Closed-form Posterior Cramér-Rao Bound for a Manoeuvring Target in the Bearings-Only Tracking Context Using Best-Fitting Gaussian Distribution

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Abstract - ***In this paper, we investigate the problem of the computation of the Posterior Cramér-Rao Bound (PCRB) in the context of Bearings-Only Tracking (BOT) for a manoeuvering target. The PCRB provides a lower bound on the mean square error. In a recent paper, Hernandez et al have proposed a new approach named Best-Fitting Gaussian (BFG) model to calculate the bound for Jump Markov Linear filtering problems with a linear measurement equation. Thanks to the linear property of the measurement equation, an exact formula for the PCRB associated to the BFG model can be obtained via a classical Riccati-like recursion. However, in the BOT framework, the measurement equation is non linear so that we do not have a closed-form formula. Consequently, the BFG-PCRB must be approximated using Monte-Carlo methods. This implies a high computational burden. We show in this paper that the BFG model associated to the BOT problem can be computed exactly using another coordinate system named Log Polar Coordinate (LPC) system.***

Keywords: Bearings-Only Tracking, Manoeuvring Target, Posterior Cramér-Rao bound, Best-Fitting Gaussian Distribution, Performance Analysis.

Notation

BOT: Bearings-Only Tracking,

LP(C): Logarithmic Polar(Coordinates),

x^* : denotes the transpose of matrix X,

x_k: is the target state in the Cartesian coordinate system,

y_k: is the target state in the LPC system,

0_n: $n \times n$ matrix composed of zero elements.

e_i: column vector where each component is zero except component i which is equal to one.

$P_{i,j}$: the matrix $P_{i,j}$ is defined by $P_{i,j} = e_i e_j^*$

1 Introduction

In many applications (submarine tracking, aircraft surveillance), a bearings-only sensor is used to collect observations about target trajectory. This problem of tracking has been of interest for the past thirty years. The aim of Bearings-Only Tracking (BOT) is to determine the target trajectory using noise-corrupted bearing measurements from a single observer. Target motion is classically described by a diffusion model[1] so that the filtering problem is composed of two stochastic equations. The first one represents the temporal evolution of the target state (position and velocity) called state equation. The second one links the bearing measurement to the target state at time k (measurement equation).

As far as performance analysis is concerned, the Posterior Cramér-Rao Bound (PCRB) proposed in [2] is widely used to assess the performance of filtering algorithms, by the tracking community ([3, 4, 5, 6]) and in particular in the bearings-only context ([7, 8, 9]). The PCRB gives a lower bound for the Error Covariance Matrix (ECM). More precisely, the PCRB is the inverse of the Fisher Information Matrix (FIM). A seminal contribution on performance analysis is the paper from Tichavský et al. [10]. Here, the authors noticed that only the right lower block of the FIM inverse was of interest for investigating tracking performance. This was the key idea for deriving a practical updating formula for the PCRB through time. Recently, the PCRB has been used for various sensor management problems like automating the deployment of sensors in [11] or determining the optimal sensor trajectory in the bearings-only context in [12]. Moreover, PCRB can be used to schedule active measurements in a system involving active and passive subsystems.

Tichavský's recursive formula is a powerful result to compute the right lower block of the FIM inverse. However, complex integrals without any closed-forms are involved in this recursion. So, these complex integrals must be approximated via Monte-Carlo methods. This approach is quite feasible but induces high computation requirements which highly reduces its suitability for complex problems

[1] see [1] for an exhaustive review on dynamic models

like sensor management. For instance, the aim of active measurement scheduling consists in optimizing the time distribution of range measurements to obtain an accurate target state estimate. It implies to perform Monte-Carlo evaluations of the PCRB for each policy, which would rapidly become infeasible. In the BOT case, Brehard et al have shown in [13] that the complex integrals required for calculating the PCRB admit closed-form expressions if the PCRB is derived in the Logarithmic Polar Coordinate (LPC) system. Remarkably, though this coordinate system is only a slight modification of the Modified Polar Coordinate (MPC) system [14], it allows instrumental simplifications in the calculation of the elementary terms of the PCRB recursion.

However, at this time, this approach is only convenient for the simplest diffusion model: the nearly constant velocity target model. The aim of this paper is to show that this approach can be extended to assess the performance of a manoeuvering target modelized by a Jump Markov linear model. The idea consists in using a general approach named Best Fitting Gaussian Distribution developed in a recent paper by Hernandez et al in [15]. In this paper, the authors investigate the computation of the PCRB for the Jump Markov Linear Model with a linear measurement equation. The idea consists in approximating this model by the best-fitting Gaussian distribution. This approach has two major advantages. First, this bound is more consistent with the performances of the Variable Structure Interacting Multiple Model (VS-IMM) tracker classically utilized. Second, the simple form of the BFG model and the linearity of the measurement equation imply that the Tichavský's recursive formula becomes a standard Riccati-like recursion so that the computation burden is small. In this paper, the BFG approach is applied to the BOT problem. However the non linearity of the measurement equation implies that some terms of Tichavský recursive formula must be approximated by Monte-Carlo methods. We show in this paper that this problem can be avoided. More precisely, this bound can be computed exactly and rapidly using a coordinate system developed by Brehard et al in [13]. More generally, this result is an extension of [13] to a more complex diffusion model.

In section 1, the specification of the model is presented in a general framework. In section 2, the problem of the computation of the PCRB is investigated. The classical method as well as the BFG approach used to compute the bound are presented. In section 3, a closed-form PCRB for the BFG model in the context of BOT tracking is proposed.

2 Specification of the model

Let x_k be the target state at time k. We consider a jump Markov linear equation given by the following equation:

$$x_{k+1} = F_k^{m_k} x_k + u_k + w_k^{m_k} \tag{1}$$

where $w_k^{m_k} \sim \mathcal{N}\left(0, Q^{r_k}\right)$ and $\{m_k\}_{k\in\mathbb{N}}$ is a finite, time-homogeneous Markov chain with transitions probabilities $\pi_{ij} \triangleq \mathbb{P}(m_{k+1} = j | m_k = i)$. Variable m_k specifies the target motion and u_k the known relative manoeuvres of the observer. Otherwise, we note z_k the measurement received at time k. The target state is related to this measurement through the following equation:

$$z_k = h(x_k) + v_k \tag{2}$$

where $v_k \sim \mathcal{N}(0, \sigma_\beta^2)$ and σ_β^2 is known. Equations (1) and (2) form a non linear filtering problem. If h is a linear function, the posterior distribution $p(x_k|z_{1:k})$ can then be estimated using a variable structure interacting multiple model [6], else a sequential Monte-Carlo method [16] should be used. The problem of the computation of the PCRB for this general model is investigated in the following section.

3 How to compute the bound ?

3.1 Tichavský's formula

The PCRB gives a lower bound for the error covariance matrix:

$$\begin{aligned} ECM_k &\triangleq \mathbb{E}\{(\hat{x}_k - x_k)(\hat{x}_k - x_k)^*\} && (3)\\ &\succcurlyeq J_k^{-1}\,. && (4)\end{aligned}$$

where $\hat{x}_k$ is the estimate and J_k is the right lower block of the FIM inverse. This classical result is proved in [2]. To compute J_k, Tichavský et al. have proposed in [10] a recursive formula:

$$J_{k+1} = D_k^{22} + D_k^{33} - D_k^{21}\left(J_k + D_k^{11}\right)^{-1} D_k^{12}, \tag{5}$$

where D_k^{11}, D_k^{12}, D_k^{21}, D_k^{22}, D_k^{33} are defined by:

$$\begin{aligned} D_k^{11} &\triangleq \mathbb{E}\{\nabla_{x_k} \ln p(x_{k+1}|x_k) \nabla_{x_k}^* \ln p(x_{k+1}|x_k)\}\,,\\ D_k^{21} &\triangleq \mathbb{E}\{\nabla_{x_{k+1}} \ln p(x_{k+1}|x_k) \nabla_{x_k}^* \ln p(x_{k+1}|x_k)\}\,,\\ D_k^{12} &\triangleq \mathbb{E}\{\nabla_{x_k} \ln p(x_{k+1}|x_k) \nabla_{x_{k+1}}^* \ln p(x_{k+1}|x_k)\}\,,\\ D_k^{22} &\triangleq \mathbb{E}\{\nabla_{x_{k+1}} \ln p(x_{k+1}|x_k) \nabla_{x_{k+1}}^* \ln p(x_{k+1}|x_k)\}\,,\\ D_k^{33} &\triangleq \mathbb{E}\{\nabla_{x_{k+1}} \ln p(z_{k+1}|x_{k+1}) \nabla_{x_{k+1}}^* \ln p(z_{t+1}|x_{k+1})\}\,. \end{aligned} \tag{6}$$

Looking at eq.(1),one can remark that the PDF associated to x_{k+1} given x_k noted $p(x_{k+1}|x_k)$ has not a simple form so that D_k^{11}, D_k^{12}, D_k^{21}, D_k^{22} do not have closed-forms. A classical solution [6] consists in conditioning on the manoeuvre sequence $m_{1:k} \triangleq \{m_1, \ldots, m_k\}$. Following this approach, we obtain

$$\begin{aligned} MCE_k &= \mathbb{E}\{\mathbb{E}\{(\hat{x}_k - x_k)(\hat{x}_k - x_k)^*|m_{1:k}\}\} && (7)\\ &\succcurlyeq \mathbb{E}\{J_k^{-1}(m_{1:k})\} && (8)\end{aligned}$$

where $J_k^{-1}(m_{1:k})$ is the right lower of the FIM inverse conditionally to a motion sequence $m_{1:k}$. Now, this quantity can be approximated using Monte-Carlo methods.

$$\mathbb{E}\{J_k^{-1}(m_{1:k})\} \approx \frac{1}{m_{1:k}^{(i)}} \sum_{i=1}^{I} J_k^{-1}(m_{1:k}^{(i)}) \tag{9}$$

where $\{m_{1:k}^{(i)}\}_{i\in\{1,\ldots,I\}}$ is a set of I motion sequence realizations. They are sampled independently using the transition probabilities π_{ij} of the Markov chain $m_{1:k}$. However, there are some hard limitations to this method. First, $J_k^{-1}(m_{1:k}^{(i)})$ must be computed for all i. The utilization of the Monte-Carlo method implies a high computational burden. Second, Hernandez et al. have shown in [15] that this bound is over-optimistic because each bound $J_k^{-1}(m_{1:k}^{(i)})$ calculated assumes that the sequence of manoeuvres is known. Consequently, Hernandez et al. have proposed a new approach to calculate J_k^{-1} for a manoeuvering target which avoids these problems, this is the BFG distribution.

3.2 Best-Fitting Gaussian Distribution

The idea of the BFG distribution consists in replacing the multiple diffusion model given by eq.(1) by a single best-fitting model so that this model has the same mean and covariance under each model. One can show that the best-fitting model associated to eq.(1) is

$$x_{k+1} = F_k x_k + u_k + w_k \tag{10}$$

where $w_k \sim \mathcal{N}(0, Q_k)$ with

$$\begin{aligned} F_k &= \sum_{m_k=1}^{M} F_k^{m_k} p_{m_k} && (11) \\ Q_k &= C_{k+1} - F_k C_k F_k^* && (12) \end{aligned}$$

and

$$\begin{aligned} C_{k+1} &= \sum_{m_k=1}^{M} [F_k^{m_k}(C_k + \epsilon_k \epsilon_k^*)(F_k^{m_k})^* + Q_k^{m_k}] \\ &\quad - F_k \epsilon_k \epsilon_k^* F_k^* && (13) \\ \epsilon_{k+1} &= F_k \epsilon_k && (14) \\ p_{m_k} &\triangleq \mathbb{P}(m_k) && (15) \end{aligned}$$

The proof of this result is given in [15]. The problem is now to compute the bound for the BFG filtering problem formed by equations (10) and (2). The idea consists in applying Tichavský's formula to this filtering problem. Contrary to Hernandez et al 's paper, we assume that the measurement equation (2) is non linear so that Tichavský's formula does not become a simple Riccati-like recursion. Now, this point is precised. First, thanks to the linear property of diffusion equation (10), D_k^{11}, D_k^{12}, D_k^{21} and D_k^{22} have closed-forms. However, in eq.(2), h is a non linear function so that no closed-form can be derived for D_k^{33}. This implies to use Monte-Carlo methods to approximate this last term and therefore induces a high computational burden. We show in the next section that this problem can be avoided in the BOT context by using another coordinate system.

4 PCRB and Bearings-Only Tracking

We show in this section that the PCRB can be computed exactly for the best-fitting Gaussian model in the bearings-only context. First of all, let us precise the filtering problem in the BOT context.

4.1 Bearings-Only Tracking

Historically, BOT is presented in the Cartesian system. Let us define target state at time k:

$$x_k = \begin{bmatrix} r_x(k) & r_y(k) & v_x(k) & v_y(k) \end{bmatrix}^*, \tag{16}$$

made of target relative velocity and position in the $x - y$ plane. Classically, the jump Markov linear model is composed of two types of diffusion models. The first one is a nearly constant velocity model $(m_k = 1)$ characterized by the following equation:

$$x_{k+1} = F_k^1 x_k + u_k + w_k^1 \tag{17}$$

where

$$\begin{aligned} F_k^1 &= \begin{pmatrix} 1 & 0 & \delta_k & 0 \\ 0 & 1 & 0 & \delta_k \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} && (18) \\ Q_k^1 &= \sigma^2 \begin{pmatrix} \frac{\delta_k^3}{3} & 0 & \frac{\delta_k^2}{2} & 0 \\ 0 & \frac{\delta_k^3}{3} & 0 & \frac{\delta_k^2}{2} \\ \frac{\delta_k^2}{2} & 0 & \delta_k & 0 \\ 0 & \frac{\delta_k^2}{2} & 0 & \delta_k \end{pmatrix} && (19) \end{aligned}$$

and a constant-turn model $(m_k = 2)$ characterized by:

$$x_{k+1} = F_k^2 x_k + u_k + w_k^2 \tag{20}$$

$$\begin{aligned} F_k^2 &= \begin{pmatrix} 1 & 0 & \frac{\sin w\delta_k}{w} & \frac{\cos w\delta_k - 1}{w} \\ 0 & 1 & \frac{1-\cos w\delta_k}{w} & \frac{\sin w\delta_k}{w} \\ 0 & 0 & \cos w\delta_k & -\sin w\delta_k \\ 0 & 0 & \sin w\delta_k & \cos w\delta_k \end{pmatrix} && (21) \\ Q_k^2 &= 0_4 && (22) \end{aligned}$$

where δ_k is the time interval and w the turn rate. The term 0_4 is defined in the notation sequence. Of course, one can consider a model with more than one constant turn model. All the results presented in this paper can be used for an arbitrary number of models.

Otherwise, z_k is the bearing measurement received at time k. The target state is related to this measurement through the following equation:

$$z_k = \arctan\left(\frac{r_x(k)}{r_y(k)}\right) + V_k \tag{23}$$

where $v_k \sim \mathcal{N}(0, \sigma_\beta^2)$ and σ_β^2 is known.

We show in this section that a closed-form PCRB for the best-fitting model in the context of the bearings-only tracking problem can be derived. The idea is to use a different coordinate system named Log Polar Coordinate system. It has been introduced in [13]:

$$y_k = \begin{bmatrix} \beta_k & \rho_k & \dot{\beta}_k & \dot{\rho}_k \end{bmatrix}^* \tag{24}$$

with

$$\rho_k = \ln r_k \tag{25}$$

where β_k and r_k are the relative bearing and range. Let f_{lp}^c and f_c^{lp} be respectively LPC-to-Cartesian and Cartesian-to-LPC state mapping functions so that:

$$x_k = \begin{cases} f_{lp}^c(y_k) \text{ if } r_y(k) > 0 \\ -f_{lp}^c(y_k) \text{ if } r_y(k) < 0 \end{cases} \tag{26}$$

with

$$f^c_{lp}(y_k) = r_k \begin{bmatrix} \sin\beta_k \\ \cos\beta_k \\ \dot\beta_k\cos\beta_k + \dot\rho_k\sin\beta_k \\ -\dot\beta_k\sin\beta_k + \dot\rho_k\cos\beta_k \end{bmatrix} \qquad (27)$$

and

$$y_k = f^{lp}_c(x_k) = \begin{bmatrix} \arctan\left(\frac{r_x(k)}{r_y(k)}\right) \\ \ln\left(\sqrt{r_x^2(k)+r_y^2(k)}\right) \\ \frac{v_x(k)r_y(k)-v_y(k)r_x(k)}{r_x^2(k)+r_y^2(k)} \\ \frac{v_x(k)r_x(k))+v_y(k)r_y(k)}{r_x^2(k)+r_y^2(k)} \end{bmatrix} . \qquad (28)$$

4.2 Calculating the Bound

From now, all the problem is expressed using the LPC system. Consequently, we calculate the lower bound for the covariance error matrix in this framework.

$$\begin{aligned} ECM_k &\triangleq \mathbb{E}\{(\hat y_k - y_k)(\hat y_k - y_k)^*\} && (29)\\ &\succcurlyeq J_k^{-1} . && (30)\end{aligned}$$

where $\hat y_k$ is the estimate and J_k is the right lower block of the FIM inverse. Tichavský's formula must also be rewritten in the LPC system.

$$J_{k+1} = D^{22}_k + D^{33}_k - D^{21}_k\left(J_k + D^{11}_k\right)^{-1} D^{12}_k, \qquad (31)$$

where D^{11}_k, D^{12}_k, D^{21}_k, D^{22}_k, D^{33}_k are defined by:

$$\begin{aligned} D^{11}_k &\triangleq \mathbb{E}\{\nabla_{y_k}\ln p(y_{k+1}|y_k)\nabla^*_{y_k}\ln p(y_{k+1}|y_k)\}\ ,\\ D^{21}_k &\triangleq \mathbb{E}\{\nabla_{y_{k+1}}\ln p(y_{k+1}|y_k)\nabla^*_{y_k}\ln p(y_{k+1}|y_k)\}\ ,\\ D^{12}_k &\triangleq \mathbb{E}\{\nabla_{y_k}\ln p(y_{k+1}|y_k)\nabla^*_{y_{k+1}}\ln p(y_{k+1}|y_k)\}\ ,\\ D^{22}_k &\triangleq \mathbb{E}\{\nabla_{y_{k+1}}\ln p(y_{k+1}|y_k)\nabla^*_{y_{k+1}}\ln p(y_{k+1}|y_k)\}\ ,\\ D^{33}_k &\triangleq \mathbb{E}\{\nabla_{y_{k+1}}\ln p(z_{k+1}|y_{k+1})\nabla^*_{y_{k+1}}\ln p(z_{k+1}|y_{k+1})\}. \end{aligned} \qquad (32)$$

We will show that all the terms in (32) allow closed-forms. There are two reasons for this peculiar property. First, the coordinate system includes β_k. Consequently $p(z_{t+1}|y_{k+1}) = p(z_{t+1}|\beta_{k+1})$ has a simple expression so that D^{33}_k has a closed-form. Second, we show in appendix A that gradients $\nabla_{y_k}\ln p(y_{k+1}|y_k)$ and $\nabla_{y_{k+1}}\ln p(y_{k+1}|y_k)$ are quadratic forms in x_k, x_{k+1}. Indeed, we have:

$$\left\{ \begin{array}{l} \nabla^*_{y_k}\ln p(y_{k+1}|y_k) = \mathcal{E}^*_{k+1}Q_k^{-1}F_k\mathcal{F}^*_{x_k}\ ,\\ \nabla^*_{y_{k+1}}\ln p(y_{k+1}|y_k) = -\mathcal{E}^*_{k+1}Q_k^{-1}\mathcal{F}^*_{x_{k+1}} + 4e_2^*\ , \end{array}\right. \qquad (33)$$

where

$$\mathcal{E}_{k+1} = x_{k+1} - F_k x_k - u_k \qquad (34)$$

and

$$\mathcal{F}^*_{x_k} \triangleq \nabla_{y_k}\{x_k\}\ . \qquad (35)$$

The term e_2 is defined in the notation section. $\mathcal{F}^*_{x_k}$ is the LPC-to-Cartesian mapping function derivatives at time k (f^c_{lp} is given by eq.(27)). This term can be expressed using the Cartesian framework:

$$\mathcal{F}^*_{x_k} = \begin{bmatrix} r_y(k) & r_x(k) & 0 & 0\\ -r_x(k) & r_y(k) & 0 & 0\\ v_y(k) & v_x(k) & r_y(k) & r_x(k)\\ -v_x(k) & v_y(k) & -r_x(k) & r_y(k)\end{bmatrix} \qquad (36)$$

Consequently, $\mathcal{F}^*_{x_k}$ is a linear operator.

The linear property is the key point to derive closed-forms. First of all, one can rewrite

$$\begin{aligned} D^{11}_k &= \mathbb{E}\left\{\mathcal{F}^*_{x_k}F^*_kQ_k^{-1}F_k\mathcal{F}_{x_k}\right\}\\ D^{12}_k &= \mathbb{E}\left\{\mathcal{F}^*_{x_k}F^*_kQ_k^{-1}\mathcal{F}_{F_kx_k}\right\} - \Upsilon^{12}_k\\ D^{22}_k &= \mathbb{E}\left\{\mathcal{F}^*_{F_kx_k}Q_k^{-1}\mathcal{F}_{F_kx_k}\right\} + \mathcal{C}_k + \Upsilon^{22}_k\\ D^{33}_k &= \begin{bmatrix}\frac{1}{\sigma^2_\beta} & 0 & 0 & 0\\ 0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \end{aligned} \qquad (37)$$

with

$$\begin{aligned} \Upsilon^{12}_k &= \mathcal{F}^*_{\mathbb{E}x_k}F^*_kQ_k^{-1}\mathcal{F}_{\mathbb{E}x_{k+1}} - \mathcal{F}^*_{\mathbb{E}x_k}F^*_kQ_k^{-1}\mathcal{F}_{F_k\mathbb{E}x_k}\ ,\\ \Upsilon^{22}_k &= \mathcal{F}^*_{\mathbb{E}x_{k+1}}Q_k^{-1}\mathcal{F}_{\mathbb{E}x_{k+1}} - \mathcal{F}^*_{F_k\mathbb{E}x_k}Q_k^{-1}\mathcal{F}_{F_k\mathbb{E}x_k}\ ,\\ \mathcal{C}_k &= \mathbb{E}\{\mathcal{F}^*_{\mathcal{E}_{k+1}}Q_k^{-1}\mathcal{E}_{k+1}\mathcal{E}^*_{k+1}Q_k^{-1}\mathcal{F}_{\mathcal{E}_{k+1}}\}\\ &\quad - \mathbb{E}\{\mathcal{F}^*_{\mathcal{E}_{k+1}}Q_k^{-1}\mathcal{E}_{k+1}\}4e_2^*\\ &\quad - 4e_2\mathbb{E}\{\mathcal{E}^*_{k+1}Q_k^{-1}\mathcal{F}_{\mathcal{E}_{k+1}}\} + 16e_2e_2^*\\ \mathcal{E}_{k+1} &= x_{k+1} - F_kx_k - u_k \end{aligned} \qquad (38)$$

and

$$\mathcal{F}^*_{\mathbb{E}x_k} \triangleq \qquad (39)$$

$$\begin{bmatrix} \mathbb{E}\{r_y(k)\} & \mathbb{E}\{r_x(k)\} & 0 & 0\\ -\mathbb{E}\{r_x(k)\} & \mathbb{E}\{r_y(k)\} & 0 & 0\\ \mathbb{E}\{v_y(k)\} & \mathbb{E}\{v_x(k)\} & \mathbb{E}\{r_y(k)\} & \mathbb{E}\{r_x(k)\}\\ -\mathbb{E}\{v_x(k)\} & \mathbb{E}\{v_y(k)\} & -\mathbb{E}\{r_x(k)\} & \mathbb{E}\{r_y(k)\}\end{bmatrix}$$

Result (37) is proved in appendix B. First, let us notice that Υ^{12}_k and Υ^{22}_k can be easily calculated. We can remark that the latter is zero if u_k is zero. If this condition is not verified, $\mathbb{E}(x_k)$ is computed for any value of k using $\mathbb{E}(x_0)$ and the relation $\mathbb{E}(x_k) = F_k\mathbb{E}(x_{k-1}) + u_{k-1}$. Now using one more time the linear property of $\mathcal{F}$, we prove in appendix C that $\mathcal{C}_k$ can computed exactly via the following formula:

$$\mathcal{C}_k = g_2(Q_k) - 4g_1(Q_k)e_2^* - 4e_2g_1^*(Q_k) + 16e_2e_2^* \qquad (40)$$

where

$$g_1(Q_k) = \sum_{i,j=1}^{4} I_i^*Q_k^{-1}e_jQ_k(i,j) \qquad (41)$$

and

$$\begin{aligned} g_2(Q_k) &= \sum_{i_1,j_1,i_2,j_2=1}^{4} I^*_{i_1}Q_k^{-1}P_{(j_1,j_2)}Q_k^{-1}I_{i_2}\\ &\quad\times\ (Q_k(i_1,j_1)Q_k(i_2,j_2) + Q_k(i_1,j_2)Q_k(i_2,j_1)) \end{aligned} \qquad (42)$$

where matrices $\{I_j\}_{j\in\{1,\ldots,4\}}$ are defined in tab.1 and $\{P^{(i,j)}\}_{i,j\in\{1,\ldots,4\}}$ in the notation section. We derive the final closed-forms for D^{11}_k, D^{12}_k, D^{22}_k in the three following sections. The final algorithm is given by figure 1.

4.2.1 Closed-form for D^{11}_k

We prove in the appendix B that D^{11}_k given by (37) can be rewritten

$$D^{11}_k = \mathbb{E}\left\{\mathcal{F}^*_{x_k}F^*_kQ_k^{-1}F_k\mathcal{F}_{x_k}\right\} \qquad (43)$$

Initialization of J_1 and $Cov(x_0)$
For k=1,...

1. Computation of F_k and Q_k using (11) and (12)
2. Computation of $Cov(x_k)$

$$Cov(x_k) = F_k Cov(x_{k-1}) F_k^* + Q_k$$

3. Computation of $\mathcal{C}_k$ using (40)
4. Computation of Υ_k^{12} and Υ_k^{22} using eq.(38)
5. Computation of

$$D_k^{11}=\sum_{i=1}^{4}\sum_{j=1}^{4} I_i^* F_k^* Q_k^{-1} F_k I_j \mathbb{E}(x_k(i)x_k(j))$$

$$D_k^{12}=\sum_{i=1}^{4}\sum_{j=1}^{4} I_i^* F_k^* Q_k^{-1} \mathcal{I}_j \mathbb{E}(x_k(i)x_k(j)) - \Upsilon_k^{12}$$

$$D_k^{22}=\sum_{i=1}^{4}\sum_{j=1}^{4} \mathcal{I}_i^* Q_k^{-1} \mathcal{I}_j \mathbb{E}(x_k(i)x_k(j)) + \mathcal{C}_k + \Upsilon_k^{22}$$

$$D_k^{33}=diag(\sigma_\beta^2, 0, 0, 0)$$

Remark : constants $\{I_1\}_{1,\dots,4}$ and $\{\mathcal{I}_1\}_{1,\dots,4}$ are respectively given in Tab.1 and 2.

6. Computation of J_{k+1} using eq.(31)

End For

Figure 1: *Computation of the PCRB*

Using the linear property of operator $\mathcal{F}$, we obtain the following decomposition:

$$\mathcal{F}_{x_k} = \sum_{i=1}^{4} I_i x_k(i) \tag{44}$$

Constants matrices I_1, I_2, I_3 and I_4 given in Tab.1 are derived from the definition of operator $\mathcal{F}$ given by eq.(36). Now D_k^{11} can be rewritten:

$$D_k^{11} = \sum_{i=1}^{4}\sum_{j=1}^{4} I_i^* F_k^* Q_k^{-1} F_k I_j \mathbb{E}(x_k(i)x_k(j)) \tag{45}$$

The terms $\mathbb{E}(x_k(i)x_k(j))$ are the elements of the covariance matrix $Cov(x_k)$. This matrix can be obtained using the recursive formula:

$$Cov(x_k) = F_k Cov(x_{k-1}) F_k^* + Q_k \tag{46}$$

derived from eq.(10).

4.2.2 Closed-form for D_k^{12}

We prove in the appendix B that D_k^{12} given by (37) can be rewritten

$$D_k^{12} = \mathbb{E}\left\{\mathcal{F}_{x_k}^* F_k^* Q_k^{-1} \mathcal{F}_{F_k x_k}\right\} - \Upsilon_k^{12} \tag{47}$$

where Υ_k^{12} is given by eq.(38). Using the linear property of operator $\mathcal{F}$, we have:

$$\mathcal{F}_{F_k x_k} = \sum_{i=1}^{4} \mathcal{I}_i x_k(i) \tag{48}$$

where $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ and $\mathcal{I}_4$ are constant matrices. Looking at eq.(48), let us remark that that the values of these constant terms depend on F_k. Tab.2 gives the values of theses constant matrices for a target that can switch between a nearly-constant velocity model and a constant turn model as presented by (18) and (21). Now using (44) and (48), D_k^{12} can be rewritten:

$$D_k^{12} = \sum_{i=1}^{4}\sum_{j=1}^{4} I_i^* F_k^* Q_k^{-1} \mathcal{I}_j \mathbb{E}(x_k(i)x_k(j)) - \Upsilon_k^{12} \tag{49}$$

The terms $\mathbb{E}(x_k(i)x_k(j))$ are the elements of the covariance matrix $Cov(x_k)$. This matrix can be obtained using the recursive formula given by (46).

4.2.3 Closed-form for D_k^{22}

We prove in the appendix B that D_k^{22} given by (37) can be rewritten:

$$D_k^{22} = \mathbb{E}\left\{\mathcal{F}_{F_k x_k}^* Q_k^{-1} \mathcal{F}_{F_k x_k}\right\} + \mathcal{C}_k + \Upsilon_k^{22} \tag{50}$$

where Υ_k^{22} and $\mathcal{C}_k$ are given by eq.(38). Now D_k^{22} can be rewritten:

$$D_k^{22} = \sum_{i=1}^{4}\sum_{j=1}^{4} \mathcal{I}_i^* Q_k^{-1} \mathcal{I}_j \mathbb{E}(x_k(i)x_k(j)) + \mathcal{C}_k + \Upsilon_k^{22} \tag{51}$$

using (48). The terms $\mathbb{E}(x_k(i)x_k(j))$ are the elements of the covariance matrix $Cov(x_k)$. This matrix can be obtained using the recursive formula given by (46).

5 Conclusion

In this paper, we have considered the problem of calculating the PCRB in the case of a manoeuvering target in the BOT context. In a recent paper, Bréhard et al have shown that a closed-form PCRB can be derived for the nearly-constant velocity model. We have proved in this paper that this approach can be extended to the manoeuvering context via the Best-Fitting Gaussian approach which has been proposed in a recent paper by Hernandez et al [15].

Along this paper, strong results were shown with regards to the PCRB calculation; namely we derived original closed-form PCRB. This power result cascades down from an original frame that consists in a new coordinate system: the Logarithmic Polar Coordinate system. Computing the PCRB then becomes an accurate and time-varying technique of particular interest for real-time sensor management issues.

Appendix A: proof of eq.(33)

First, it is necessary to derive $p(y_{k+1}|y_k)$. Bréhard et al have shown in [13] that

$$p(y_{k+1}|y_k) = r_{k+1}^4 p(x_{k+1}|x_k)\alpha(y_k)\ . \tag{52}$$

where:

$$\alpha(y_k) = \mathbb{P}(r_y(k) > 0|y_k)\mathbb{1}_{\{r_y(k)>0\}} + \mathbb{P}(r_y(k) < 0|y_k)\mathbb{1}_{\{r_y(k)<0\}} . \quad (53)$$

Remarking that $\nabla_{y_k}\alpha(y_k) = 0$, we obtain:

$$\begin{aligned} \nabla_{y_k} \ln p(y_{k+1}|y_k) &= \mathcal{F}^*_{x_k} A^* Q_k^{-1} \mathcal{E}_{k+1} , \\ \nabla_{y_{k+1}} \ln p(y_{k+1}|y_k) &= -\mathcal{F}^*_{x_{k+1}} Q_k^{-1} \mathcal{E}_{k+1} + 4e_2 . \end{aligned} \quad (54)$$

where

$$\mathcal{F}^*_{x_k} \triangleq \nabla_{y_k}\{x_k\} , \quad (55)$$

$$\mathcal{E}_{k+1} \triangleq x_{k+1} - F_k x_k - u_k \quad (56)$$

Appendix B: proof of eq.(37)

Considering at D_k^{11}, D_k^{12} and D_k^{22} formulas given by eq.(32), incorporating $\nabla_{y_k} \ln p(y_{k+1}|y_k)$, $\nabla_{y_{k+1}} \ln p(y_{k+1}|y_k)$ given by (33), we obtain:

$$\begin{cases} D_k^{11} = \mathbb{E}\left\{\mathcal{F}^*_{x_k} F_k^* Q_k^{-1}\mathcal{E}_{k+1}\mathcal{E}^*_{k+1}Q_k^{-1}F_k\mathcal{F}_{x_k}\right\} , \\ D_k^{12} = -\mathbb{E}\left\{\mathcal{F}^*_{x_k} F_k^* Q_k^{-1}\mathcal{E}_{k+1}\mathcal{E}^*_{k+1}Q_k^{-1}\mathcal{F}_{x_{k+1}}\right\} , \\ D_k^{22} = \mathbb{E}\left\{\mathcal{F}^*_{x_{k+1}} Q_k^{-1}\mathcal{E}_{k+1}\mathcal{E}^*_{k+1}Q_k^{-1}\mathcal{F}_{x_{k+1}}\right\} \\ \quad - 4\mathbb{E}\left\{\mathcal{F}^*_{x_{k+1}} Q_k^{-1}\mathcal{E}_{k+1}\right\} e_2 \\ \quad - 4e_2^*\mathbb{E}\left\{\mathcal{E}^*_{k+1}Q_k^{-1}\mathcal{F}_{x_{k+1}}\right\} + 16 e_2 e_2^* . \end{cases} \quad (57)$$

Now, we are dealing with the calculation of each elementary term of eq.(57) separately.

D_k^{11} formula

Let us rewrite D_k^{11} as given by eq.(57), we have:

$$\begin{aligned} D_k^{11} &= \mathbb{E}\left\{\mathcal{F}^*_{x_k} A^* Q_k^{-1}\mathcal{E}_{k+1}\mathcal{E}^*_{k+1}Q_k^{-1}F_k\mathcal{F}_{x_k}\right\} , \quad (58) \\ &= \mathbb{E}\{\mathcal{F}^*_{x_k} F_k^* Q_k^{-1} \underbrace{\mathbb{E}\left\{\mathcal{E}_{k+1}\mathcal{E}^*_{k+1}|x_k\right\}}_{=Q_k} Q_k^{-1} A\mathcal{F}_{x_k}\} . \end{aligned}$$

Then using the statistical property of x_{k+1} given x_k i.e. $\mathcal{N}\left(F_k x_k + u_k k, Q_k\right)$ given by eq.(10), we obtain D_k^{11} formula as given by eq.(37).

D_k^{12} formula

Our aim is now to render explicit D_k^{12} given by eq.(57). Let us first use the linear property of $\mathcal{F}_.$:

$$\begin{aligned} D_k^{12} &= -\overbrace{\mathbb{E}\left\{\mathcal{F}^*_{x_k} F_k^* Q_k^{-1}\mathcal{E}_{k+1}\mathcal{E}^*_{k+1}Q_k^{-1}\mathcal{F}_{\mathcal{E}_{k+1}}\right\}}^{=0} \quad (59) \\ &- \mathbb{E}\left\{\mathcal{F}^*_{x_k} F_k^* Q_k^{-1}\mathcal{E}_{k+1}\mathcal{E}^*_{k+1}Q_k^{-1}\mathcal{F}_{F_k x_k + u_k k}\right\} . \end{aligned}$$

Using the statistical property of x_{k+1} i.e x_{k+1} given x_k is a $\mathcal{N}\left(F_k x_k + u_k k, Q_k\right)$, we obtain:

$$D_k^{12} = -\mathbb{E}\left\{\mathcal{F}^*_{x_k} A^* Q_k^{-1}\mathcal{F}_{F_k x_k}\right\} - \mathcal{F}^*_{\mathbb{E}x_k} F_k^* Q_k^{-1}\mathcal{F}_{u_k k} . \quad (60)$$

Now remarking that $u_k = \mathbb{E}x_{k+1} - F_k x_k$ and the linearity of operator $\mathcal{F}$, we obtain D_k^{12} expression given by eq.(37).

D_k^{22} formula

Starting from D_k^{22} given by eq.(57) and using again the linearity of $F_.$:

$$\begin{aligned} D_k^{22} &= \overbrace{\mathbb{E}\left\{\mathcal{F}^*_{F_k x_k + u_k} Q_k^{-1}\mathcal{E}_{k+1}\mathcal{E}^*_{k+1}Q_k^{-1}\mathcal{F}_{\mathcal{E}_{k+1}}\right\}}^{=0} , \\ &+ \mathbb{E}\left\{\mathcal{F}^*_{F_k x_k + u_k} Q_k^{-1}\mathcal{E}_{k+1}\mathcal{E}^*_{k+1}Q_k^{-1}\mathcal{F}_{F_k x_k + u_k}\right\} \\ &+ \mathcal{C}_k \end{aligned} \quad (61)$$

with $\mathcal{C}_k$ is defined by eq.(38). Now, using again the statistical property of x_{k+1}, we obtain:

$$\begin{aligned} D_k^{22} &= \mathbb{E}\left\{\mathcal{F}^*_{F_k x_k + u_k k} Q_k^{-1}\mathcal{E}_{k+1}\mathcal{E}^*_{k+1}Q_k^{-1}\mathcal{F}_{F_k x_k + u_k}\right\} \\ &+ \mathcal{C}_k . \end{aligned} \quad (62)$$

To end the proof, the linearity of the operator $\mathcal{F}$ and the equality $u_k = \mathbb{E}x_{k+1} - F_k x_k$ allow us to infer eq.(37).

6 Appendix C: proof of eq.(40)

We derive here a closed-form expression for $\mathcal{C}_k$. First, let us define Ω_k:

$$\Omega_k \triangleq \mathcal{F}^*_{\mathcal{E}_{k+1}} Q_k^{-1}\mathcal{E}_{k+1} \quad (63)$$

Using this definition, $\mathcal{C}_k$ given by eq.(38) can rewritten:

$$\mathcal{C}_k = g_2(Q_k) - 4g_1(Q_k)e_2^* - 4e_2 g_1^*(Q_k) + 16 e_2 e_2^* \quad (64)$$

where

$$g_1(Q_k) = \mathbb{E}\{\Omega_k\} \quad (65)$$

$$g_2(Q_k) = \mathbb{E}\{\Omega_k\Omega_k^*\} \quad (66)$$

We can see that the main problem is to compute the two first moments of the random variable Ω_k. Using the linear property of operator $\mathcal{F}$ given by (44), Ω_k can be rewritten:

$$\Omega_k = \sum_{i=1}^{4} I_i^* Q_k^{-1}\epsilon_{k+1}(i)\epsilon_{k+1} \quad (67)$$

where constant matrices I_1, I_2, I_3, I_4 are given in Tab.1. Now using the following decomposition:

$$\epsilon_{k+1} = \sum_{i=1}^{4} e_i \epsilon_{k+1}^{(i)} \quad (68)$$

where e_i is defined in notation section. We obtain:

$$\Omega_k = \sum_{i,j=1}^{4} I_i^* Q_k^{-1} e_j \epsilon_{k+1}^{(i)}\epsilon_{k+1}^{(j)} \quad (69)$$

Now using the statistical properties of ϵ_{k+1} defined by eq.(34), we derive the first moment of Ω_k.

$$g_1(Q_k) = \sum_{i,j=1}^{4} I_i^* Q_k^{-1} e_j Q_k(i,j) \quad (70)$$

Now let us consider the second moment. Using eq.(68), we obtain:

$$\begin{aligned} g_2(Q_k) &= \sum_{i_1,j_1,i_2,j_2=1}^{4} I_{i_1}^* Q_k^{-1} P_{(j_1,j_2)} Q_k^{-1} I_{i_2} \\ &\times \mathbb{E}\{\epsilon_{k+1}^{(i_1)}\epsilon_{k+1}^{(j_1)}\epsilon_{k+1}^{(i_2)}\epsilon_{k+1}^{(j_2)}\} \end{aligned} \quad (71)$$

where $P^{(i,j)}$ is defined in notation section. We have now to calculate the fourth moment of a normal distribution. One can show

$$\mathbb{E}\{\epsilon_{k+1}^{(i_1)}\epsilon_{k+1}^{(j_1)}\epsilon_{k+1}^{(i_2)}\epsilon_{k+1}^{(j_2)}\} = Q_k(i_1,j_1)Q_k(i_2,j_2) + Q_k(i_1,j_2)Q_k(i_2,j_1) \quad (72)$$

This is a classical result which can be found in [17]. We obtain the final expression for the second moment.

Table 1: Constants I_1, I_2, I_3 and I_4.

$I_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	$I_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$I_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$	$I_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

Table 2: Constants $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ and $\mathcal{I}_4$.

$\mathcal{I}_1 = I_1$
$\mathcal{I}_2 = I_2$
$\mathcal{I}_3 = (I_1\delta_k + I_3)p_1 + \begin{pmatrix} \frac{1-\cos w\delta_k}{w} & \frac{\sin w\delta_k}{w} & 0 & 0 \\ -\frac{\sin w\delta_k}{w} & \frac{1-\cos w\delta_k}{w} & 0 & 0 \\ \sin w\delta_k & \cos w\delta_k & 0 & 0 \\ -\cos w\delta_k & \sin w\delta_k & 0 & 0 \end{pmatrix} p_2$
$\mathcal{I}_4 = (I_2\delta_k + I_4)p_1 + \begin{pmatrix} \frac{\sin w\delta_k}{w} & \frac{\cos w\delta_k - 1}{w} & 0 & 0 \\ -\frac{\cos w\delta_k - 1}{w} & \frac{\sin w\delta_k}{w} & 0 & 0 \\ \cos w\delta_k & \sin w\delta_k & 0 & 0 \\ \sin w\delta_k & \cos w\delta_k & 0 & 0 \end{pmatrix} p_2$

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