Closed-form Posterior Cramér-Rao Bound for Active Measurement Scheduling

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***Abstract*— We here address the classical bearings-only tracking problem (BOT) for a single object, which belongs to the general class of nonlinear filtering problems. Recently, algorithms based on sequential Monte Carlo methods (particle filtering) have been proposed. As far as performance analysis is concerned, the Posterior Cramér-Rao Bound (PCRB) provides a lower bound on the mean square error. Classically, the PCRB is given by the inverse Fisher Information Matrix (FIM). The latter is computed using Tichavský's recursive formula via Monte Carlo methods. Recently an exact algorithm to compute the PCRB via Tichavský's recursive formula without using Monte-Carlo methods have been derived by Bréhard et al. This result is based on a new coordinates system named Logarithmic Polar Coordinates (LPC) system. This paper illustrates that PCRB can now be computed accurately and quickly, making it suitable for active measurement scheduling.**

NOTATION

X_t: is the target state in the Cartesian coordinates system,

Y_t: is the target state in the LPC system,

n_y: size of the target state ($n_y = 4$),

$\succcurlyeq$: inequality $R \succcurlyeq S$ means that $R - S$ is a positive semi-definite matrix,

Id_n: $n \times n$ identity matrix,

$0_{n\times m}$: $n \times m$ matrix composed of zero element,

$\otimes$: Kronecker product,

X^*: denotes the transpose of matrix X.

δ: Dirac delta function,

Δ: Laplacian operator,

∇: gradient operator,

I. INTRODUCTION

In many applications (submarine tracking, aircraft surveillance), a bearings-only sensor is used to collect observations about target trajectory. This problem of tracking has been of interest for the past thirty years. The aim of Bearings-Only Tracking (BOT) is to determine the target trajectory using noise-corrupted bearing measurements from a single observer. Target motion is classically described by a diffusion model[1] so that the filtering problem is composed of two stochastic equations. The first one represents the temporal evolution of the target state (position and velocity) called state equation. The second one links the bearing measurement to the target state at time t (measurement equation).

One of the characteristics of the problem is the nonlinearity of the measurement equation so that the classical Kalman filter is not convenient in this case. We can find in literature two kinds of solutions to this problem. The first one, proposed

[1] see [1] for an exhaustive review on dynamic models

by Lindgren and Gong in [2], consists of deriving a pseudo-linear measurement equation. Then, a Kalman filter can be used to solve the problem. The stochastic stability analysis of the estimates is addressed by Song and Speyer in [3]. However, Aidala and Nardone show in [4] that this approach produces bias range estimate which can be reduced if the observer executes a maneuver. Consequently, bias range can be estimated as soon as it becomes observable [5]. A second idea consists of using the Extended Kalman Filter (EKF) in Cartesian coordinates system to solve the problem. However, simulations show that this algorithm is often divergent due to the weak observability of range ([6], [7], [8]). To remedy this problem, Aidala and Hammel in [9] proposed an EKF using another system named Modified Polar Coordinates (MPC) system whose one salient feature is that range is not coupled with the observable components. This constitutes a neat improvement. Another solution proposed by Peach in [10] is a range-parametrized EKF, in which a number of EKF trackers parametrized by range run in parallel. Recently, particle filtering algorithms have been proposed in this context ([11], [12], [13]). In [14], Arulampalam and Ristic compare the particle filter with the range-parametrized and EKF in MPC system; while a comprehensive overview of the state of art can be found in [15].

As far as performance analysis is concerned, the Posterior Cramér-Rao Bound (PCRB) proposed in [16] is widely used to assess the performance of filtering algorithms, by the tracking community ([17], [18], [19], [20]) and in particular in the bearings-only context ([15], [21], [22]). Moreover, PCRB has been recently used for various sensor management problems like automating the deployment of sensors in [23] or determining the optimal sensor trajectory in the bearings-only context in [24].

The PCRB gives a lower bound for the Error Covariance Matrix (ECM). More precisely, under a technical assumption, the PCRB is the inverse of the Fisher Information Matrix (FIM). A seminal contribution on performance analysis is the paper from Tichavský et al. [25]. Here, the authors noticed that only the right lower block of the FIM inverse was of interest for investigating tracking performance. This was the key idea for deriving a practical updating formula for the PCRB. However, complex integrals without any closed-forms are involved in this recursion. So, these complex integrals must be approximated via Monte Carlo methods. This approach is quite feasible but induces high computation requirements which highly reduces its suitability for complex problems like sensor management. For instance, optimal measurement scheduling (OMS) would imply to consider a large number of active measurement sequences and to perform Monte-Carlo evaluations of the PCRB for each sequence, which would rapidly become infeasible. Recently Bréhard et al have shown in [26] that the complex integrals required for calculating the PCRB admit closed-form expressions if the PCRB is derived in an original coordinates system named Logarithmic Polar Coordinates (LPC) system.

We investigate in this paper how this new result can be used to schedule active measurements in a system involving active and passive subsystems. Concerning OMS, The general problem of optimizing the time-distribution of measurements has a long history. Avitzour et al. in [27] have proposed an algorithm to optimize the time-distribution of measurements when estimating a scalar random variable by solving a nonquadratic minimization problem. This result has been extended by Shakeri et al in [28] to discrete-time stochastic processes. However, this previous approach is devoted to linear systems when the BOT is highly nonlinear. Then, Le Cadre has proposed to use the CRB to solve the problem in [29] for nonlinear systems where the state equation is deterministic. We show in this paper that a closed-form PCRB can be derived for active measurement scheduling based on Bréhard et al framework.

In section II, the BOT problem is presented in the Cartesian coordinates system and then in the LPC system. This original coordinates system is the key point to derive a closed-form for the PCRB. Calculation of closed-form expressions of the right lower block of the FIM inverse via Tichavský's recursive formula is addressed in section III, in the LPC setting. Then, the closed-form PCRB is investigated for scheduling active measurements in section IV. In section V, simulation results are presented.

II. From Cartesian to LPC system

A. Cartesian framework for BOT

Historically, BOT is presented in the Cartesian system. Let us define target state at time t:

$$X_t = \left[\begin{array}{cccc} r_x(t) & r_y(t) & v_x(t) & v_y(t) \end{array} \right]^*, \tag{1}$$

made of target relative velocity and position in the $x-y$ plane. It is assumed that the target follows a nearly constant-velocity model. The discretized state equation[2] is given by:

$$X_{t+1} = AX_t + HU_t + \sigma W_t \ , \tag{2}$$

where:

$$\left| \begin{array}{l} W_t \sim \mathcal{N}(0, Q) \ , \\ A = Id_4 + \delta_t B \text{ with } B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes Id_2, \\ H = \begin{bmatrix} \delta_t \\ 1 \end{bmatrix} \otimes Id_2 \ , \\ Q = \Sigma \otimes Id_2 \text{ with } \Sigma = \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_1 \end{bmatrix} \ . \end{array} \right.$$

δ_t is the elementary time period and U_t is the known difference between observer velocity at time $t+1$ and t.

Otherwise, we note Z_t the bearing measurement received at time t. The target state is related to this measurement through the following equation:

$$Z_t = \arctan\left(\frac{r_x(t)}{r_y(t)}\right) + V_t, \tag{3}$$

where $V_t \sim \mathcal{N}(0, \sigma_\beta^2)$ and σ_β^2 is known.

The system (2-3) has two components : a linear state equation (3) and a nonlinear measurement equation (3). Particle filter techniques (see [30],[31]) are, thus, particularly appealing. Otherwise, practical implementations of EKF-based algorithms ([9] and [10]) use a specific coordinates system, namely Modified Polar Coordinates (MPC). Indeed, if the target follows a deterministic trajectory (i.e. $W_t = 0 \quad \forall t \in \{0, \ldots, T\}$ in eq.(2)), Nardone and Aidala have demonstrated in [7] that no information on range exists as long as the observer is not maneuvering. So the idea consists of using a coordinates system for which unobservable component (range) is not coupled with the observable part. This is also the motivation of Aidala and Hammel [9] for defining the MPC system:

$$\left[\begin{array}{cccc} \beta_t & \frac{1}{r_t} & \dot{\beta}_t & \frac{\dot{r}_t}{r_t} \end{array} \right]^* . \tag{4}$$

Thus, the target state at time t is defined by eq.(4), where β_t and r_t are the relative bearing and target range. We present in the following section a slight modification of the MPC system, named **Logarithmic** Polar Coordinates (LPC) system. The only difference is that the second component is not $\frac{1}{r_t}$ but $\mathbf{ln}(r_t)$. Even if this tiny difference appears very minor, it is instrumental for deriving a closed-form of the PCRB. Let us now derive BOT equations given by eqs.(2,3) in the LPC framework.

B. LPC framework for BOT

We consider now that the system state Y_t is expressed in the Logarithmic Polar Coordinates (LPC) system, i.e. :

$$Y_t = \left[\begin{array}{cccc} \beta_t & \ln r_t & \dot{\beta}_t & \frac{\dot{r}_t}{r_t} \end{array} \right]^* . \tag{5}$$

As between Cartesian and MP system, we do not have a direct bijection between Cartesian and LPC system due to arctan function definition. We just have f_{lp}^c and f_c^{lp} respectively LPC-to-Cartesian and Cartesian-to-LPC state mapping functions such that:

$$X_t = \begin{cases} f_{lp}^c(Y_t) \text{ if } r_y(t) > 0 \\ -f_{lp}^c(Y_t) \text{ if } r_y(t) < 0 \end{cases} \tag{6}$$

with

$$f_{lp}^c(Y_t) = r_t \begin{bmatrix} \sin\beta_t \\ \cos\beta_t \\ \dot{\beta}_t \cos\beta_t + \frac{\dot{r}_t}{r_t}\sin\beta_t \\ -\dot{\beta}_t \sin\beta_t + \frac{\dot{r}_t}{r_t}\cos\beta_t \end{bmatrix} \tag{7}$$

[2] For a general review of dynamic models for target tracking see [1].

and

$$Y_t = f_c^{lp}(X_t) = \begin{bmatrix} \arctan\left(\frac{r_x(t)}{r_y(t)}\right) \\ \ln\left(\sqrt{r_x^2(t)+r_y^2(t)}\right) \\ \frac{v_x(t)r_y(t)-v_y(t)r_x(t)}{r_x^2(t)+r_y^2(t)} \\ \frac{v_x(t)r_x(t))+v_y(t)r_y(t)}{r_x^2(t)+r_y^2(t)} \end{bmatrix} . \quad (8)$$

Thus, using eqs.(6-8), the stochastic system given by eqs.(2,3) becomes:

$$\begin{aligned} Y_{t+1} &= \begin{cases} f_c^{lp}\left(Af_{lp}^c(Y_t)+HU_t+\sigma W_t\right) & \text{if } r_y(t)>0, \\ f_c^{lp}\left(-Af_{lp}^c(Y_t)+HU_t+\sigma W_t\right) & \text{if } r_y(t)<0. \end{cases} \\ Z_t &= \beta_t + V_t. \end{aligned} \quad (9)$$

III. PCRB FOR STATE ESTIMATION

In this section, ”usual” PCRB given by the inverse Fisher Information Matrix (FIM) is presented. However, this result is true under a technical hypothesis named ”*asymptotic unbiasedness assumption*”. Recently, Bréhard et al have shown in [26] that this assumption can be replaced by a more concrete one named “*side assumption*”. This new result is reminded here.

Let Y_t and $Z_{1:t}$ be the target state and the set of bearing measurements up to time t. Let $\hat{Y}_t$ be an estimator of Y_t which is a function of $Z_{1:t}$. We focus here on the Error Covariance Matrix (ECM) at time t which is $n_y \times n_y$-matrix, defined by:

$$\text{ECM}_t = \|\hat{Y}_t - Y_t\|^2 . \quad (10)$$

First, let us recall the Fisher Information Matrix (FIM) definition.

Definition 1 (FIM): *For the filtering problem given by eq.(9); the FIM ,at time* t*, is denoted* J_t *and defined as:*

$$J_t = \mathbb{E}\left\{\nabla_{Y_t} \ln p(Z_{1:t}, Y_{0:t}) \nabla_{Y_t}^* \ln p(Z_{1:t}, Y_{0:t})\right\} , \quad (11)$$

where $p(Z_{1:t}, Y_{0:t})$ *is the joint probability density function of* $Z_{1:t}$ *and* $Y_{0:t}$*.*

Assumption 1 (Side assumption): *For a filtering problem given by eq.(9), the* side assumption *is defined as:*

$$\lim_{\beta_t \to \frac{\pi}{2}} p(\beta_t) = 0 , \quad (12)$$

where $p(\beta_t)$ *is the probability density function of* β_t*.*

Roughly speaking, this assumption is valid if bearing measurements are ”sufficiently” far from $\frac{\pi}{2}$.

Proposition 1 (PCRB): *Under assumption 1,*

$$ECM_t \succcurlyeq J_t^{-1} . \quad (13)$$

IV. CLOSED-FORM FORMULATION FOR TICHAVSKÝ'S FORMULA IN THE LPC COORDINATES SYSTEM

We have presented in the previous section a PCRB adapted to the BOT context, given by eq.(13). Now it is necessary to estimate J_t^{-1}. The classical approach consists of using J_t^{-1} recursive formula proposed by Tichavský's et al. However, some terms involved in this formula must be estimated using Monte Carlo methods. Bréhard et al in [26] have shown that all these terms have closed-form expressions if the PCRB is derived using the LPC system, so that J_t^{-1} can be computed exactly via Tichavský's formula. In section A, Tichavský's recursive formula is reminded. We remark that no closed-form expressions for the terms involved in this formula can be obtained using Cartesian or MPC framework. Then we present in section B the closed-form PCRB algorithm in the LPC system.

A. Tichavský's formula

Tichavský et al. proposed a recursive formula in [25] for the right lower block of the FIM inverse noted J_t^{-1}.

Proposition 2 (Tichavský's formula): *For a filtering problem given by eq.(9), the right lower block of the FIM inverse noted* J_t^{-1} *has a recursive formula:*

$$J_{t+1}^{-1} = D_t^{22} + D_t^{33} - D_t^{21}\left(J_t^{-1} + D_t^{11}\right)^{-1} D_t^{12}, \quad (14)$$

where D_t^{11}, D_t^{12}, D_t^{21}, D_t^{22}, D_t^{33} *are defined by:*

$$\begin{cases} D_t^{11} = \mathbb{E}\{\nabla_{Y_t} \ln p(Y_{t+1}|Y_t) \nabla_{Y_t}^* \ln p(Y_{t+1|Y_t})\} , \\ D_t^{21} = \mathbb{E}\{\nabla_{Y_{t+1}} \ln p(Y_{t+1}|Y_t) \nabla_{Y_t}^* \ln p(Y_{t+1}|Y_t)\} , \\ D_t^{12} = \mathbb{E}\{\nabla_{Y_t} \ln p(Y_{t+1}|Y_t) \nabla_{Y_{t+1}}^* \ln p(Y_{t+1}|Y_t)\} , \\ D_t^{22} = \mathbb{E}\{\nabla_{Y_{t+1}} \ln p(Y_{t+1}|Y_t) \nabla_{Y_{t+1}}^* \ln p(Y_{t+1}|Y_t)\} , \\ D_t^{33} = \mathbb{E}\{\nabla_{Y_{t+1}} \ln p(Z_{t+1}|Y_{t+1}) \nabla_{Y_{t+1}}^* \ln p(Z_{t+1}|Y_{t+1})\} . \end{cases}$$

Proposition 2 is proved in [25]. The problem now is to compute D_t^{11}, D_t^{12}, D_t^{22} and D_t^{33}. Ristic et al. in [15] have derived the PCRB in the Cartesian coordinates system. Matrices D_t^{11},

D_t^{12}, D_t^{22} and D_t^{21} have closed-form expressions using this system. However D_t^{33} has no closed-form, so that the authors assumed that the process noise makes a very small effect on the PCRB (i.e. $W_t = 0$) for approximating D_t^{33}. Otherwise, the classical PCRB has not been derived in MP coordinates system yet. It seems that no closed-form for D_t^{11}, D_t^{12}, D_t^{22} and D_t^{21} can be expected, though a closed-form of D_t^{33} exists under assumption 1. In the LPC system, Bréhard et al have shown that all the terms have a closed-form expressions using the LPC system. This result is presented now.

B. An algorithm for calculating a closed-form PCRB, in the LPC system

The closed-form PCRB in the non manoeuvring case as given in [26] is presented in fig.2. We can see that calculation of D_t^{11}, D_t^{12} and D_t^{22} is splited in two steps. In step 1, the auxiliary matrices Γ_t^{11}, Γ_t^{12} and Γ_t^{22} are computed via a linear system. Then, D_t^{11}, D_t^{12} and D_t^{22} are extracted from Γ_t^{11}, Γ_t^{12}, Γ_t^{22} in step 2. Moreover, we can see D_t^{33} is easily computed using step 3.a.

V. About active measurements scheduling for state estimation

We assume now that additionally to (passive) bearing measurements, there is an other sub-system which can produce a noise corrupted range measurement at time t noted d_t:

$$d_t = r_t + \eta_t \quad \text{where } \eta_t \sim \mathcal{N}\left(0, \sigma_r^2\right) \ . \tag{15}$$

where σ_r is the range standard deviation. However, active measurements have a cost. The aim of OMS is to optimize the time-distribution of active measurements to obtain an accurate target state estimate. The following problem is considered:

Problem 1: *Can we find a policy $\mathcal{U}_{0:t}$ such that:*

$$MSE_{\ln r_l} < s \quad \forall l \in \{1, \ldots, T\} \tag{16}$$

where $MSE_{\ln r_l}$ is the mean square error related to $\ln r_l$ and s is a fixed threshold.

Fig. 1 illustrates a policy which could be obtained. In fig. 1(a), $MSE_{\ln r_t}$ is below a threshold s all along the scenario.

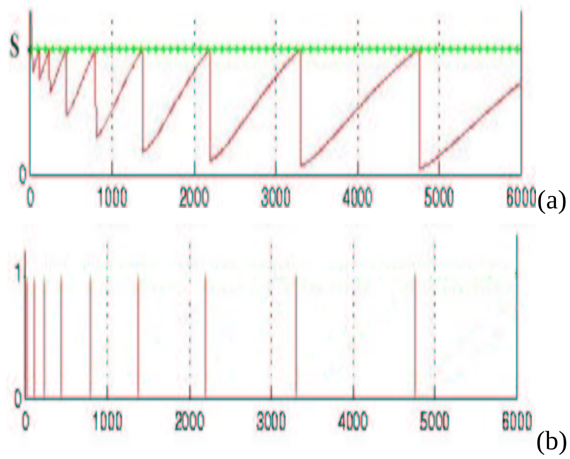

Fig. 1. *(a)$MSE_{\ln r_t}$ in red (b) policy, "$\mathcal{U}_t = 1$" if an active measurement is produced.*

The policy $\mathcal{U}_{0:t}$ is given by fig. 1(b). At different moments, a range measurement is produced such that $MSE_{\ln r_t}$ remains below the threshold.

The problem is that $MSE_{\ln r_l}$ can not be computed for the different policies such that we solve the suboptimal problem based on proposition 1.

Problem 1 bis: *Can we find a policy $\mathcal{U}_{0:t}$ such that:*

$$J^{-1}_{\ln r_l} < s \quad \forall l \in \{1, \ldots, T\} \tag{17}$$

where $J^{-1}_{\ln r_l}$ is the PCRB related to $\ln r_l$ and s is a fixed threshold.

The problem is now to compute the PCRB for the different policies. In the previous section, a closed-form PCRB has been derived for bearings-only measurements. What happens if a range measurement is included ? The fact is that the PCRB has still a closed-form. First, looking at eq.(15), we can see that only D_t^{33} depends on the measurement equation. Then, only the latter has to be modified. If the sensor produces a range measurement at time t, one can show that:

$$D_t^{33} = \begin{bmatrix} \frac{1}{\sigma_\beta^2} & 0 & 0 & 0 \\ 0 & \frac{\mathbb{E} r_{t+1}^2}{\sigma_r^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

where

$$\begin{cases} \mathbb{E}r_{t+1}^2 & = & \begin{bmatrix}1 & 0 & 0\end{bmatrix} \Gamma_t^{33} \,, \\ \Gamma_t^{33} & = & \Omega^{33} \,+\, \Phi \;\, \Gamma_{t-1}^{33} \end{cases} \tag{18}$$

We can see that $\mathbb{E}r_{t+1}^2$ is the first component of Γ_t^{33} and we have a simple recursive formula for Γ_t^{33}. The computation of D_t^{33} when a range measurement is produced, is presented in step3.a' in 2 such that the algorithm presented in fig.2 is able to calculate the PCRB for the different policies. Fig.3 show how the policy $\mathcal{U}_{0:t}$ related to problem 1' is derived from the PCRB.

Let us now make some remarks about the OMS algorithm summed up in fig.2 and 3. First, the policy only depends on the characteristics of the filtering problem given by eq.(9): the covariance of target state σ, the measurement covariance σ_β and the initial probability density function of the state $p(Y_0)$. You do not need the measurements to compute the policy. Second, this algorithm is presented in the non-manoeuvring case but can be extended using the closed-form PCRB in the case of a manoeuvring observer presented in [26]. Finally, this algorithm may be extended for a manoeuvring target using [32].

VI. Simulations

We illustrate in this section the OMS algorithm proposed in section V with the scenario presented in fig. 4. For the sake of completeness, all the constants involved in the scenario are presented in tab.I. For this scenario, the standard deviation of the process noise in the state equation σ is fixed to 0.05 ms^{-1} so that target trajectory strongly departs from a straight line. An example of trajectory is presented in fig. 4.

For these simulations, the range standard deviation is set to $\sigma_r = 100$ m. Fig. 5 presents the closed-form PCRB with active measurement scheduling as depicted in fig.3. As expected the PCRB $\ln r_t$ is under the threshold fixed to 0.01. Moreover, we can see that PCRB falls lower the thereshold when the policy suggests to produce a range measurement.

VII. Conclusion

Along this paper, the closed-form PCRB proposed by Bréhard et al has been applied to active measurement scheduling such that PCRB is becoming an accurate and time-varying technique of particular interest in this context. The next step will be to consider more complex sensor management strategies like considering that the total active measurements budget is fixed. Moreover, realistic sensor management algorithm will needs extension of the closed-form PCRB to the three dimensions case.

Appendix

We refer to eq.(2), for a definition of the various terms $\{A, B, Q, \alpha_1, \alpha_2, \alpha_3\}$ involved in this closed form.

$$\Psi = \begin{pmatrix} 1 & \delta_t & \delta_t & \delta_t^2 \\ 0 & 1 & 0 & \delta_t \\ 0 & 0 & 1 & \delta_t \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes Id_4 \quad , \Phi = \begin{bmatrix} 1 & 2\delta_t & \delta_t^2 \\ 0 & 1 & \delta_t \\ 0 & 0 & 1 \end{bmatrix} \tag{19}$$

$$\Omega^{11} = \begin{pmatrix} 2\alpha_3 A^* Q^{-1} A + 2\alpha_1 B A^* Q^{-1} A B^*, \\ \qquad +2\alpha_2 B A^* Q^{-1} A + 2\alpha_2 A^* Q^{-1} A B^* \\ 2\alpha_1 B A^* Q^{-1} A + 2\alpha_2 A^* Q^{-1} A \\ 2\alpha_1 A^* Q^{-1} A B^* + 2\alpha_2 A^* Q^{-1} A \\ 2\alpha_1 A^* Q^{-1} A \end{pmatrix} \tag{20}$$

$$\Omega^{12} = \begin{pmatrix} 2(\alpha_3 + \delta_t \alpha_2) A^* Q^{-1} + 2\alpha_1 B A^* Q^{-1} B^*, \\ \qquad +2(\alpha_2 + \delta_t \alpha_1) B A^* Q^{-1} + 2\alpha_2 A^* Q^{-1} B^* \\ 2\alpha_1 B A^* Q^{-1} + 2\alpha_2 A^* Q^{-1} \\ 2\alpha_1 A^* Q^{-1} B^* + 2(\alpha_2 + \delta_t \alpha_1) A^* Q^{-1} \\ 2\alpha_1 A^* Q^{-1} \end{pmatrix} \tag{21}$$

$$\Omega^{22} = \begin{pmatrix} 2(\alpha_3 + 2\delta_t \alpha_2 + \delta_t^2 \alpha_1) Q^{-1}, \\ \qquad +2\alpha_1 B Q^{-1} B^* + 2(\alpha_2 + \delta_t \alpha_1) \left(B Q^{-1} + Q^{-1} B^* \right) \\ 2\alpha_1 B Q^{-1} + 2(\alpha_2 + \delta_t \alpha_1) Q^{-1} \\ 2\alpha_1 Q^{-1} B^* + 2(\alpha_2 + \delta_t \alpha_1) Q^{-1} \\ 2\alpha_1 Q^{-1} \end{pmatrix} \tag{22}$$

$$\Omega^{33} = 2\sigma^2 \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \end{bmatrix}^* \tag{23}$$

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_1 & 0 & 0 & 0 \\ 0 & 16 + \mathcal{C}_1 & 0 & \mathcal{C}_3 \\ 0 & 0 & \mathcal{C}_2 & 0 \\ 0 & \mathcal{C}_3 & 0 & \mathcal{C}_2 \end{pmatrix} \tag{24}$$

$$\begin{cases} \mathcal{C}_1 = \frac{576\alpha_3^2}{\delta_t^6} + \frac{672\alpha_2^2}{\delta_t^4} + \frac{64\alpha_1^2}{\delta_t^2} - \frac{1152\alpha_3\alpha_2}{\delta_t^5} + \frac{288\alpha_3\alpha_1}{\delta_t^4} - \frac{384\alpha_2\alpha_1}{\delta_t^3} \,, \\ \mathcal{C}_2 = \frac{144\alpha_3^2}{\delta_t^4} + \frac{32\alpha_2^2}{\delta_t^2} - \frac{192\alpha_3\alpha_2}{\delta_t^3} + \frac{32\alpha_3\alpha_1}{\delta_t^2} \,, \\ \mathcal{C}_3 = -\frac{288\alpha_3^2}{\delta_t^5} - \frac{192\alpha_2^2}{\delta_t^3} + \frac{480\alpha_3\alpha_2}{\delta_t^4} - \frac{96\alpha_3\alpha_1}{\delta_t^3} + \frac{64\alpha_2\alpha_1}{\delta_t^2} \end{cases} \tag{25}$$

For $t = 1$ to T

1) Calculation of auxiliary matrices Γ_t^{11}, Γ_t^{12}, Γ_t^{22} and Γ_t^{33}

$$\begin{cases} \Gamma_t^{11} &= \Omega^{11} + \Psi \ \Gamma_{t-1}^{11} \ , \\ \Gamma_t^{12} &= \Omega^{12} + \Psi \ \Gamma_{t-1}^{12} \ , \\ \Gamma_t^{22} &= \Omega^{22} + \Psi \ \Gamma_{t-1}^{22} \ , \\ \Gamma_t^{33} &= \Omega^{33} + \Phi \ \Gamma_{t-1}^{33} \ . \end{cases} .$$

2) Calculation of D_t^{11}, D_t^{12}, D_t^{22}

a) $$\begin{cases} D_t^{11} &= \begin{bmatrix} Id_{n_y \times n_y} & 0_{n_y \times 3n_y} \end{bmatrix} \Gamma_t^{11} \ , \\ D_t^{12} &= - \begin{bmatrix} Id_{n_y \times n_y} & 0_{n_y \times 3n_y} \end{bmatrix} \Gamma_t^{12} \ , \\ D_t^{22} &= \begin{bmatrix} Id_{n_y \times n_y} & 0_{n_y \times 3n_y} \end{bmatrix} \Gamma_t^{22} + \mathcal{C} \ . \end{cases}$$

3) Calculation of D_t^{33}

a) *if passive meas.*

$$D_t^{33} = \begin{pmatrix} \frac{1}{\sigma_\beta^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

a') *if active meas.* + *passive meas.*

$$D_t^{33} = \begin{pmatrix} \frac{1}{\sigma_\beta^2} & 0 & 0 & 0 \\ 0 & \frac{\mathbb{E}r_{t+1}^2}{\sigma_r^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with $\mathbb{E}r_{t+1}^2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \Gamma_t^{33}$.

4) Calculate J_{t+1}^{-1} using Tichavský's formula:

$$J_{t+1}^{-1} = D_t^{22} + D_t^{33} - D_t^{21} \left(J_t^{-1} + D_t^{11} \right)^{-1} D_t^{12} \ ,$$

End for

Fig. 2. Closed-form calculation of the PCRB for "passive meas." and "passive meas.+ active meas". ($\Omega^{11}, \Omega^{12}, \Omega^{22}, \Omega^{33}, \mathcal{C}, \Psi$ *and* Φ *are given in appendix)*

For $t = 1$ to T

1) Compute $\tilde{J}_t^{-1}$ if passive meas.
2) if $\tilde{J}_{\ln r_l}^{-1} > s$ then $\mathcal{U}_t = 1$ and compute J_t^{-1} if active meas.+ passive meas.
3) if $\tilde{J}_{\ln r_l}^{-1} < s$ then $\mathcal{U}_t = 0$ and $J_t^{-1} = \tilde{J}_t^{-1}$.

End for

Fig. 3. Active measurement scheduling

	Scenario
$duration$	6000 s
$r_x^{obs}(0)$	$3,5$ km
$r_y^{obs}(0)$	0 km
$v_x^{obs}(0)$	10 ms^{-1}
$v_y^{obs}(0)$	-2 ms^{-1}
$r_x^{cib}(0)$	0 km
$r_y^{cib}(0)$	$3,5$ km
$v_x^{cib}(0)$	6 ms^{-1}
$v_y^{cib}(0)$	3 ms^{-1}
δ_t	6 s
σ	0.05 ms^{-1}
σ_β	0.05 rad (about 3 deg.)
σ_{r_0}	2 km
σ_{v_0}	1 ms^{-1}
σ_{β_0}	0.05 rad (about 3 deg.)

TABLE I

SCENARIO CONSTANTS

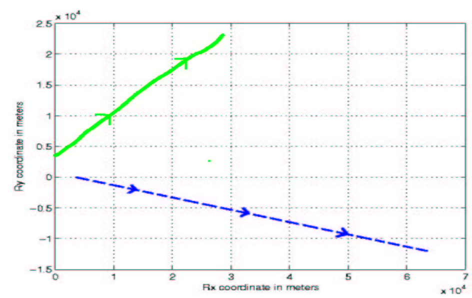

Fig. 4. **Scenario***:(a) example of trajectory of the target (solid line) and the observer (dashed line) (b)*

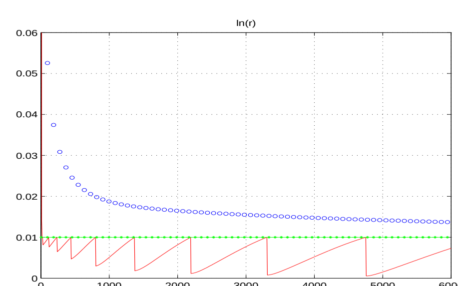

Fig. 5. *Closed-form PCRB related to* $\ln r_t$ *with range measurement scheduling (solid line) versus closed-form PCRB related to* $\ln r_t$ *without range measurements (dashed line).*

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