ (1-p_t)^{q-1}p_t, \cdots, (1-p_t){p_t}^{q-1},\ {p_t}^q\right] . \tag{53}$$

State $\tilde{q}$ corresponds to q consecutive detections (i.e. event: $\{\varepsilon_{n+1} > t, \varepsilon_{n+2} > t, \cdots, \varepsilon_{n+q} > t\}$). Considering the properties of the P_q matrix, the mean intervisit time for this state is calculated by the same way, yielding:

$$m_{\tilde{q},\tilde{q}} = \frac{1}{{p_t}^q} . \tag{54}$$

So, we see that the threshold can be considerably lowered by considering greater values of q, which could be interesting for improving early maneuver detection.

5.2 Multi-level detection/confirmation

Consider now that the threshold exceeding is quantized on multiple levels [10]. For instance, we consider two thresholds t_1 and t_2 ($t_1 < t_2$) and define the following events:

$$\left| \begin{array}{l} [0]\ \ 0 < \varepsilon_n < t_1 \quad ; [1]\ \ t_1 < \varepsilon_n < t_2 \ ; \\ [2]\ \ t_2 < \varepsilon_n \ . \end{array} \right. \tag{55}$$

Dealing with two consecutive threshold exceeds, we model the detection/ confirmation process by a 9 states DTMC, as described below:

$$\begin{array}{ll} \text{state: } (1) : [0,0] & \text{state: } (5) : [2,0] \\ \text{state: } (2) : [0,1] & \text{state: } (6) : [1,1] \\ \text{state: } (3) : [0,2] & \text{state: } (7) : [1,2] \\ \text{state: } (4) : [1,0] & \text{state: } (8) : [2,1] \\ \text{state: } (9) : [2,2] & \end{array} \tag{56}$$

State 9 may be viewed as previously, but states (3) or (7) may have a specific meaning. For instance, these states may be associated with various levels of prealerts which may be useful for tasks like sensor management or threat evaluation. Knowing the distribution of ε_n under H_0 and considering that they are i.i.d. under this hypothesis, the following probabilities are easily calculated:

$$\begin{array}{l} p_1 \stackrel{\Delta}{=} P(\ \{\ t_1 < \varepsilon_n < t_2\ \}\ ) = P([1])\ , \\ p_2 \stackrel{\Delta}{=} P(\ \{\ t_2 < \varepsilon_n\ \}\ ) = P([2])\ ,\ \ p \stackrel{\Delta}{=} p_1 + p_2\ . \end{array} \tag{57}$$

With this definition, the DTMC transition matrix P is easily calculated yielding (first row):

$$\left\{ \begin{array}{ll} \mathsf{P}(1,1) = 1 - p_1 - p_2 & \mathsf{P}(1,2) = p_1 \\ \mathsf{P}(1,3) = p_2 & \mathsf{P}(1,4) = \cdots = \mathsf{P}(1,9) = 0\ . \end{array} \right. \tag{58}$$

Other rows are calculated in the same way. More importantly, the powers of the matrix P still exhibit a remarkable property. More precisely, we have:

$$\begin{array}{rcl} \mathsf{P}^4 & = & [(1-p)^2\ \mathbb{1}, p_1(1-p)\ \mathbb{1}, p_2(1-p)\ \mathbb{1}, \\ & & p_1(1-p)\ \mathbb{1}, p_2(1-p)\ \mathbb{1}, p_1^2\ \mathbb{1}, \\ & & p\, p_2(1-p_2)\ \mathbb{1}, p_1p_2\ \mathbb{1},\ p\, p_2^2\ \mathbb{1}]\ . \end{array} \tag{59}$$

Thus, P^4 is again a rank 1 matrix, admitting the factorization $\mathsf{P}^4 = \mathsf{V}\ \mathsf{W}^T$. Remarkably, we have also:

$$\begin{array}{l} \mathsf{W}^T\ \mathsf{P} = \mathsf{W}\ . \\ \text{so, that:} \\ \mathsf{P}^5 = \left(\mathsf{V}\ \mathsf{W}^T\right)\ \mathsf{P} = \mathsf{V}\left(\mathsf{W}^T\mathsf{P}\right) = \mathsf{P}^4\ . \end{array} \tag{60}$$

Considering eq. 60, we see that the sequence P^n is stationary for $n \geq 4$, which allows us to straightforwardly extend the previous results to this multi-level detection/confirmation modeling.

6 Conclusion

A novel architecture for detecting target maneuvers has been presented. This architecture is defined at the system level and can use any tracking approach. Main advantages are robustness and satisfaction of operational requirements. It basically exploits temporal trends in threshold exceeds. The DTMC framework is the backbone of our analysis and is sufficiently general to handle a variety of extensions.

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