Closed-Form Posterior Cramér-Rao Bounds for Bearings-Only Tracking

T. BREHARD

J.-P. LE CADRE
IRISA/CNRS
France

We address the classical bearings-only tracking problem (BOT) for a single object, which belongs to the general class of nonlinear filtering problems. Recently, algorithms based on sequential Monte-Carlo methods (particle filtering) have been proposed. As far as performance analysis is concerned, the posterior Cramér-Rao bound (PCRB) provides a lower bound on the mean square error. Classically, under a technical assumption named "asymptotic unbiasedness assumption," the PCRB is given by the inverse Fisher information matrix (FIM). The latter is computed using Tichavský's recursive formula via Monte-Carlo methods. Two major problems are studied here. First, we show that the asymptotic unbiasedness assumption can be replaced by an assumption which is more meaningful. Second, an exact algorithm to compute the PCRB is derived via Tichavský's recursive formula without using Monte-Carlo methods. This result is based on a new coordinate system named logarithmic polar coordinate (LPC) system. Simulation results illustrate that PCRB can now be computed accurately and quickly, making it suitable for sensor management applications.

Manuscript received December 20, 2003; revised March 4 and December 9, 2005; released for publication February 25, 2006.

IEEE Log No. T-AES/42/4/890174.

Refereeing of this contribution was handled by B. La Scala.

Authors' address: IRISA/CNRS, Campus de Beaulieu, 35042 Rennes Cedex, France. E-mail: (thomas.brehard@inria.fr).

NOTATION

LP(C)	Logarithmic polar coordinates
MP(C)	Modified polar coordinates
BOT	Bearings-only tracking
X_t	Target state in Cartesian coordinate system
Y_t	Target state in LPC system
n_y	Size of target state ($n_y = 4$)
$\succcurlyeq$	Inequality $R \succcurlyeq S$ means that $R - S$ is positive semi-definite matrix
Id_n	$n \times n$ identity matrix
$0_{n\times m}$	$n \times m$ matrix composed of zero element
$\otimes$	Kronecker product
X^*	Denotes transpose of matrix X
$\\|X\\|_Q^2$	$= \mathbb{E}\{X^* Q^{-1} X\}$ where X is column vector
δ	Dirac delta function
Δ	Laplacian operator
∇	Gradient operator
det(X)	Determinant of matrix X
pdf	Probability density function
A	$= Id_4 + \delta_t B$ with $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes Id_2$
H	$= \begin{pmatrix} \delta_t \\ 1 \end{pmatrix} \otimes Id_2$
Q	$= \Sigma \otimes Id_2$ with $\Sigma = \begin{pmatrix} \alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_1 \end{pmatrix}$.

I. INTRODUCTION

In many applications (submarine tracking, aircraft surveillance), a bearings-only sensor is used to collect observations about target trajectory. This problem of tracking has been of interest for the past thirty years. The aim of bearings-only tracking (BOT) is to determine the target trajectory using noise-corrupted bearing measurements from a single observer. Target motion is classically described by a diffusion model[1] so that the filtering problem is composed of two stochastic equations. The first one represents the temporal evolution of the target state (position and velocity) called state equation. The second one links the bearing measurement to the target state at time t (measurement equation).

One of the characteristics of the problem is the nonlinearity of the measurement equation so that the classical Kalman filter is not convenient in this case. We can find in literature two kinds of solutions to this problem. The first one, proposed by Lindgren and Gong in [2], consists of deriving a pseudolinear measurement equation. Then, a Kalman filter can be used to solve the problem. The stochastic stability analysis of the estimates had been addressed by Song and Speyer in [3]. However, Aidala and Nardone show in [4] that this approach produces bias range estimates which can be reduced if the observer

[1] See [1] for an exhaustive review on dynamic models.

executes a maneuver. Consequently, bias range can be estimated as soon as it becomes observable [5]. A second idea consists of using the extended Kalman filter (EKF) in a Cartesian coordinate system to solve the problem. However, simulations show that this algorithm is often divergent due to the weak observability of range [6–8]. To remedy this problem, Aidala and Hammel in [9] proposed an EKF using another system named modified polar coordinate (MPC) system whose one salient feature is that range is not coupled with the observable components. This constitutes a neat improvement. Another solution proposed by Peach in [10] is a range-parametrized EKF, in which a number of EKF trackers parametrized by range run in parallel. Recently, particle filtering algorithms have been proposed in this context [11–13]. In [14], Arulampalam and Ristic compare the particle filter with the range-parametrized and EKF in MPC system; while a comprehensive overview of the state of art can be found in [15].

As far as performance analysis is concerned, the posterior Cramér-Rao bound (PCRB) proposed in [16] is widely used to assess the performance of filtering algorithms, by the tracking community [17–20] and in particular in the bearings-only context [15, 21, 22]. The PCRB gives a lower bound for the error covariance matrix (ECM). More precisely, under a technical assumption, the PCRB is the inverse of the Fisher information matrix (FIM). A seminal contribution on performance analysis is the paper from Tichavský, et al. [23]. Here, the authors noticed that only the right lower block of the FIM inverse was of interest for investigating tracking performance. This was the key idea for deriving a practical updating formula for the PCRB. Recently, PCRB has been used for various sensor management problems like automating the deployment of sensors in [24] or determining the optimal sensor trajectory in the bearings-only context in [25]. Moreover, PCRB can be used to schedule active measurements in a system involving active and passive subsystems. This application is addressed in the simulation section.

However, some problems remain to be solved. In this paper, two major issues of the PCRB are addressed. First, under a technical assumption named "asymptotic unbiasedness assumption," the PCRB is the FIM inverse. However, the validity of this assumption has not been thoroughly investigated in the BOT context yet. Here, our approach consists of deriving the PCRB in an original coordinate system named logarithmic polar coordinate (LPC) system. Using this coordinate system, it is shown that the asymptotic unbiasedness assumption can be replaced with another one, more meaningful in the BOT context. Second, Tichavský's recursive formula is a powerful result to compute the right lower block of the FIM inverse. However, complex integrals without any closed forms are involved in this recursion. So, these complex integrals must be approximated via Monte-Carlo methods. This approach is quite feasible but induces high computation requirements which highly reduces its suitability for complex problems like sensor management. For instance, the aim of active measurement scheduling consists in optimizing the time distribution of range measurements to obtain an accurate target state estimate. It implies to perform Monte-Carlo evaluations of the PCRB for each policy, which would rapidly become infeasible.

To avoid this problem, Ristic, et al. in [15] assume that the target process noise is zero. In the general case, we show that the complex integrals required for calculating the PCRB admit closed-form expressions if the PCRB is derived in the LPC system. Remarkably, though this coordinate system is only a slight modification of the MPC [9], it allows instrumental simplifications in the calculation of the elementary terms of the PCRB recursion. Applications to active measurement scheduling is briefly considered in a simulation framework.

In Section II, the BOT problem is presented in the Cartesian coordinate system and then in the LPC system. This original coordinate system is the key point to derive a closed form for the PCRB. In Section III, the classical PCRB is presented. A close examination of the asymptotic unbiasedness assumption is achieved so as to prove the validity of the "usual" PCRB, as given by the FIM inverse. We study this assumption and derive a more meaningful condition. In particular, conditions ensuring its validity are examined in the BOT context. Calculation of closed-form expressions of the right lower block of the FIM inverse via Tichavský's recursive formula is addressed in Section IV, in the LPC setting. Then, the closed-form PCRB is investigated for scheduling active measurements in Section V. In Section VI, simulation results present a comparison between the closed-form PCRB and the classical one (i.e., where the terms involved in Tichavský's formula are approximated by Monte-Carlo methods). Finally, the closed-form PCRB is used for investigating scheduling of passive and active measurements.

II. FROM CARTESIAN TO LPC SYSTEM

A. Cartesian Framework for BOT

Historically, BOT is presented in the Cartesian system. Let us define target state at time t:

$$X_t = [r_x(t)\ \ r_y(t)\ \ v_x(t)\ \ v_y(t)]^* \tag{1}$$

made of target relative velocity and position in the x-y plane. It is assumed that the target follows a nearly constant-velocity model. The discretized state

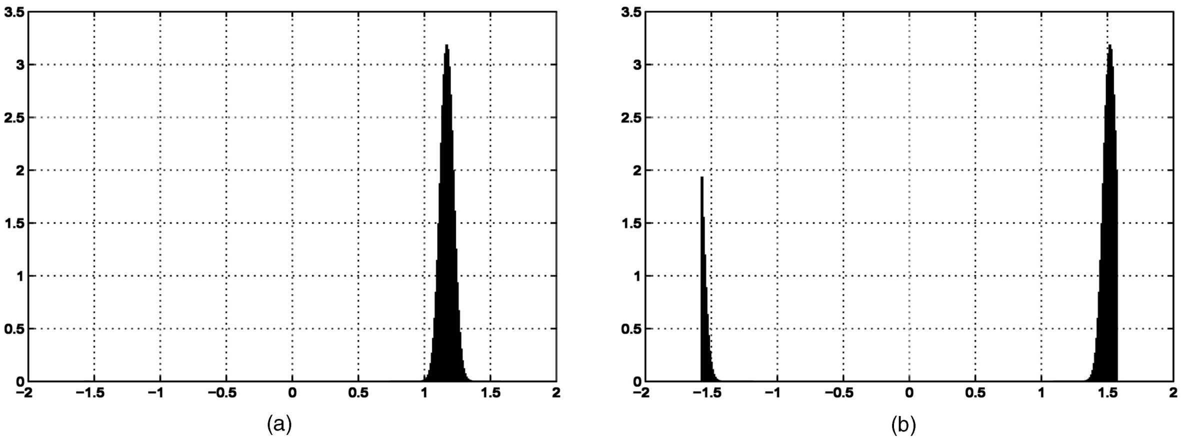


Fig. 1. Two examples of pdf of Z_t given X_t. (a) If Z_t is far from the bounds. (b) If Z_t is close to $\pi/2$.

equation[2] is given by

$$X_{t+1} = AX_t + U_t + \sigma W_t \tag{2}$$

where

$$W_t \sim \mathcal{N}(0,Q)$$

$$A = Id_4 + \delta_t B \qquad \text{with} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes Id_2$$

$$Q = \Sigma \otimes Id_2 \qquad \text{with} \quad \Sigma = \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_1 \end{bmatrix}.$$

and δ_t is the elementary time period and $-U_t$ is the known difference between observer velocity at time $t+1$ and t. The state covariance σ is unknown. However we assume classically that $\sigma < \sigma_{\max}$, so that we use in practice the following equation:

$$X_{t+1} = AX_t + U_t + \sigma_{\max} W_t. \tag{3}$$

Otherwise, we note Z_t the bearing measurement received at time t. The target state is related to this measurement through the following equation:

$$Z_t = \arctan\left(\frac{r_x(t)}{r_y(t)}\right) + V_t + \underbrace{\sum_{k\in\mathbb{Z}} k\pi \mathbf{1}_{-\pi/2<\arctan(r_x(t)/r_y(t))+V_t+k\pi<(\pi/2)}}_{(\star)} \tag{4}$$

where $V_t \sim \mathcal{N}(0,\sigma_\beta^2)$ and σ_β^2 is known. Let us notice that the term $(\star)$ is usually omitted. However, it is necessary to consider that measurement Z_t is restricted to a part of the space. This is the case if symmetry of the receiver (e.g. linear array) leads to considering measurements belonging in the interval $]-\pi/2,\pi/2[$, so that the additional term $(\star)$ in (4) is necessary. Two examples of probability density function (pdf) of Z_t given X_t are presented in Fig. 1 to enlighten the importance of the additional term $(\star)$. In Fig. 1(b), the bearing measurement is close to $\pi/2$ so that there is an overlapping phenomena.

[2]For a general review of dynamic models for target tracking see [1].

The system (3)–(4) has two components: a linear state equation (3) and a nonlinear measurement equation (4). Particle filter techniques [26, 27] are, thus, particularly appealing. Otherwise, practical implementations of EKF-based algorithms [9, 10] use a specific coordinate system, namely MPC. Indeed, if the target follows a deterministic trajectory (i.e., $W_t = 0\ \forall t \in \{0,\ldots,T\}$ in (3)), Nardone and Aidala have demonstrated in [7] that no information on range exists as long as the observer is not maneuvering. So the idea consists of using a coordinate system for which unobservable component (range) is not coupled with the observable part. This is also the motivation of Aidala and Hammel [9] for defining the MPC system:

$$\left[\beta_t \ \ \frac{1}{r_t} \ \ \dot{\beta}_t \ \ \frac{\dot{r}_t}{r_t}\right]^*. \tag{5}$$

Thus, the target state at time t is defined by (5), where β_t and r_t are the relative bearing and target range. We propose in the following section a slight modification of the MPC system, named the LPC system. The only difference is that the second component is not $1/r_t$ but $\ln(r_t)$. Even if this tiny difference appears very minor, it will be shown that it is instrumental for deriving a closed form of the PCRB. Let us now derive BOT equations given by (3) and (4) in the LPC framework.

B. LPC Framework for BOT

We consider now that the system state Y_t is expressed in the LPC system, i.e.,

$$Y_t = [\beta_t \ \ \rho_t \ \ \dot{\beta}_t \ \ \dot{\rho}_t]^* \tag{6}$$

where

$$\rho_t = \ln r_t.$$

As between Cartesian and modified polar (MP) system, we do not have a direct bijection between the Cartesian and the LPC system due to arctan function definition. We just have f_{lp}^c and f_c^{lp}, respectively LPC-to-Cartesian and Cartesian-to-LPC state mapping functions such that

$$X_t = \begin{cases} f_{lp}^c(Y_t) & \text{if } r_y(t) > 0 \\ -f_{lp}^c(Y_t) & \text{if } r_y(t) < 0 \end{cases} \quad \text{with}$$

$$f_{lp}^c(Y_t) = r_t \begin{bmatrix} \sin\beta_t \\ \cos\beta_t \\ \dot{\beta}_t \cos\beta_t + \dot{\rho}_t \sin\beta_t \\ -\dot{\beta}_t \sin\beta_t + \dot{\rho}_t \cos\beta_t \end{bmatrix} \tag{7}$$

and

$$Y_t = f_c^{lp}(X_t) = \begin{bmatrix} \arctan\left(\dfrac{r_x(t)}{r_y(t)}\right) \\ \ln\left(\sqrt{r_x^2(t) + r_y^2(t)}\right) \\ \dfrac{v_x(t)r_y(t) - v_y(t)r_x(t)}{r_x^2(t) + r_y^2(t)} \\ \dfrac{v_x(t)r_x(t)) + v_y(t)r_y(t)}{r_x^2(t) + r_y^2(t)} \end{bmatrix}. \tag{8}$$

Thus, using (7) and (8), the stochastic system given by (3) and (4) becomes

$$Y_{t+1} = \begin{cases} f_c^{lp}(Af_{lp}^c(Y_t) + U_t + \sigma_{\max} W_t) & \text{if} \quad r_y(t) > 0 \\ f_c^{lp}(-Af_{lp}^c(Y_t) + U_t + \sigma_{\max} W_t) & \text{if} \quad r_y(t) < 0 \end{cases}$$

$$Z_t = \beta_t + V_t + \sum_{k\in\mathbb{Z}} k\pi \mathbf{1}_{-\pi/2<\beta_t+V_t+k\pi<\pi/2}. \tag{9}$$

Though it seems that the LPC increases the complexity of the BOT problem, it has also the advantage of highlighting the multi-modality associated with the two solutions corresponding to $r_y(t) > 0$ and $r_y(t) < 0$, respectively.

III. PCRB FOR STATE ESTIMATION

In this section, "usual" PCRB given by the inverse FIM is presented. Notably, in subsection A, we present the proof of this classical result. The role of a technical hypothesis named asymptotic unbiasedness assumption is thus highlighted, especially in the LPC system. Then, we show in subsection B that this hypothesis is not always satisfied in the BOT context and we propose to replace it by an original extension. Finally, it is shown that the usual PCRB as given by FIM inverse is valid if bearing measurements are sufficiently far from $-\pi/2$ and $\pi/2$. Let us remark that the PCRB is not derived in the Cartesian framework for two reasons. First, the asymptotic unbiasedness assumption seems rather difficult to address in this setting. Second, it is shown that a closed form exists in LPC but not in the classical coordinate systems (Cartesian or MPC).

A. Classical PCRB

Let $Y_{0:t}$ and $Z_{1:t}$ be the trajectory and the set of bearing measurements up to time t. They are random vectors of size $n_y(t+1)$ and t, respectively. Let $\hat{Y}_{0:t}$ be an estimator of $Y_{0:t}$ which is a function of $Z_{1:t}$. We focus here on the ECM at time t which is $n_y(t+1) \times n_y(t+1)$-matrix, defined by

$$\text{ECM}_{0:t} = \|\hat{Y}_{0:t} - Y_{0:t}\|^2. \tag{10}$$

First, let us recall the FIM and bias definitions.

DEFINITION 1 (FIM) For the filtering problem given by (9), the FIM, at time t, is denoted $J_{0:t}$ and defined as

$$J_{0:t} = \mathbb{E}\{\nabla_{Y_{0:t}} \ln p(Z_{1:t}, Y_{0:t}) \nabla^*_{Y_{0:t}} \ln p(Z_{1:t}, Y_{0:t})\} \tag{11}$$

where $p(Z_{1:t}, Y_{0:t})$ is the joint pdf of $Z_{1:t}$ and $Y_{0:t}$.

DEFINITION 2 (Bias) For the filtering problem described by (9), estimation bias related to the estimated trajectory $\hat{Y}_{0:t}$ is defined as:

$$B(Y_{0:t}) = \mathbb{E}\{\hat{Y}_{0:t} - Y_{0:t} \mid Y_{0:t}\}. \tag{12}$$

$Y_{0:t}$ is a $n_y(t+1)$ vector so that $B(Y_{0:t})$ is a $n_y(t+1)$ vector too. The estimator of the trajectory $\hat{Y}_{0:t}$ is unbiased if vector $B(Y_{0:t})$ is almost surely equal to zero. This choice of the bias definition is justified in Appendix A. Proposition 1 ensures that the FIM gives a lower bound for the ECM under a specific assumption called asymptotic unbiasedness assumption. Before introducing this technical assumption let us introduce a notation to simplify the presentation:

Notation 1 For a function $F : \mathbb{R}^d \to \mathbb{R}^n$, U and $\mathcal{U}$ two $\mathbb{R}^d$-vectors such that $U = [U_1, \ldots, U_d]^*$ and $\mathcal{U} = [\mathcal{U}_1, \ldots, \mathcal{U}_d]^*$, we define

$$\lim_{U\to\mathcal{U}} F(U) = \begin{bmatrix} \lim_{U_1\to\mathcal{U}_1}(F(U))_1 & \cdots & \lim_{U_d\to\mathcal{U}_d}(F(U))_1 \\ \vdots & & \vdots \\ \lim_{U_1\to\mathcal{U}_1}(F(U))_n & \cdots & \lim_{U_d\to\mathcal{U}_d}(F(U))_n \end{bmatrix} \tag{13}$$

where $(F(U))_i$ is the ith component of vector $F(U)$.

Let us notice that $\lim_{U_1\to\mathcal{U}_1}(F(U))_1$ is a function which depends on variables $\mathcal{U}_1$ and $\{U_2, \ldots, U_d\}$ so that $\lim_{U\to\mathcal{U}} F(U)$ depends on variables $\mathcal{U}$ and U. We will see that Notation 1 is defined unambiguously in Proposition 1 proof and is helpful in presenting the following assumption.

Assumption 1 (Asymptotic unbiasedness) For the filtering problem given by (9), the asymptotic unbiasedness assumption is defined as:

$$\forall \quad k \in \{1, \ldots, t\}, \qquad \lim_{Y_k\to\mathcal{Y}_k^+} B(Y_{0:t})p(Y_{0:t}) = \lim_{Y_k\to\mathcal{Y}_k^-} B(Y_{0:t})p(Y_{0:t}) \tag{14}$$

where $\mathcal{Y}_k$ is the (connected) domain of Y_k, $k \in \{1,\ldots,t\}$, while $\{\mathcal{Y}_k^-, \mathcal{Y}_k^+\}$ are its bounds.

Looking at the definition of LPC given by (6), we have $\mathcal{Y}_l^- = [-\pi/2, -\infty, -\infty, -\infty]^*$ and $\mathcal{Y}_l^+ = [\pi/2, +\infty, +\infty, +\infty]^*$. Moreover, $B(Y_{0:t})p(Y_{0:t})$ is a $n_y(t+1)$ vector following Notation 1, $\lim_{Y_k \to \mathcal{Y}_k^+} B(Y_{0:t})p(Y_{0:t})$ is an $n_y(t+1) \times n_y$ matrix. After introducing Assumption 1, we can now present the classical result on the PCRB.

PROPOSITION 1 (PCRB) *For a filtering problem given by* (9)

$$\mathrm{ECM}_{0:t} \succcurlyeq C_{0:t} J_{0:t}^{-1} C_{0:t}^* \qquad \textit{with}$$

$$C_{0:t} \triangleq \mathbb{E}\{(\hat{Y}_{0:t} - Y_{0:t}) \nabla_{Y_{0:t}}^* \ln p(Z_{1:t}, Y_{0:t})\}. \tag{15}$$

Moreover, under Assumption 1, $C_{0:t}$ *is the identity matrix.*

Proposition 1 ensures that the FIM inverse gives a lower bound for the ECM conditionally to the validity of the technical Assumption 1 named asymptotic unbiasedness assumption. Classically, Assumption 1 is true if the estimator $\hat{Y}_{0:t}$ is unbiased when $Y_k \approx \mathcal{Y}_k^-$ and $Y_k \approx \mathcal{Y}_k^+$. However, this point is relatively complex to verify in the bearings-only context. We propose to study Assumption 1 to find a more concrete one. First, let us present a proof of the rather classical Proposition 1. For the sake of completeness, the following lemma is reviewed.

LEMMA 1 *Let S be a symmetric matrix defined as*

$$S = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix} \tag{16}$$

where

A is a nonnegative real symmetric matrix
B is a positive real symmetric matrix
C is a real matrix
then $S \succcurlyeq 0$ *implies* $A - CB^{-1}C^* \succcurlyeq 0$.

PROOF OF LEMMA 1 This lemma is a classical algebraic result given in [28].

PROOF OF PROPOSITION 1 Using Lemma 1, we build the S matrix such that

$$S = \begin{bmatrix} A_{0:t} & C_{0:t} \\ C_{0:t}^* & B_{0:t} \end{bmatrix}$$

where

$$\begin{aligned} A_{0:t} &\triangleq \mathrm{ECM}_{0:t} \\ B_{0:t} &\triangleq J_{0:t} \\ C_{0:t} &\triangleq \mathbb{E}\{(\hat{Y}_{0:t} - Y_{0:t}) \nabla_{Y_{0:t}}^* \ln p(Z_{1:t}, Y_{0:t})\}. \end{aligned} \tag{17}$$

From this definition, S is a nonnegative matrix. Using Lemma 1, one remarks that we just have to prove that $C_{0:t}$ is equal to the identity matrix. The asymptotic unbiasedness assumption is used to do so. First, let us notice that $C_{0:t}$ can be rewritten as

$$C_{0:t} = \int (\hat{Y}_{0:t} - Y_{0:t}) \nabla_{Y_{0:t}}^* p(Z_{1:t}, Y_{0:t}) d(Z_{1:t}, Y_{0:t}). \tag{18}$$

$C_{0:t}$ is an $n_y(t+1) \times n_y(t+1)$ matrix made of $(t+1) \times (t+1)$ elementary blocks. We study each of these elementary blocks (denoted $C_{0:t}(k,l)$):

$$C_{0:t}(k,l) = \int (\hat{Y}_k - Y_k) \nabla_{Y_l}^* p(Z_{1:t}, Y_{0:t}) d(Z_{1:t}, Y_{0:t}), \quad k \in \{1,\ldots,n_y\}, \quad l \in \{1,\ldots,n_y\}. \tag{19}$$

Before integrating by parts, let us introduce the following notation:

Notation 2 For a function $F : \mathbb{R}^d \to \mathbb{R}^n$, $U, \mathcal{U}^-$ and $\mathcal{U}^+$ three $\mathbb{R}^d$-vectors such that $U = [U_1,\ldots,U_d]^*$, $\mathcal{U}^- = [\mathcal{U}_1^-,\ldots,\mathcal{U}_d^-]^*$ and $\mathcal{U}^+ = [\mathcal{U}_1^+,\ldots,\mathcal{U}_d^+]^*$, then we can define

$$[F(U)]_{\mathcal{U}^-}^{\mathcal{U}^+} = \lim_{U \to \mathcal{U}^+} F(U) - \lim_{U \to \mathcal{U}^-} F(U) \tag{20}$$

where $\lim_{U \to \mathcal{U}^+} F(U)$ and $\lim_{U \to \mathcal{U}^-} F(U)$ are defined using Notation 1.

Integrating by parts and using the previous notation, a matrix element of $C_{0:t}$ given by (19) can be rewritten

$$C_{0:t}(k,l) = Id_{n_y} \delta_{k=l} + \int [(\hat{Y}_k - Y_k) p(Z_{1:t}, Y_{0:t})]_{\mathcal{Y}_l^-}^{\mathcal{Y}_l^+} d(Z_{1:t}, Y_{0:t}^{-\{l\}}) \tag{21}$$

where $Y_{0:t}^{-\{l\}}$ is a whole target trajectory except the term Y_l. Now, if limit and integral operators can be reversed, we have

$$C_{0:t}(k,l) = Id_{n_y} \delta_{k=l} + \int \left[\int (\hat{Y}_k - Y_k) p(Z_{1:t}, Y_{0:t}) dZ_{1:t} \right]_{\mathcal{Y}_l^-}^{\mathcal{Y}_l^+} dY_{0:t}^{-\{l\}}. \tag{22}$$

Using bias notation previously introduced, we finally obtain

$$C_{0:t}(k,l) = Id_{n_y} \delta_{k=l} + \int [B(Y_{0:t}) p(Y_{0:t})]_{\mathcal{Y}_l^-}^{\mathcal{Y}_l^+} dY_{0:t}^{-\{l\}}. \tag{23}$$

Thus, under Assumption 1, $C_{0:t}$ is the identity matrix.

Then we can apply Proposition 1 to the BOT problem if asymptotic unbiasedness assumption is satisfied. More precisely, this assumption ensures that the term $C_{0:t}$ is the identity matrix. Let us now study the validity of this hypothesis in the BOT context.

B. Validity of Asymptotic Unbiasedness Assumption in BOT Context

First let us remind that by Proposition 1 the PCRB is given by the inverse FIM if a technical assumption named asymptotic unbiasedness assumption is true. According to the previous section, $C_{0:t}$ given by (15) is not the identity matrix if this assumption is not verified. The following proposition shows that the asymptotic unbiasedness assumption is not always true in the BOT context.

PROPOSITION 2 (PCRB) *For a filtering problem given by* (9),

$$\mathrm{ECM}_{0:t} \succcurlyeq C_{0:t} J_{0:t}^{-1} C_{0:t}^{*}$$

where $C_{0:t}$ is an $n_y(t+1) \times n_y(t+1)$ block diagonal matrix where diagonal terms are expressed as follows:

$$C_{0:t}(l,l) = \begin{bmatrix} 1 - \pi p(\beta_l)|_{\pi/2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \forall \quad l \in \{0,\ldots,t\} \tag{24}$$

where $p(\beta_l)$ is the pdf *of β_l.*

More precisely, Proposition 2 gives a more simple formula for $C_{0:t}$. This result is quite intuitive. When bearing measurements are close to a bound (i.e., $-\pi/2$ or $\pi/2$) there is an overlapping phenomenon due to the arctan definition as the underlying pdf is not Gaussian but something like that function represented in Fig. 1. Finally let us notice that $p(\beta_l)$ is not defined in $\pi/2$ because β_l is in $]-\pi/2, \pi/2[$. However, the limit exists.

PROOF OF PROPOSITION 2 The complete proof of Proposition 2 is given in Appendix B with two intermediate results skipped in Subappendices B1 and B2. The idea of the proof consists of studying $C_{0:t}$ using the formula given by (22) in Proposition 1 proof. To study (22), the pdf of Y_{t+1} given Y_t, i.e., $p(Y_{t+1} \mid Y_t)$ is derived in Appendix B1. Then, a technical lemma allows us to end the proof.

In the filtering context, we are generally not interested in $\mathrm{ECM}_{0:t}$ but only in the right lower block $\mathrm{ECM}_t = \|\hat{Y}_t - Y_t\|^2$. Thus, it is not the whole matrix $C_{0:t} J_{0:t}^{-1} C_{0:t}^{*}$ which is of interest but just the right lower block. As $C_{0:t}$ is a diagonal matrix according to Proposition 2, we have

$$\mathrm{ECM}_t \succcurlyeq C_t J_t^{-1} C_t^{*}$$

with

$$C_t = \begin{bmatrix} 1 - \pi p(\beta_t)|_{\pi/2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{25}$$

Matrix J_t^{-1} is the right lower block of $J_{0:t}$-inverse, given by (11). Now from a practical point of view, the problem is to be able to estimate J_t^{-1} and C_t. Concerning the first one, J_t^{-1} is classically obtained by means of Tichavský's recursive formula via Monte-Carlo methods. Looking at (25), we can see that C_t only modifies the PCRB linked to the first component of the target state β_t. The PCRB associated to this component is overestimated because $p(\beta_t)|_{\pi/2}$ is not zero all the time. When bearing measurements are sufficiently far from the bounds $-\pi/2$ and $\pi/2$, C_t is the identity matrix, so that the classical PCRB is given by the FIM inverse.

Assumption 2 (Side assumption) For a filtering problem given by (9), the side assumption is defined as

$$p(\beta_l)|_{\pi/2} = 0, \qquad \forall \quad l \in \{0,\ldots,T\} \tag{26}$$

where $p(\beta_l)$ is the pdf of β_l.

PROPOSITION 3 (PCRB) *Under Assumption* 2*,*

$$\mathrm{ECM}_t \succcurlyeq J_t^{-1}. \tag{27}$$

PROOF OF PROPOSITION 3 Proposition 3 is easily derived from Proposition 2.

IV. CLOSED-FORM FORMULATION FOR TICHAVSKÝ'S FORMULA IN LPC COORDINATE SYSTEM

We have derived in the previous section a PCRB adapted to the BOT context, given by (27). Now it is necessary to estimate J_t^{-1}. The classical approach consists of using J_t^{-1} recursive formula proposed by Tichavský's et al. However, some terms involved in this formula must be estimated using Monte-Carlo methods. We demonstrate here that all these terms have closed-form expressions if the PCRB is derived using the LPC system, so that J_t^{-1} can be computed exactly via Tichavský's formula. In subsection A, Tichavský's recursive formula is reminded. We remark in subsection B that no closed-form expressions for the terms involved in this formula can be obtained using Cartesian or MPC framework. Then we show in subsection C that closed-form calculation can be derived in the new LPC system.

A. Tichavský's Formula

Tichavský, et al. proposed a recursive formula in [23] for the right lower block of the FIM inverse noted J_t^{-1}.

PROPOSITION 4 (Tichavský's formula) *For a filtering problem given by* (9)*, the right lower block of the FIM inverse noted J_t^{-1} has a recursive formula:*

$$J_{t+1} = D_t^{22} + D_t^{33} - D_t^{21}(J_t + D_t^{11})^{-1} D_t^{12}$$

TABLE I
Closed Forms In Different Coordinate Systems

	Cartesian	Modified Polar	Logarithmic Polar
D_t^{11}	Yes	No	Yes
D_t^{12}	Yes	No	Yes
D_t^{21}	Yes	No	Yes
D_t^{22}	Yes	No	Yes
D_t^{33}	No	Yes	Yes

where D_t^{11}, D_t^{12}, D_t^{21}, D_t^{22}, D_t^{33} *are defined by*

$$
\begin{aligned}
D_t^{11} &\triangleq \mathbb{E}\{\nabla_{Y_t} \ln p(Y_{t+1} \mid Y_t) \nabla_{Y_t}^{*} \ln p(Y_{t+1} \mid Y_t)\} \\
D_t^{21} &\triangleq \mathbb{E}\{\nabla_{Y_{t+1}} \ln p(Y_{t+1} \mid Y_t) \nabla_{Y_t}^{*} \ln p(Y_{t+1} \mid Y_t)\} \\
D_t^{12} &\triangleq \mathbb{E}\{\nabla_{Y_t} \ln p(Y_{t+1} \mid Y_t) \nabla_{Y_{t+1}}^{*} \ln p(Y_{t+1} \mid Y_t)\} \qquad (28) \\
D_t^{22} &\triangleq \mathbb{E}\{\nabla_{Y_{t+1}} \ln p(Y_{t+1} \mid Y_t) \nabla_{Y_{t+1}}^{*} \ln p(Y_{t+1} \mid Y_t)\} \\
D_t^{33} &\triangleq \mathbb{E}\{\nabla_{Y_{t+1}} \ln p(Z_{t+1} \mid Y_{t+1}) \nabla_{Y_{t+1}}^{*} \ln p(Z_{t+1} \mid Y_{t+1})\}.
\end{aligned}
$$

Proposition 4 is proved in [23]. However, for the BOT context, even if pdf $p(Y_{l+1} \mid Y_l)$ and $p(Z_t \mid Y_t)$ are known and simple, D_t^{11}, D_t^{12}, D_t^{21}, D_t^{22}, and D_t^{33} do not have closed-form expressions altogether. We show now that existence of closed-form expressions is a characteristic of the LPC system, introduced in Section IIB.

B. Closed-Form Expressions of D_t^{11}, D_t^{12}, D_t^{22}, D_t^{21}, and D_t^{33} in Different Coordinate Systems

Ristic, et al. in [15] have derived the PCRB in the Cartesian coordinate system. Matrices D_t^{11}, D_t^{12}, D_t^{22} and D_t^{21} have closed-form expressions using this system. However D_t^{33} has no closed form, so that the authors assumed that the process noise makes a very small effect on the PCRB (i.e., $W_t = 0$) for approximating D_t^{33}. Otherwise, the classical PCRB has not been derived in MPC system yet. It seems that no closed form for D_t^{11}, D_t^{12}, D_t^{22}, and D_t^{21} can be expected, though a closed form of D_t^{33} exists. These results are summed up in Table I.

Now the question is whether we can find a coordinate system allowing closed forms for all terms. First, it seems that the coordinate system must include β_t so that under Assumption 2, D_t^{33} has a closed form as in the MPC system. Second, in the Cartesian framework, it seems that the existence of closed forms for D_t^{11}, D_t^{12}, D_t^{22}, and D_t^{21} in (28) are inherited from the linear property of $\nabla_{X_t} \ln p(X_{t+1} \mid X_t)$ and $\nabla_{X_{t+1}} \ln p(X_{t+1} \mid X_t)$. First, considering LPC definition given by (6), we can see that β_t is one of the components of the state. Second, we can show that gradients $\nabla_{Y_t} \ln p(X_{t+1} \mid X_t)$ and $\nabla_{Y_{t+1}} \ln p(X_{t+1} \mid X_t)$ are quadratic forms in X_t, X_{t+1}. Indeed, we have

$$\nabla_{Y_t}^{*} \ln p(X_{t+1} \mid X_t) = \frac{1}{\sigma_{\max}^2}(X_{t+1} - AX_t - U_t)^{*} Q^{-1} A \nabla_{Y_t}\{X_t\} \qquad (29)$$

$$\nabla_{Y_{t+1}}^{*} \ln p(X_{t+1} \mid X_t) = -\frac{1}{\sigma_{\max}^2}(X_{t+1} - AX_t - U_t)^{*} Q^{-1} \nabla_{Y_{t+1}}\{X_{t+1}\}$$

where $\nabla_{Y_t}\{X_t\}$ and $\nabla_{Y_t+1}\{X_{t+1}\}$ are LPC-to-Cartesian mapping function derivatives at time t and $t+1$ (LPC-to-Cartesian mapping function is given by (7)). These two terms can be expressed using the Cartesian framework:

$$\nabla_{Y_t}\{X_t\} = \begin{bmatrix} r_y(t) & r_x(t) & 0 & 0 \\ -r_x(t) & r_y(t) & 0 & 0 \\ v_y(t) & v_x(t) & r_y(t) & r_x(t) \\ -v_x(t) & v_y(t) & -r_x(t) & r_y(t) \end{bmatrix} \qquad (30)$$

$$\nabla_{Y_t+1}\{X_{t+1}\} = \begin{bmatrix} r_y(t+1) & r_x(t+1) & 0 & 0 \\ -r_x(t+1) & r_y(t+1) & 0 & 0 \\ v_y(t+1) & v_x(t+1) & r_y(t+1) & r_x(t+1) \\ -v_x(t+1) & v_y(t+1) & -r_x(t+1) & r_y(t+1) \end{bmatrix}.$$

so that $\nabla_{Y_t}\{X_t\}$ and $\nabla_{Y_t+1}\{X_{t+1}\}$ given by (30) are linear operators in X_t, X_{t+1}.

C. An Algorithm for Calculating a Closed-Form PCRB, in the LPC System

Based on previous sections, 1, 2, 3, and 4 below give closed forms for D_t^{11}, D_t^{12}, D_t^{22}, and D_t^{33} in the LPC framework. Moreover, we show that these closed-forms can be written in a recursive manner. The algorithm that calculates the closed-form PCRB is summed up in Fig. 2. We can see that calculation of D_t^{11}, D_t^{12}, and D_t^{22} is split in two steps. In step 1, the auxiliary matrices Γ_t^{11}, Γ_t^{12}, and Γ_t^{22}, defined by (35), (38), and (41), are computed via a linear system. Then, D_t^{11}, D_t^{12}, and D_t^{22} are extracted from Γ_t^{11}, Γ_t^{12}, Γ_t^{22} in step 2. This algorithm is compared in the simulations section with the classical PCRB summed up in Fig. 3.

1) D_t^{11} *Closed Form*: We show in Appendix D that D_t^{11} can be expressed as an expectation of a simple function in the Cartesian coordinate system:

$$D_t^{11} = \frac{1}{\sigma_{\max}^2} \mathbb{E}\{F_{X_t}^{*} A^{*} Q^{-1} A F_{X_t}\} \qquad \text{with} \quad F_{X_t} = \nabla_{Y_t}\{X_t\}. \qquad (31)$$

The problem is now to compute this expectation. We show now that no "direct" recursive formula can be derived for D_t^{11} but the latter can be obtained as the by-product of a general linear system in Proposition 5.1. First let us investigate the nonmaneuvering case. In this case, using the statistical properties of X_{t+1} given X_t and the linear property of F, (31) can be rewritten as

- Initialization of J_0^{-1} using the initial error covariance matrix given by eq.(51).
- Initialization of Γ_0^{11}, Γ_0^{12} and Γ_0^{22} by Monte-Carlo method.
- J_1^{-1} is calculated using only step 2 and 3 with $t = 0$.
- For $t = 1$ to T
  1) Calculation of auxiliary matrices Γ_t^{11}, Γ_t^{12} and Γ_t^{22}
     a) Calculate Λ_{t-1}^{11}, Λ_{t-1}^{12} and Λ_{t-1}^{22} using eqs.(36,39,42) if observer maneuvers (else these terms are equal to zero).
     b) $\begin{cases} \Gamma_t^{11} = \Omega^{11} + \Psi\ \Gamma_{t-1}^{11} + \boldsymbol{\Lambda_{t-1}^{11}} , \\ \Gamma_t^{12} = \Omega^{12} + \Psi\ \Gamma_{t-1}^{12} + \boldsymbol{\Lambda_{t-1}^{12}} , \\ \Gamma_t^{22} = \Omega^{22} + \Psi\ \Gamma_{t-1}^{22} + \boldsymbol{\Lambda_{t-1}^{22}} . \end{cases}$
     Remark : Ω^{11}, Ω^{12} *and* Ω^{22} *are given by eqs.(36,39,42).* Ψ *is given by eq.(36).*
  2) Calculation of D_t^{11}, D_t^{12} and D_t^{22}
     a) If observer maneuvers, compute Υ_t^{12} and Υ_t^{22} using eq.(37) and eq.(40) (else these terms are equal to zero).
     b) $\begin{cases} D_t^{11} = \begin{bmatrix} Id_{n_y} & 0_{n_y \times 3n_y} \end{bmatrix} \Gamma_t^{11} , \\ D_t^{12} = - \begin{bmatrix} Id_{n_y} & 0_{n_y \times 3n_y} \end{bmatrix} \Gamma_t^{12} - \boldsymbol{\Upsilon_t^{12}} , \\ D_t^{22} = \begin{bmatrix} Id_{n_y} & 0_{n_y \times 3n_y} \end{bmatrix} \Gamma_t^{22} + \mathcal{C} + \boldsymbol{\Upsilon_t^{22}} . \end{cases}$
     Remark : $\mathcal{C}$ *is given by eq.(40) and* D_t^{21} *is given by the relation* $D_t^{21} = (D_t^{12})^*$.
     c) D_t^{33} is given by eq.(43).
  3) Calculate J_{t+1}^{-1} using Tichavský's formula:

$$J_{t+1} = D_t^{22} + D_t^{33} - D_t^{21} \left(J_t + D_t^{11}\right)^{-1} D_t^{12} .$$

Fig. 2. Closed-form calculation of PCRB.

$$D_t^{11} = \underbrace{\frac{1}{\sigma_{\max}^2} \mathbb{E}\{F_{X_t - AX_{t-1}}^* A^* Q^{-1} A F_{X_t - AX_{t-1}}\}}_{\text{constant}} + \frac{1}{\sigma_{\max}^2} \mathbb{E}\{F_{AX_{t-1}}^* A^* Q^{-1} A F_{AX_{t-1}}\}. \tag{32}$$

The first term can be calculated remarking that $X_t - AX_{t-1} \sim \mathcal{N}(0, \sigma_{\max}^2 Q)$ and F is a linear operator. We derived in Appendix D from the linear property of F that

$$\begin{cases} F_{AX_t} = F_{X_t} + \delta_t G_{X_t} \\ G_{AX_t} = G_{X_t} \end{cases} \quad \text{where} \quad \begin{cases} F_{X_t} = \nabla_{Y_t}\{X_t\} \\ G_{X_t} = Id_2 \otimes \begin{pmatrix} v_y(t) & v_x(t) \\ -v_x(t) & v_y(t) \end{pmatrix}. \end{cases} \tag{33}$$

Incorporating (33) in (32), we obtain

$$\begin{aligned} D_t^{11} = \text{constant} &+ \underbrace{\frac{1}{\sigma_{\max}^2} \mathbb{E}\{F_{X_{t-1}}^* A^* Q^{-1} A F_{X_{t-1}}\}}_{=D_{t-1}^{11}} \\ &+ \frac{\delta_t^2}{\sigma_{\max}^2} \mathbb{E}\{G_{X_{t-1}}^* A^* Q^{-1} A G_{X_{t-1}}\} \\ &+ \frac{\delta_t}{\sigma_{\max}^2} \mathbb{E}\{F_{X_{t-1}}^* A^* Q^{-1} A G_{X_{t-1}}\} \\ &+ \frac{\delta_t}{\sigma_{\max}^2} \mathbb{E}\{G_{X_{t-1}}^* A^* Q^{-1} A F_{X_{t-1}}\}. \end{aligned} \tag{34}$$

Looking at (34), it seems that no "direct" recursive formula can be derived for D_t^{11}. However, we can propose an original recursive formula for D_t^{11} via a joint matrix Γ_t^{11} formed with the four terms involved in (34) which is valid in the general case including the maneuvering case:

- Initialisation of J_0^{-1} using the initial error covariance matrix given by eq.(51).
- For $t = 0$ to T
  1) Approximation of D_t^{11}, D_t^{12} and D_t^{22} by Monte-Carlo method.
  2) D_t^{21} is given by the relation $D_t^{21} = (D_t^{12})^*$ and D_t^{33} is given by eq.(43).
  3) Compute J_{t+1}^{-1} using Tichavský's formula:

$$J_{t+1}^{-1} = D_t^{22} + D_t^{33} - D_t^{21} \left(J_t + D_t^{11}\right)^{-1} D_t^{12} .$$

Fig. 3. Classical computation of PCRB.

$$D_t^{11} = [Id_{n_y} \ \ 0_{n_y \times 3n_y}] \Gamma_t^{11},$$

$$\Gamma_t^{11} = \frac{1}{\sigma_{\max}^2} \begin{pmatrix} \mathbb{E}\{F_{X_t}^* A^* Q^{-1} A F_{X_t}\} \\ \mathbb{E}\{F_{X_t}^* A^* Q^{-1} A G_{X_t}\} \\ \mathbb{E}\{G_{X_t}^* A^* Q^{-1} A F_{X_t}\} \\ \mathbb{E}\{G_{X_t}^* A^* Q^{-1} A G_{X_t}\} \end{pmatrix} \tag{35}$$

where F_{X_t} and G_{X_t} are defined by (33).

We can see that D_t^{11} is just one block of Γ_t^{11}. Now the following proposition assumes that we have a recursive formula for Γ_t^{11}, so that D_t^{11} is obtained as a by product.

PROPOSITION 5.1 (Γ_t^{11} formula) *For a filtering problem given by* (9)*, we have the following recursive formula for* Γ_t^{11}*:*

$$\Gamma_t^{11} = \Omega^{11} + \Psi \Gamma_{t-1}^{11} + \boldsymbol{\Lambda}_{t-1}^{11}$$

where

$$\Psi = \begin{pmatrix} 1 & \delta_t & \delta_t & \delta_t^2 \\ 0 & 1 & 0 & \delta_t \\ 0 & 0 & 1 & \delta_t \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes Id_4$$

$$\Omega^{11} = \begin{pmatrix} 2\alpha_3 A^* Q^{-1} A + 2\alpha_1 BA^* Q^{-1} AB^* + 2\alpha_2 BA^* Q^{-1} A + 2\alpha_2 A^* Q^{-1} AB^* \\ 2\alpha_1 BA^* Q^{-1} A + 2\alpha_2 A^* Q^{-1} A \\ 2\alpha_1 A^* Q^{-1} AB^* + 2\alpha_2 A^* Q^{-1} A \\ 2\alpha_1 A^* Q^{-1} A \end{pmatrix}$$

and

$$\Lambda_{t-1}^{11} = \begin{cases} 0_{4n_y \times n_y} & \text{if} \quad U_{t-1} = 0, \\ \frac{1}{\sigma_{\max}^2} \begin{pmatrix} F_{\mathbb{E}X_t}^* A^* Q^{-1} A F_{\mathbb{E}X_t} - F_{A\mathbb{E}X_{t-1}}^* A^* Q^{-1} A F_{A\mathbb{E}X_{t-1}} \\ F_{\mathbb{E}X_t}^* A^* Q^{-1} A G_{\mathbb{E}X_t} - F_{A\mathbb{E}X_{t-1}}^* A^* Q^{-1} A G_{A\mathbb{E}X_{t-1}} \\ G_{\mathbb{E}X_t}^* A^* Q^{-1} A F_{\mathbb{E}X_t} - G_{A\mathbb{E}X_{t-1}}^* A^* Q^{-1} A F_{A\mathbb{E}X_{t-1}} \\ G_{\mathbb{E}X_t}^* A^* Q^{-1} A G_{\mathbb{E}X_t} - G_{A\mathbb{E}X_{t-1}}^* A^* Q^{-1} A G_{A\mathbb{E}X_{t-1}} \end{pmatrix} & \text{if} \quad U_{t-1} \neq 0. \end{cases} \tag{36}$$

We refer to (2)*, for a definition of the various terms* $\{A, B, Q, \alpha_1, \alpha_2, \alpha_3\}$ *involved in this closed form. For definitions of F and G see* (33).

Let us now make some remarks about the previous proposition. We can see that the recursive formula for Γ_t^{11} given by (36) is just a simple linear equation, where all the terms have closed-form expressions. Moreover, if the maneuvering term U_{t-1} is zero, then $\mathbb{E}X_t = A\mathbb{E}X_{t-1}$. As a consequence, Λ_{t-1}^{11} is zero if the maneuvering term U_{t-1} is zero. If this condition does not hold, Λ_{t-1}^{11} can be computed exactly using $\mathbb{E}(X_0)$ and the recursion $\mathbb{E}(X_t) = A\mathbb{E}(X_{t-1}) + U_{t-1}$. Finally, Γ_0^{11} can be initialized by Monte-Carlo method.

2) D_t^{12} *Closed Form*: Using the same approach as in the previous section, we show in Appendix D that

$$D_t^{12} = \underbrace{-\frac{1}{\sigma_{\max}^2} \mathbb{E}\{F_{X_t}^* A^* Q^{-1} F_{AX_t}\}}_{(\star)} - \Upsilon_t^{12}$$

with

$$\Upsilon_t^{12} = \begin{cases} 0_{n_y \times n_y} & \text{if} \quad U_t = 0 \\ \frac{1}{\sigma_{\max}^2} (F_{\mathbb{E}X_t}^* A^* Q^{-1} F_{EX_{t+1}} - F_{\mathbb{E}X_t}^* A^* Q^{-1} F_{AEX_t}) & \text{if} \quad U_t \neq 0 \end{cases} \tag{37}$$

where operator F is defined by (33). Comparing (37) with (31), we can notice that we have now two terms to compute. The term Υ_t^{12} can be easily calculated. We can remark that the latter is zero if U_t is zero. If this condition is not verified, $\mathbb{E}(X_t)$ is computed for any value of t using $\mathbb{E}(X_0)$ and the relation $\mathbb{E}(X_t) = A\mathbb{E}(X_{t-1}) + U_{t-1}$. Otherwise, $(\star)$ can be computed recursively using the same approach as for D_t^{11}. D_t^{12} is deduced from Γ_t^{12} via

$$D_t^{12} = -[Id_{n_y} \;\; 0_{n_y \times 3n_y}]\Gamma_t^{12} - \Upsilon_t^{12}$$

$$\Gamma_t^{12} = \frac{1}{\sigma_{\max}^2} \begin{pmatrix} \mathbb{E}\{F_{X_t}^* A^* Q^{-1} F_{AX_t}\} \\ \mathbb{E}\{F_{X_t}^* A^* Q^{-1} G_{AX_t}\} \\ \mathbb{E}\{G_{X_t}^* A^* Q^{-1} F_{AX_t}\} \\ \mathbb{E}\{G_{X_t}^* A^* Q^{-1} G_{AX_t}\} \end{pmatrix} \tag{38}$$

where operators F and G are given by (33). Again, we have a recursive formula for Γ_t^{12}, yielding D_t^{12} as a by-product.

PROPOSITION 5.2 (Γ_t^{12} formula) *For a filtering problem given by* (9)*, we have the following recursive formula for* Γ_t^{12}

$$\Gamma_t^{12} = \Omega^{12} + \Psi\Gamma_{t-1}^{12} + \Lambda_{t-1}^{12}$$

where

$$\Omega^{12} = \begin{pmatrix} 2(\alpha_3 + \delta_t\alpha_2)A^* Q^{-1} + 2\alpha_1 BA^* Q^{-1} B^* + 2(\alpha_2 + \delta_t\alpha_1)BA^* Q^{-1} + 2\alpha_2 A^* Q^{-1} B^* \\ 2\alpha_1 BA^* Q^{-1} + 2\alpha_2 A^* Q^{-1} \\ 2\alpha_1 A^* Q^{-1} B^* + 2(\alpha_2 + \delta_t\alpha_1)A^* Q^{-1} \\ 2\alpha_1 A^* Q^{-1} \end{pmatrix}$$

and

$$\boldsymbol{\Lambda}_{t-1}^{12} = \begin{cases} 0_{4n_y \times n_y} & \text{if} \quad U_{t-1} = 0, \\ \dfrac{1}{\sigma_{\max}^2} \begin{pmatrix} F_{\mathbb{E}X_t}^* A^* Q^{-1} F_{A\mathbb{E}X_t} - F_{A\mathbb{E}X_{t-1}}^* A^* Q^{-1} F_{A^2\mathbb{E}X_{t-1}} \\ F_{\mathbb{E}X_t}^* A^* Q^{-1} G_{A\mathbb{E}X_t} - F_{A\mathbb{E}X_{t-1}}^* A^* Q^{-1} G_{A^2\mathbb{E}X_{t-1}} \\ G_{\mathbb{E}X_t}^* A^* Q^{-1} F_{A\mathbb{E}X_t} - G_{A\mathbb{E}X_{t-1}}^* A^* Q^{-1} F_{A^2\mathbb{E}X_{t-1}} \\ G_{\mathbb{E}X_t}^* A^* Q^{-1} G_{A\mathbb{E}X_t} - G_{A\mathbb{E}X_{t-1}}^* A^* Q^{-1} G_{A^2\mathbb{E}X_{t-1}} \end{pmatrix} & \text{if} \quad U_{t-1} \neq 0. \end{cases} \tag{39}$$

Ψ *is given by* (36). *We refer to* (2), *for a definition of the various terms* $\{A, B, Q, \alpha_1, \alpha_2, \alpha_3\}$ *involved in this closed form. For definitions of* F *and* G *see* (33).

Again, the recursion giving Γ_t^{12} is linear and has a closed form. Similarly to Γ_t^{11} recursion, $\boldsymbol{\Lambda}_{t-1}^{12}$ is zero if no maneuver occurs ($\mathbb{E}X_t = A\mathbb{E}X_{t-1}$). Else, $\boldsymbol{\Lambda}_{t-1}^{12}$ is updated from $\mathbb{E}(X_0)$. Considering the initialization of the Γ_t^{12} recursion, Γ_0^{12} can be approximated using the Monte-Carlo method.

3) *D_t^{22} Closed Form*: Using the same approach as in the previous section, we show in Appendix D that

$$D_t^{22} = \underbrace{\frac{1}{\sigma_{\max}^2} \mathbb{E}\{F_{AX_t}^* Q^{-1} F_{AX_t}\}}_{(\star)} + \mathcal{C} + \boldsymbol{\Upsilon}_t^{22}$$

where

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 2\dfrac{\alpha_3^2}{\alpha_3\alpha_1 - \alpha_2^2} & 0 \\ 0 & 0 & 0 & 2\dfrac{\alpha_3^2}{\alpha_3\alpha_1 - \alpha_2^2} \end{pmatrix}$$

and

$$\boldsymbol{\Upsilon}_t^{22} = \begin{cases} 0_{n_y \times n_y} & \text{if} \quad U_t = 0, \\ \dfrac{1}{\sigma_{\max}^2}(F_{\mathbb{E}X_{t+1}}^* Q^{-1} F_{\mathbb{E}X_{t+1}} - F_{A\mathbb{E}X_t}^* Q^{-1} F_{A\mathbb{E}X_t}) & \text{if} \quad U_t \neq 0 \end{cases} \tag{40}$$

where the operator F is defined by (33). As we can see above, $\mathcal{C}$ is just a constant term and $\boldsymbol{\Upsilon}_t^{22}$ is a maneuvering term which can be calculated using the same approach as for $\boldsymbol{\Upsilon}_t^{12}$ in Section B2. Otherwise, $(\star)$ in (40) can be calculated recursively. The matrix D_t^{22} is deduced from Γ_t^{22} via

$$D_{t+1}^{22} = [Id_{n_y \times n_y} \quad 0_{n_y \times 3n_y}]\Gamma_{t+1}^{22} + \mathcal{C} + \boldsymbol{\Upsilon}_t^{22}$$

$$\Gamma_t^{22} = \frac{1}{\sigma_{\max}^2} \begin{pmatrix} \mathbb{E}\{F_{AX_t}^* Q^{-1} F_{AX_t}\} \\ \mathbb{E}\{F_{AX_t}^* Q^{-1} G_{AX_t}\} \\ \mathbb{E}\{G_{AX_t}^* Q^{-1} F_{AX_t}\} \\ \mathbb{E}\{G_{AX_t}^* Q^{-1} G_{AX_t}\} \end{pmatrix} \tag{41}$$

where operators F and G are given by (33). Again, the following proposition yields a closed-form recursive formula for Γ_t^{22}, and for D_t^{22} as a by-product.

PROPOSITION 5.3 (Γ_t^{22} formula) *For a filtering problem given by (9), a closed-form recursive formula for* Γ_t^{22} *is given by*

$$\Gamma_t^{22} = \Omega^{22} + \Psi\Gamma_{t-1}^{22} + \boldsymbol{\Lambda}_{t-1}^{22}$$

where

$$\Omega^{22} = \begin{pmatrix} 2(\alpha_3 + 2\delta_t\alpha_2 + \delta_t^2\alpha_1)Q^{-1} + 2\alpha_1 BQ^{-1}B^* + 2(\alpha_2 + \delta_t\alpha_1)(BQ^{-1} + Q^{-1}B^*) \\ 2\alpha_1 BQ^{-1} + 2(\alpha_2 + \delta_t\alpha_1)Q^{-1} \\ 2\alpha_1 Q^{-1}B^* + 2(\alpha_2 + \delta_t\alpha_1)Q^{-1} \\ 2\alpha_1 Q^{-1} \end{pmatrix}$$

and

$$\boldsymbol{\Lambda}_{t-1}^{22} = \begin{cases} 0_{n_y \times n_y} & \text{if} \quad U_{t-1} = 0, \\ \dfrac{1}{\sigma_{\max}^2} \begin{pmatrix} F_{A\mathbb{E}X_t}^* Q^{-1} F_{A\mathbb{E}X_t} - F_{A^2\mathbb{E}X_{t-1}}^* Q^{-1} F_{A^2\mathbb{E}X_{t-1}} \\ F_{A\mathbb{E}X_t}^* Q^{-1} G_{A\mathbb{E}X_t} - F_{A^2\mathbb{E}X_{t-1}}^* Q^{-1} G_{A^2\mathbb{E}X_{t-1}} \\ G_{A\mathbb{E}X_t}^* Q^{-1} F_{A\mathbb{E}X_t} - G_{A^2\mathbb{E}X_{t-1}}^* Q^{-1} F_{A^2\mathbb{E}X_{t-1}} \\ G_{A\mathbb{E}X_t}^* Q^{-1} G_{A\mathbb{E}X_t} - G_{A^2\mathbb{E}X_{t-1}}^* Q^{-1} G_{A^2\mathbb{E}X_{t-1}} \end{pmatrix} & \text{if} \quad U_{t-1} \neq 0. \end{cases} \tag{42}$$

4) D_t^{33} *Closed Form:* We show in Appendix D that D_t^{33} is simply

$$D_t^{33} = \begin{pmatrix} \frac{1}{\sigma_\beta^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{43}$$

V. PCRB FOR PASSIVE AND ACTIVE MEASUREMENTS

We assume now that additionally to (passive) bearing measurements, there is another subsystem which can produce a noise-corrupted range measurement at time t noted d_t:

$$d_t = r_t + \eta_t \qquad \text{where} \quad \eta_t \sim \mathcal{N}(0,\sigma_r^2) \tag{44}$$

where σ_r is the range measurement standard deviation. However, active measurements have a cost so that the total active measurements budget is fixed. The aim of measurement scheduling is to optimize the time distribution of active measurements to obtain an accurate target state estimate.

The general problem of optimizing the time distribution of measurements has a long history. Avitzour, et al. in [29] have proposed an algorithm to optimize the time-distribution of measurements when estimating a scalar random variable by solving a nonquadratic minimization problem. This result has been extended by Shakeri, et al. in [30] to discrete-time stochastic processes. However, this approach is devoted to linear systems when the BOT is highly nonlinear. Then, Le Cadre has proposed to use the CRB to solve the problem in [31] for nonlinear systems where the state equation is deterministic. We show in this section that a closed-form PCRB derived can be used for active measurement scheduling.

In the previous section, a closed-form PCRB has been derived for bearings-only measurements. What happens if range measurements are included ? We show in this section that the PCRB still has a closed form. First, looking at (28), we can see that only D_t^{33} depends on the measurement equation. Then, only the latter has to be modified. If the sensor produces a range measurement at time t, then:

$$\mathcal{D}_t^{33} = \mathbb{E}\{\nabla_{Y_{t+1}} \ln p(Z_{t+1}, d_{t+1} \mid Y_{t+1}) \nabla^*_{Y_{t+1}} \ln p(Z_{t+1}, d_{t+1} \mid Y_{t+1})\}. \tag{45}$$

Using the independence property between bearings and range measurements, (45) can be rewritten

$$\mathcal{D}_t^{33} = \underbrace{\mathbb{E}\{\nabla_{Y_{t+1}} \ln p(Z_{t+1} \mid Y_{t+1}) \nabla^*_{Y_{t+1}} \ln p(Z_{t+1} \mid Y_{t+1})\}}_{=D_t^{33}} + \mathbb{E}\{\nabla_{Y_{t+1}} \ln p(d_{t+1} \mid Y_{t+1}) \nabla^*_{Y_{t+1}} \ln p(d_{t+1} \mid Y_{t+1})\}. \tag{46}$$

Using D_t^{33} given by (43) and range measurement equation given by (44), we obtain

$$\mathcal{D}_t^{33} = \begin{bmatrix} \frac{1}{\sigma_\beta^2} & 0 & 0 & 0 \\ 0 & \frac{\mathbb{E}r_{t+1}^2}{\sigma_r^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{47}$$

Consequently, the problem is to compute $\mathbb{E}r_{t+1}^2$. We show now that there is no "direct" recursive formula to calculate $\mathbb{E}r_{t+1}^2$ but the latter can be obtained as a by-product of a linear system. First let us address the nonmaneuvering case. Using the state equation given by (3) and the statistical properties of W_t, elementary calculations yield

$$\begin{aligned} \mathbb{E}r_{t+1}^2 &= \mathbb{E}\{r_x^2(t+1) + r_y^2(t+1)\} \\ &= 2\sigma_{\max}^2 \alpha_3 + \underbrace{\mathbb{E}\{r_x^2(t) + r_y^2(t)\}}_{=\mathbb{E}r_t^2} \\ &\quad + 2\delta_t \mathbb{E}\{v_x(t) r_x(t) + v_y(t) r_y(t)\} \\ &\quad + \delta_t^2 \mathbb{E}\{v_x^2(t) + v_y^2(t)\}. \end{aligned} \tag{48}$$

Then looking at (48), It seems that no "direct" recursive formula can be derived for $\mathbb{E}r_{t+1}^2$. However, we can propose an original recursive formula for the latter via a joint matrix Γ_t^{33} formed with the three terms involved in (48) which is valid in the general case including the maneuvering case:

$$\mathbb{E}\mathrm{r}_{\mathrm{t}+1}^2 = [1 \ 0 \ 0]\Gamma_t^{33} \tag{49}$$

$$\Gamma_t^{33} = \begin{bmatrix} \mathbb{E}\{r_x^2(t+1) + r_y^2(t+1)\} \\ \mathbb{E}\{v_x(t+1) r_x(t+1) + v_y(t+1) r_y(t+1)\} \\ \mathbb{E}\{v_x^2(t+1) + v_y^2(t+1)\} \end{bmatrix}.$$

We can see that $\mathbb{E}r_{t+1}^2$ is the first component of Γ_t^{33}. We have a simple recursive formula for Γ_t^{33} given by Proposition 6.

PROPOSITION 6 (Γ_t^{33} formula)

$$\Gamma_t^{33} = \Omega^{33} + \Phi \Gamma_{t-1}^{33} + \mathbf{\Lambda}_{t-1}^{33}$$

where

$$\Omega^{33} = 2\sigma_{\max}^2 \begin{bmatrix} \alpha_3 \\ \alpha_2 \\ \alpha_1 \end{bmatrix}$$

$$\Phi = \begin{bmatrix} 1 & 2\delta_t & \delta_t^2 \\ 0 & 1 & \delta_t \\ 0 & 0 & 1 \end{bmatrix}$$

- Initialization of J_0^{-1} using the initial error covariance matrix given by eq.(51).
- Initialization of Γ_0^{11}, Γ_0^{12}, Γ_0^{22} and Γ_0^{33} by Monte-Carlo method.
- J_1^{-1} is calculated using only step 2 and 3 with $t = 0$.
- For $t = 1$ to T

  1) Calculation of auxiliary matrices Γ_t^{11}, Γ_t^{12}, Γ_t^{22} and Γ_t^{33}

     a) Calculate Λ_{t-1}^{11}, Λ_{t-1}^{12}, Λ_{t-1}^{22} and Λ_{t-1}^{33} using eqs.(36,39,42) if observer maneuvers (else these terms are null).

     b) $$\begin{cases} \Gamma_t^{11} = \Omega^{11} + \Psi\ \Gamma_{t-1}^{11} + \boldsymbol{\Lambda_{t-1}^{11}} , \\ \Gamma_t^{12} = \Omega^{12} + \Psi\ \Gamma_{t-1}^{12} + \boldsymbol{\Lambda_{t-1}^{12}} , \\ \Gamma_t^{22} = \Omega^{22} + \Psi\ \Gamma_{t-1}^{22} + \boldsymbol{\Lambda_{t-1}^{22}} , \\ \Gamma_t^{33} = \Omega^{33} + \Phi\ \Gamma_{t-1}^{33} + \boldsymbol{\Lambda_{t-1}^{33}} . \end{cases}$$

     Remark : *Ω^{11}, Ω^{12}, Ω^{22} and Ω^{33} are given by eqs.(36,39,42). Ψ and Φ are given by eq.(36) and eq.(49).*

  2) Calculation of D_t^{11}, D_t^{12}, D_t^{22}, D_t^{33} and $\mathcal{D}_t^{33}$

     a) If observer maneuvers, compute Υ_t^{12} and Υ_t^{22} using eq.(37) and eq.(40) (else these terms are null).

     b) $$\begin{cases} D_t^{11} = \begin{bmatrix} Id_{n_y \times n_y} & 0_{n_y \times 3n_y} \end{bmatrix} \Gamma_t^{11} , \\ D_t^{12} = - \begin{bmatrix} Id_{n_y \times n_y} & 0_{n_y \times 3n_y} \end{bmatrix} \Gamma_t^{12} - \boldsymbol{\Upsilon_t^{12}} , \\ D_t^{22} = \begin{bmatrix} Id_{n_y \times n_y} & 0_{n_y \times 3n_y} \end{bmatrix} \Gamma_t^{22} + \mathcal{C} + \boldsymbol{\Upsilon_t^{22}} . \end{cases}$$

     Remark : *$\mathcal{C}$ is given by eq.(40) and D_t^{21} is given by the relation $D_t^{21} = \left(D_t^{12}\right)^*$.*

     c) Calculation of D_t^{33} using eq.(43) *(passive meas.)*.

     d) Calculation of $\mathcal{D}_t^{33}$ is given by eq.(47) *(active meas. + passive meas.)*

     Remark : *$\mathbb{E}r_{t+1}^2$ is calculated by using eq.(49) and Γ_t^{33}.*

  3) Calculate J_{t+1}^{-1} using Tichavský's formula:

     $$J_{t+1} = \begin{cases} D_t^{22} + D_t^{33} - D_t^{21}\left(J_t + D_t^{11}\right)^{-1} D_t^{12} & \textit{(passive meas.)} , \\ D_t^{22} + \mathcal{D}_t^{33} - D_t^{21}\left(J_t + D_t^{11}\right)^{-1} D_t^{12} & \textit{(active meas. + passive meas.)} \end{cases}$$

Fig. 4. Closed-form calculation of PCRB for active measurements scheduling.

and

$$\boldsymbol{\Lambda}_{t-1}^{33} = \begin{bmatrix} 2\delta_t \begin{bmatrix} \mathbb{E}r_x(t) \\ \mathbb{E}r_y(t) \end{bmatrix}^* U_t + 2\delta_t^2 \begin{bmatrix} \mathbb{E}v_x(t) \\ \mathbb{E}v_y(t) \end{bmatrix}^* U_t + \delta_t^2 U_t^* U_t \\ \begin{bmatrix} \mathbb{E}r_x(t) \\ \mathbb{E}r_y(t) \end{bmatrix}^* U_t + 2\delta_t \begin{bmatrix} \mathbb{E}v_x(t) \\ \mathbb{E}v_y(t) \end{bmatrix}^* U_t + \delta_t U_t^* U_t \\ 2 \begin{bmatrix} \mathbb{E}r_x(t) \\ \mathbb{E}r_y(t) \end{bmatrix}^* U_t + U_t^* U_t \end{bmatrix} . \quad (50)$$

We refer to (2)*, for a definition of the various terms* $\{\alpha_1, \alpha_2, \alpha_3\}$ *involved in this closed form.*

PROOF OF PROPOSITION 6 We incorporate the diffusion equation given by (3) in Γ_t^{33} given by (49). Finally, we obtain (50) using the statistical properties of W_t.

Λ_{t-1}^{33} is zero if no maneuver occurs. Concerning the initialization, Γ_0^{33} can be approximated by Monte-Carlo method. The algorithm is summed up in Fig. 4 and is illustrated by simulation results in the following section.

VI. SIMULATIONS

We have shown in the Section IV that under Assumption 2, the PCRB has a closed form. We have presented the algorithm in Fig. 2. The aim of this section is double. First, we show that these original formulas are valid and allow to compute accurately the PCRB without high computation load. Second, this bound can be used for optimal scheduling of active measurements in a sensor management context.

To check formulas, the closed-form PCRB is compared with the classical one using two scenarios. In the first one, the observer goes straight line while in the second one, the observer maneuvers. For the sake of completeness, all the constants involved in the two scenarios are presented in Table II. For these two scenarios, the standard deviation of the process noise in the state equation $\sigma_{\max}$ is fixed to 0.05 ms^{-1} so that target trajectory strongly departs from a straight line. The classical PCRB algorithm is reviewed in Fig. 3 (the sample size to approximate D_t^{11}, D_t^{12}, D_t^{22}, and D_t^{21} by Monte-Carlo methods is 1000). For all the algorithms, the initial FIM inverse is computed using the initial ECM. The latter is computed using Monte-Carlo methods. More precisely, N initial target states in LPC, noted $\{Y_0^{(i)}\}_{i\in\{1,\ldots,N\}}$, are sampled by using the initial range, bearing, and speed standard deviations which are, respectively, set to σ_{r_0} = 2 km, σ_{β_0} = 0.05 rad (about 3 deg), and σ_s = 1 ms^{-1}. Then, we obtain J_0^{-1} using the following approximation:

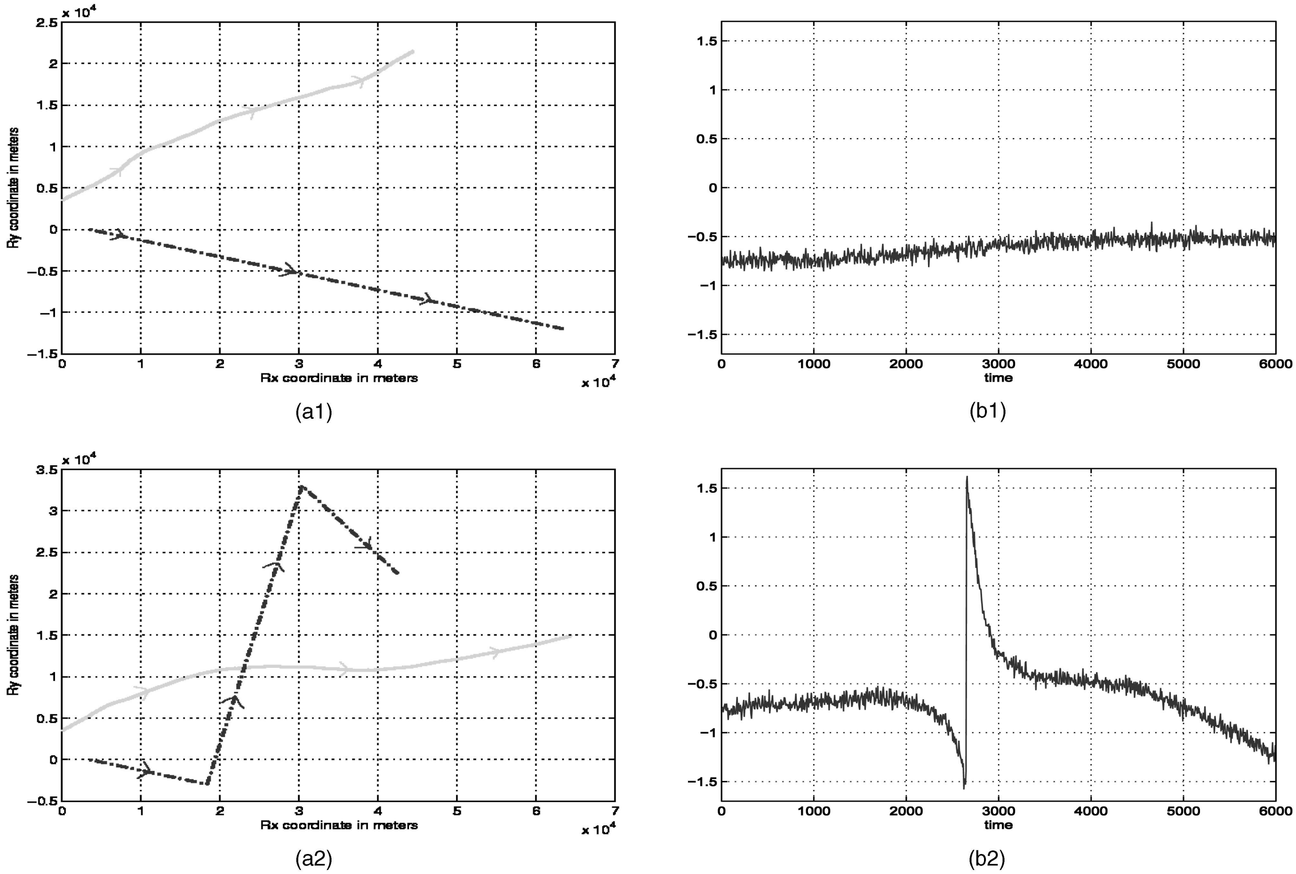


Fig. 5. Scenario 1. (a1) Example of trajectory of target (solid line) and observer (dashed line). (b1) Set of bearings measurements. Scenario 2. (a2) Example of trajectory of target (solid line) and observer (dashed line). (b2) Set of bearings measurements.

$$J_0^{-1} \approx \mathbb{E}\{(Y_0 - \mathbb{E}\{Y_0\})(Y_0 - \mathbb{E}\{Y_0\})^*\} \approx \frac{1}{N}\sum_{i=1}^{N}(Y_0^{(i)} - Y_0)^*(Y_0^{(i)} - Y_0). \tag{51}$$

The first scenario is presented in Fig. 5. An example of trajectory is presented in Fig. 5(a1), while the set of bearing measurements is presented in Fig. 5(b1). Fig. 6 presents the comparison of PCRB obtained by the algorithms given by Fig. 2 and Fig. 3 for the four components of the target state. The closed-formed PCRB and the classical one produce the same results which verify formulas. Moreover, the computation load difference between the two methods is important. The approximated PCRB takes about 600 sec when closed-form PCRB takes about 3 sec. Now looking at ρ_t's bound given Fig. 6(b), it is a bit surprising to see that the two PCRBs decrease while r_t is weakly observable. The fact is that ρ_t is not a meaningful component such that the bound given Fig. 6(b) for ECM_{ρ_t} (i.e., the ECM related to ρ_t) is not intuitive. A bound for ECM_{r_t} (i.e., the ECM related to r_t) would be more meaningful. Using a Taylor series, we can demonstrate that

$$\mathrm{ECM}_{r_t} \approx e^{2\mathbb{E}(\rho_t)}\mathrm{ECM}_{\rho_t} \tag{52}$$

TABLE II
Scenarios Constants

Duration	Scenario 1 6000 s	Scenario 2 6000 s
$r_x^{\text{obs}}(0)$	3, 5 km	3, 5 km
$r_y^{\text{obs}}(0)$	0 km	0 km
$v_x^{\text{obs}}(0)$	10 ms^{-1}	10 ms^{-1}
$v_y^{\text{obs}}(0)$	−2 ms^{-1}	−2 ms^{-1}
$r_x^{\text{cib}}(0)$	0 km	0 km
$r_y^{\text{cib}}(0)$	3, 5 km	3, 5 km
$v_x^{\text{cib}}(0)$	6 ms^{-1}	6 ms^{-1}
$v_y^{\text{cib}}(0)$	3 ms^{-1}	3 ms^{-1}
δ_t	6 s	6 s
$\sigma_{\max}$	0.05 ms^{-1}	0.05 ms^{-1}
σ_β	0.05 rad (about 3 deg)	0.05 rad (about 3 deg)
σ_{r_0}	2 km	2 km
σ_{v_0}	1 ms^{-1}	1 ms^{-1}
σ_{β_0}	0.05 rad (about 3 deg)	0.05 rad (about 3 deg)

so that

$$\mathrm{ECM}_{r_t} \geq e^{2\mathbb{E}(\rho_t)}\mathrm{FIM}_{\rho_t}. \tag{53}$$

Consequently, we can use the PCRB related to ρ_t to derive a bound for the ECM related to r_t. The problem

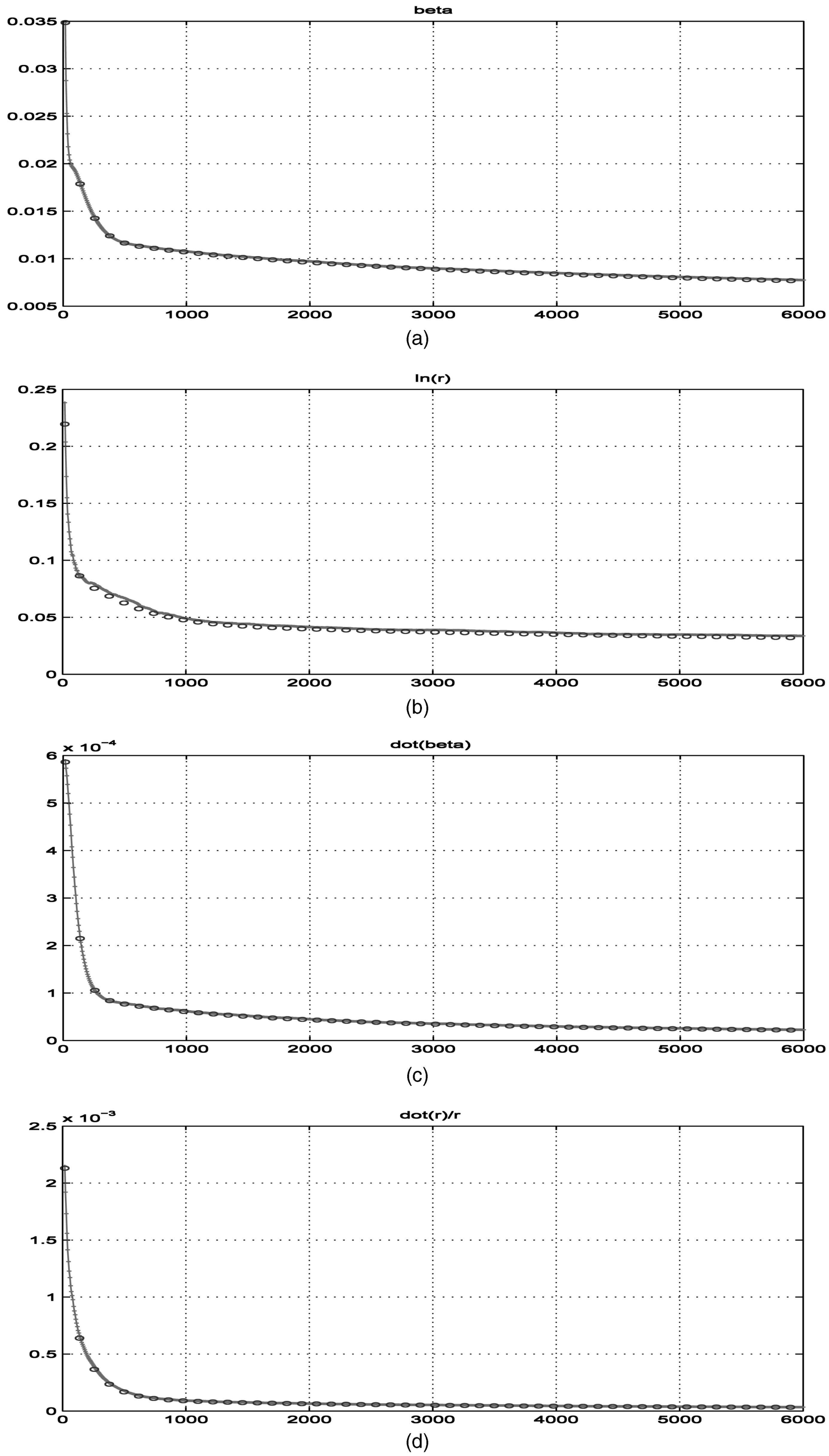


Fig. 6. PCRB for (a) β_t, (b) ρ_t, (c) $\dot{\beta}_t$, (d) $\dot{\rho}_t$.

is that $\mathbb{E}(\rho_t)$ is generally weakly observable. We have computed in Fig. 9 the bound given by (53) using the true r_t. We can see that the bound increases over time which matches theoretical observability results.

In the second scenario, the closed-form PCRB is checked when maneuvering terms appear. We consider that the observer follows a leg-by-leg trajectory. Its velocity vector is constant on each leg:

$$\begin{aligned} 1500 \le t \le 4500 \begin{pmatrix} v_x^{\mathrm{obs}}(t) \\ v_y^{\mathrm{obs}}(t) \end{pmatrix} &= \begin{pmatrix} 4 \text{ ms}^{-1} \\ 12 \text{ ms}^{-1} \end{pmatrix} \\ 4500 \le t \le \text{end} \begin{pmatrix} v_x^{\mathrm{obs}}(t) \\ v_y^{\mathrm{obs}}(t) \end{pmatrix} &= \begin{pmatrix} 8 \text{ ms}^{-1} \\ -7 \text{ ms}^{-1} \end{pmatrix}. \end{aligned} \tag{54}$$

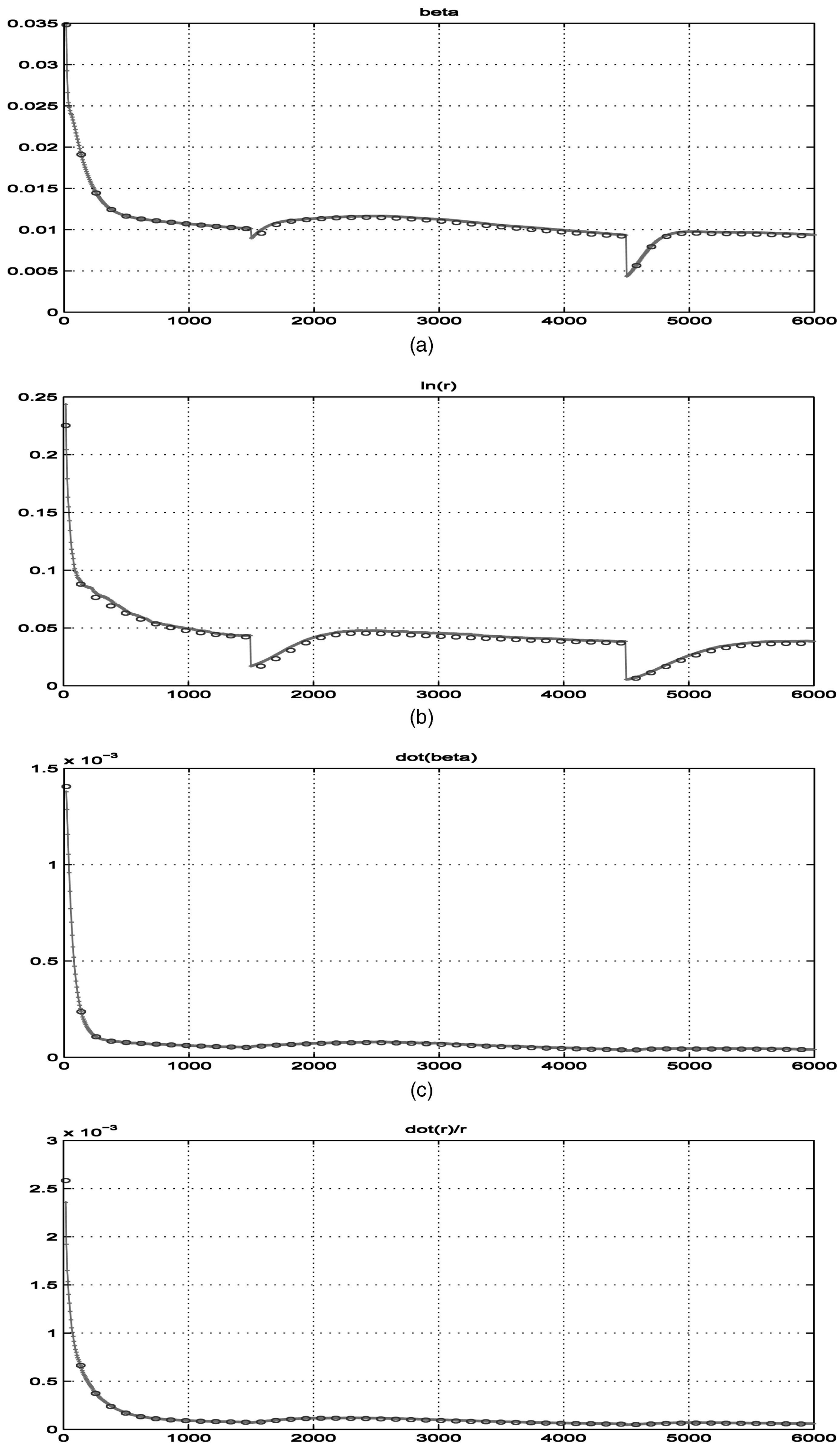


Fig. 7. PCRB for (a) β_t, (b) ρ_t, (c) $\dot{\beta}_t$, (d) $\dot{\rho}_t$ with scenario 2: closed-form PCRB (dashed line) versus approximated PCRB (solid line).

An example of trajectory for the second scenario is presented in Fig. 5(a2), while the set of bearing measurements is presented in Fig. 5(b2). Fig. 7 presents a comparison of PCRB obtained by the algorithms given in Fig. 2 and Fig. 3. We obtain the same results. Then the closed-form PCRB is valid in the maneuvering case. As for the previous scenario, we compute the bound given by (53) which is given by Fig. 10. As expected, the PCRB dramatically decreases when the observer maneuvers at time periods 1500 and 4500.

Consequently, we can now compute the PCRB accurately and quickly, making it suitable for sensor management applications. We have

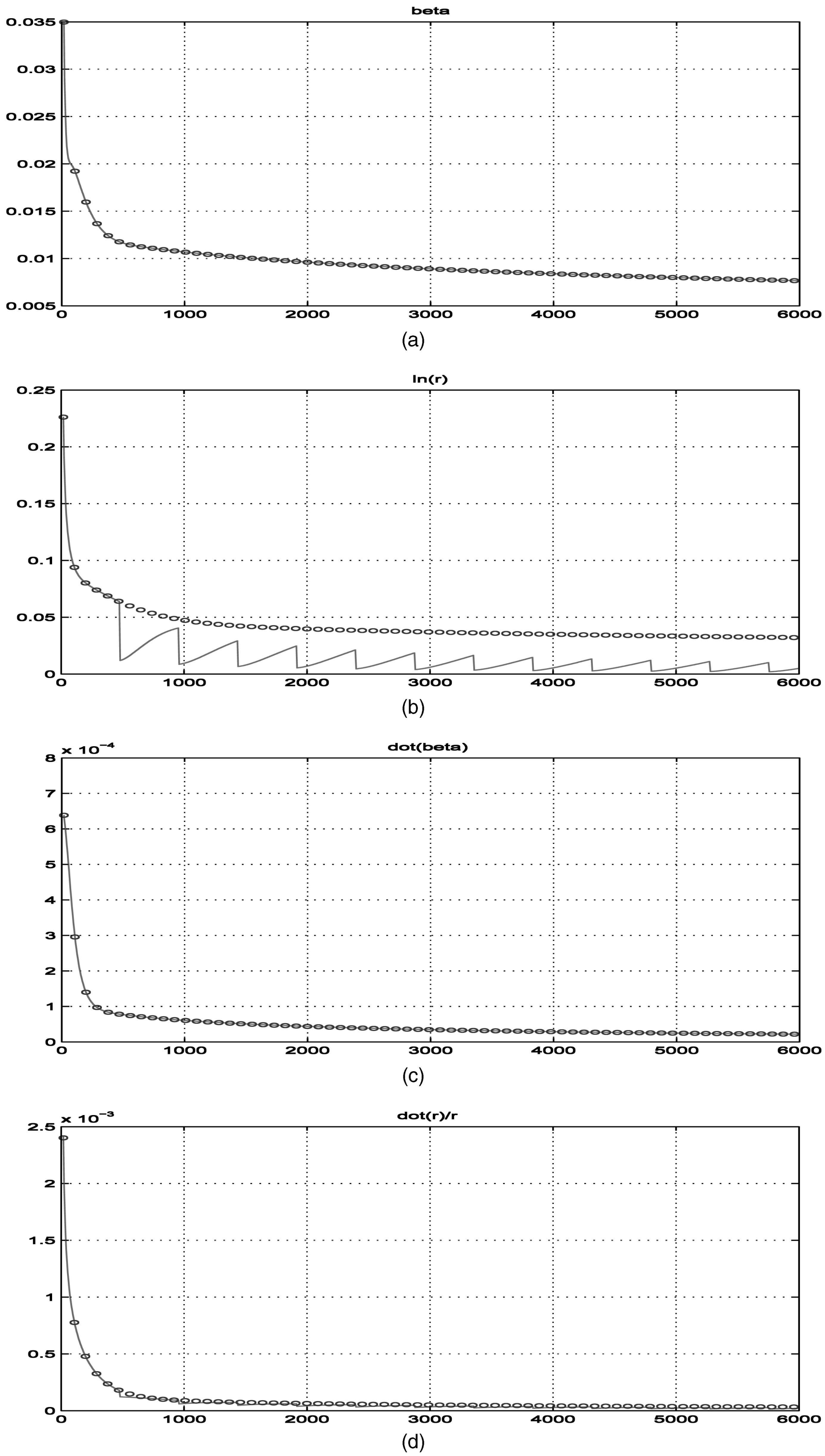


Fig. 8. Closed-form PCRB with range measurements scheduling (solid line) versus closed-form PCRB without range measurements (dashed line). (a) β_t, (b) ρ_t, (c) $\dot{\beta}_t$, (d) $\dot{\rho}_t$.

proposed in Section V an algorithm given by Fig. 4 which calculates the closed form PCRB for active measurement scheduling application. Fig. 8 presents a comparison based on the first scenario of the closed-form PCRB with active measurements produced every 80 sec with the closed-form when no active measurements are produced. In simulations, The range measurement standard deviation is set to σ_r = 100 m. As we can see in Fig. 8(b). ρ_t bound falls when the sensor produces a range measurement. Fig. 11 presented the related bounds for r_t given by (53).

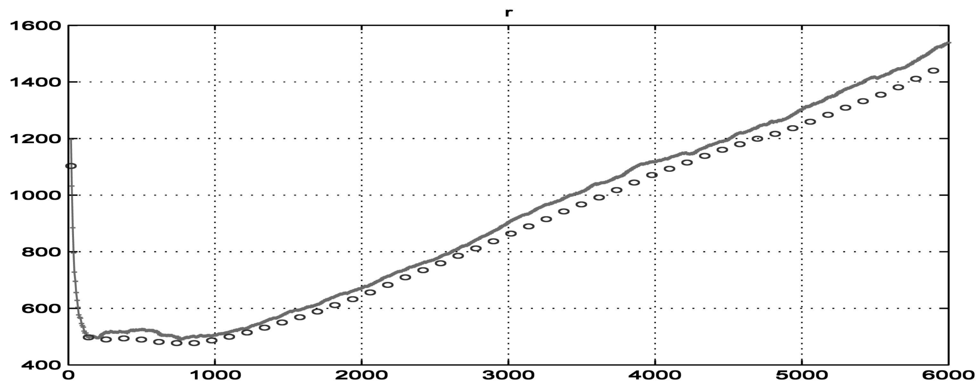


Fig. 9. PCRB for r_t with scenario 1: closed-form PCRB (dashed line) versus approximated PCRB (solid line).

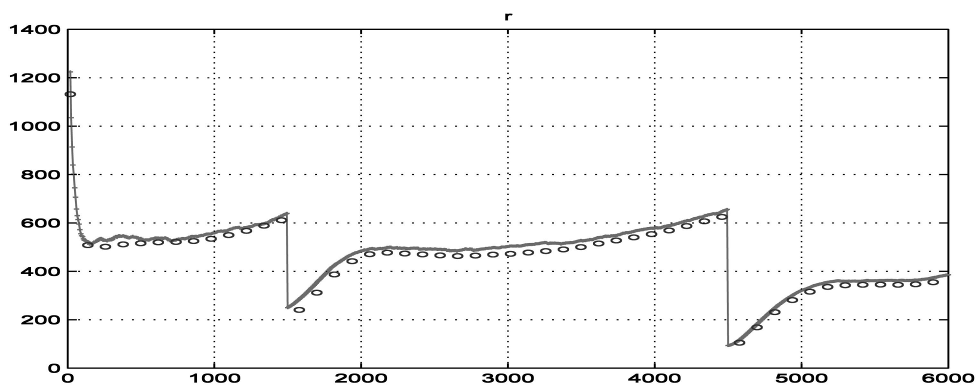


Fig. 10. PCRB for scenario 2: closed-form PCRB for r_t (dashed line) versus approximated PCRB for r_t (dashed line).

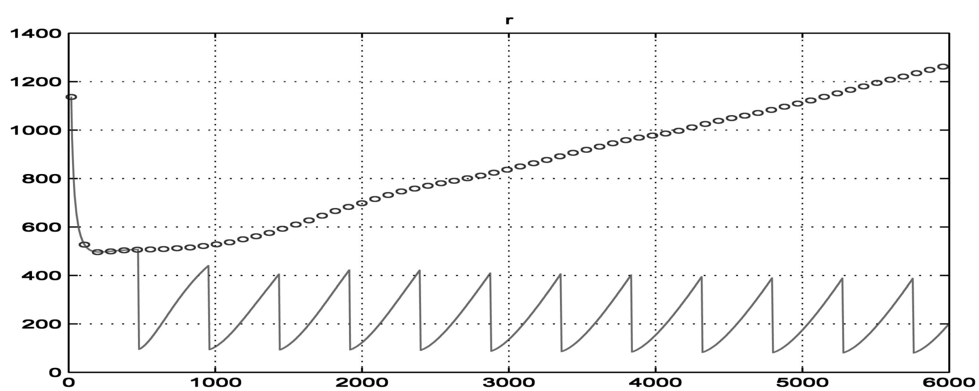


Fig. 11. Closed-form PCRB with range measurements scheduling for r_t (solid line) versus closed-form PCRB for r_t without range measurements (dashed line).

VII. CONCLUSION

An innovative analysis of the PCRB in the bearings-only context has been presented. In particular, strong results were shown with regards to the PCRB calculation; namely we derived an original closed-form PCRB. This power result, asserted by various simulations, cascades down from an original frame that consists in a new coordinate system: the LPC system. Computing the PCRB then becomes an accurate and time-varying technique of particular interest for real-time sensor management issues.

APPENDIX A. ABOUT THE BIAS

Bias definition as given by (12) may appear surprising at first. A more natural definition could be $\mathbb{E}\{\hat{Y}_{0:t} - Y_{0:t}\}$ where $\hat{Y}_{0:t}$ is an estimator of $Y_{0:t}$ and function of $Z_{1:t}$. It is this point of view we are now going to explain through a decomposition of the mean square error related to the estimation of $Y_{0:t}$. When estimating a deterministic parameter, the mean square error can be classically decomposed in estimation variance and bias. However, in the stochastic case, using (10), we only have the following relation:

$$\mathrm{ECM}_{0:t} = \|Y_{0:t} - \mathbb{E}\{\hat{Y}_{0:t} \mid Y_{0:t}\}\|^2 + \|\mathbb{E}\{\hat{Y}_{0:t} \mid Y_{0:t}\} - \hat{Y}_{0:t}\|^2. \tag{55}$$

The mean square error is then equal to the covariance estimation error if and only if

$$\|Y_{0:t} - \mathbb{E}\{\hat{Y}_{0:t} \mid Y_{0:t}\}\|^2 = 0. \tag{56}$$

Assumption (56) is equivalent to

$$\mathbb{E}\{Y_{0:t} - \hat{Y}_{0:t} \mid Y_{0:t}\} = 0, \qquad \text{for almost } Y_{0:t} \tag{57}$$

which is the retained definition of an unbiased estimator.

APPENDIX B. PROOF OF PROPOSITION 2

Proposition 2 is adapted from Proposition 1 to BOT context. More precisely, Proposition 2 gives a more simple formula for $C_{0:t}$. The idea of proof is to study this term. Looking at (22) in Proposition 1 proof, each $n_y \times n_y$-matrix term of $C_{0:t}$ can be rewritten

$$C_{0:t}(k,l) = Id_{n_y\times n_y}\delta_{k=l} + \int \Theta(k,l)d(Z_{1:t}, Y_{0:t}^{-\{l\}})$$

where

$$\Theta(k,l) = [(\hat{Y}_k - Y_k)p(Z_{1:t}, Y_{0:t})]_{\mathcal{Y}_l^-}^{\mathcal{Y}_l^+}. \tag{58}$$

Remark that $\mathcal{Y}_l^-$ and $\mathcal{Y}_l^+$ are n_y-vectors, so that $\Theta(k,l)$ is an $n_y \times n_y$-matrix (notation $[\ ]_{\mathcal{Y}_l^-}^{\mathcal{Y}_l^+}$ defined in (20)). First, let us rewrite $\Theta(k,l)$ using the statistical property of stochastic system (9). The idea is to use the following relation:

$$p(Z_{1:t}, Y_{0:t}) = \prod_{j=1}^{t}\{p(Z_j \mid Y_j)p(Y_j \mid Y_{j-1})\}p(Y_0) \tag{59}$$

which is true under two assumptions. First, the measurement at time t depends only on the target state at time t. Second, $\{Y_t\}_{t\in\mathbb{N}}$ is a Markovian process. These two assumptions are easily deduced from the formulation of the BOT problem given by (9). Then using (59), (58) is equivalent to

$$\Theta(k,l) = \left[(\hat{Y}_k - Y_k)\prod_{j=1}^{t}\{p(Z_j \mid Y_j)(Y_j \mid Y_{j-1})\}p(Y_0)\right]_{\mathcal{Y}_l^-}^{\mathcal{Y}_l^+}. \tag{60}$$

Now, one can see that some terms in (60) do not depend on Y_l so that they can be factorized. Then we obtain

$$\Theta(k,l) = \begin{cases} \theta(k,l)p(Z_{l+1:t}, Y_{l+2:t} \mid Y_{l+1}) & \text{if} \quad l = 0, \\ \theta(k,l)p(Z_{l+1:t}, Y_{l+2:t} \mid Y_{l+1})p(Y_{l-1}) & \text{if} \quad l = 1 \\ \theta(k,l)p(Z_{l+1:t}, Y_{l+2:t} \mid Y_{l+1})p(Z_{1:l-1}, Y_{0:l-1}) & \text{if} \quad 1 < l < t \\ \theta(k,l)p(Z_{1:l-1}, Y_{0:l-1}) & \text{if} \quad l = t \end{cases}$$

where

$$\theta(k,l) = \begin{cases} [(\hat{Y}_k - Y_k)p(Y_{l+1} \mid Y_l)p(Y_l)]_{\mathcal{Y}_l^-}^{\mathcal{Y}_l^+} & \text{if} \quad l = 0 \\ [(\hat{Y}_k - Y_k)p(Z_l \mid Y_l)p(Y_{l+1} \mid Y_l)p(Y_l \mid Y_{l-1})]_{\mathcal{Y}_l^-}^{\mathcal{Y}_l^+} & \text{if} \quad 0 < l < t \\ [(\hat{Y}_k - Y_k)p(Z_l \mid Y_l)p(Y_l \mid Y_{l-1})]_{\mathcal{Y}_l^-}^{\mathcal{Y}_l^+} & \text{if} \quad l = t. \end{cases} \tag{61}$$

We are thus reduced to calculate $\theta(k,l)$. Thus, the following limits must be studied:

$$\begin{aligned} &\lim_{Y_l\to\mathcal{Y}_l^+} p(Y_l \mid Y_{l-1}), \qquad \lim_{Y_l\to\mathcal{Y}_l^-} p(Y_l \mid Y_{l-1}) \\ &\lim_{Y_l\to\mathcal{Y}_l^+} p(Y_{l+1} \mid Y_l), \qquad \lim_{Y_l\to\mathcal{Y}_l^-} p(Y_{l+1} \mid Y_l) \\ &\lim_{Y_l\to\mathcal{Y}_l^+} p(Z_l \mid Y_l), \qquad \lim_{Y_l\to\mathcal{Y}_l^-} p(Z_l \mid Y_l). \end{aligned} \tag{62}$$

To study the first four limits, $p(Y_{l+1} \mid Y_l)$ derived in Appendix B1 is needed:

$$p(Y_{t+1} \mid Y_t) = r_{t+1}^4 p(X_{t+1} \mid X_t)\alpha(Y_t)$$

where

$$\begin{aligned} p(X_{t+1} \mid X_t) &= \frac{1}{4\pi^2\sqrt{\det(Q)}}e^{-\frac{1}{2}\|X_{t+1}-AX_t-U_t\|_Q^2}, \\ \alpha(Y_t) &= \mathbb{P}(r_y(l) > 0 \mid Y_l)\mathbf{1}_{\{r_y(l)>0\}} \\ &\quad + \mathbb{P}(r_y(l) < 0 \mid Y_l)\mathbf{1}_{\{r_y(l)<0\}}. \end{aligned} \tag{63}$$

We can notice that in (63), $p(X_{t+1} \mid X_t)$ is just the pdf of the diffusion process given by (3). The pdf of Y_{t+1} given Y_t is less simple than in Cartesian coordinate system because we do not have a direct bijection between the two coordinate systems.

Now let us remark that Y_l takes its values in $]-\pi/2, \pi/2[\times\mathbb{R}^3$ so that $\mathcal{Y}_l^- = [-\pi/2, -\infty, -\infty, -\infty]$ and $\mathcal{Y}_l^+ = [\pi/2, +\infty, +\infty, +\infty]$. According to (62), we must study $\lim_{Y_l\to\mathcal{Y}_l^-} p(X_{t+1} \mid X_t)$ and $\lim_{Y_l\to\mathcal{Y}_l^+} p(X_{t+1} \mid X_t)$ to derive the first four limits of (62). Using f_{lp}^c definition given by (7), we can obtain $\lim_{Y_l\to\mathcal{Y}_l^-} X_t$ and $\lim_{Y_l\to\mathcal{Y}_l^+} X_t$ via $\lim_{Y_l\to\mathcal{Y}_l^-} f_{lp}^c(Y_l)$ and $\lim_{Y_l\to\mathcal{Y}_l^+} f_{lp}^c(Y_l)$ and finally derive

$$\begin{aligned}\lim_{Y_l\to\mathcal{Y}_l^-} p(X_t \mid X_{t-1}) &= [p(X_t \mid X_{t-1})|_{\beta_l=-\pi/2}\ \ 0\ \ 0\ \ 0]\\ \lim_{Y_l\to\mathcal{Y}_l^+} p(X_t \mid X_{t-1}) &= [p(X_t \mid X_{t-1})|_{\beta_l=\pi/2}\ \ 0\ \ 0\ \ 0]\\ \lim_{Y_l\to\mathcal{Y}_l^-} p(X_{t+1} \mid X_t) &= [p(X_{t+1} \mid X_t)|_{\beta_l=-\pi/2}\ \ 0\ \ 0\ \ 0]\\ \lim_{Y_l\to\mathcal{Y}_l^+} p(X_{t+1} \mid X_t) &= [p(X_{t+1} \mid X_t)|_{\beta_l=\pi/2}\ \ 0\ \ 0\ \ 0].\end{aligned} \tag{64}$$

Now using (64) and notice that $\mathbb{P}(r_y(l) > 0 \mid Y_l)$ and $\mathbb{P}(r_y(l) < 0 \mid Y_l)$ are bounded functions, we obtain

$$\begin{aligned}\lim_{Y_l\to\mathcal{Y}_l^+} p(Y_l \mid Y_{l-1}) &= [p(Y_l \mid Y_{l-1})|_{\beta_l=\pi/2}\ \ 0\ \ 0\ \ 0]\\ \lim_{Y_l\to\mathcal{Y}_l^-} p(Y_l \mid Y_{l-1}) &= [p(Y_l \mid Y_{l-1})|_{\beta_l=-\pi/2}\ \ 0\ \ 0\ \ 0]\\ \lim_{Y_l\to\mathcal{Y}_l^+} p(Y_{l+1} \mid Y_l) &= [p(Y_{l+1} \mid Y_l)|_{\beta_l=\pi/2}\ \ 0\ \ 0\ \ 0]\\ \lim_{Y_l\to\mathcal{Y}_l^-} p(Y_{l+1} \mid Y_l) &= [p(Y_{l+1} \mid Y_l)|_{\beta_l=-\pi/2}\ \ 0\ \ 0\ \ 0].\end{aligned} \tag{65}$$

We have studied the four first limits of (62). Now, let us turn toward the two last ones. According to (4):

$$p(Z_l \mid Y_l) = p(Z_l \mid \beta_l). \tag{66}$$

We deduce from (66) that

$$\begin{aligned}\lim_{Y_l\to\mathcal{Y}_l^+} p(Z_l \mid Y_l) &= [p(Z_l \mid \beta_l)|_{\beta_l=\pi/2}\ \ p(Z_l \mid \beta_l)\ \ p(Z_l \mid \beta_l)\ \ p(Z_l \mid \beta_l)]\\ \lim_{Y_l\to\mathcal{Y}_l^-} p(Z_l \mid Y_l) &= [p(Z_l \mid \beta_l)|_{\beta_l=-\pi/2}\ \ p(Z_l \mid \beta_l)\ \ p(Z_l \mid \beta_l)\ \ p(Z_l \mid \beta_l)].\end{aligned} \tag{67}$$

Using limits given by (65) and (67), $\theta(k,l)$ given by (61) can be rewritten

$$\theta(k,l) = \begin{cases} [[(\hat{Y}_k - Y_k)p(Y_{l+1} \mid Y_l)p(Y_l)]_{-\pi/2}^{\pi/2}\quad 0_{n_y\times(n_y-1)}] & \text{if} \quad l = 0\\ [[(\hat{Y}_k - Y_k)p(Z_l \mid Y_l)p(Y_{l+1} \mid Y_l)p(Y_l \mid Y_{l-1})]_{-\pi/2}^{\pi/2}\quad 0_{n_y\times(n_y-1)}] & \text{if} \quad 1 < l < t\\ [[(\hat{Y}_k - Y_k)p(Z_l \mid Y_l)p(Y_l \mid Y_{l-1})]_{-\pi/2}^{\pi/2}\quad 0_{n_y\times(n_y-1)}] & \text{if} \quad l = t. \end{cases} \tag{68}$$

Consequently, lots of terms in $\theta(k,l)$ are equal to zero without any technical assumption. The problem is now to study more precisely the first column of $\theta(k,l)$. The following result assures a more simple formulation for this column.

LEMMA 2 *For a filtering problem given by* (9)

$$\begin{aligned}\lim_{\beta_l\to-\pi/2} p(Z_l \mid Y_l) &= \lim_{\beta_l\to\pi/2} p(Z_l \mid Y_l)\\ \lim_{\beta_l\to-\pi/2} p(Y_l \mid Y_{l-1}) &= \lim_{\beta_l\to\pi/2} p(Y_l \mid Y_{l-1})\\ \lim_{\beta_l\to-\pi/2} p(Y_{l+1} \mid Y_l) &= \lim_{\beta_l\to\pi/2} p(Y_{l+1} \mid Y_l).\end{aligned} \tag{69}$$

Lemma 2 is proved in Appendix B2. Using previous lemma, $\theta(k,l)$ formula given by (68) becomes

$$\theta(k,l) = \delta_{\{k=l\}} \begin{bmatrix} -\pi\zeta(l) & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where

$$\zeta(l) = \begin{cases} p(Y_{l+1} \mid Y_l)p(Y_l)|_{\beta_l=\pi/2} & \text{if} \quad l = 0,\\ p(Z_l \mid Y_l)p(Y_{l+1} \mid Y_l)p(Y_l \mid Y_{l-1})|_{\beta_l=\pi/2} & \text{if} \quad 0 < l < t,\\ p(Z_{1:t}, Y_{0:t})|_{\beta_l=\pi/2} & \text{if} \quad l = t. \end{cases} \tag{70}$$

Incorporating $\theta(k,l)$ new formula given by (70) in $\Theta(k,l)$ formulation given by (61), yields

$$\Theta(k,l) = \delta_{\{k=l\}} \begin{bmatrix} -\pi p(Z_{1:t}, Y_{0:t})|_{\beta_l=\pi/2} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{71}$$

Putting the new expression of $\Theta(k,l)$ given by (71) in $C_{0:t}$ formula given by (58), we deduce that $C_{0:t}$ is a diagonal matrix with diagonal element:

$$C_{0:t}(l,l) = \begin{bmatrix} 1 - \pi p(\beta_l)|_{\pi/2} & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{72}$$

APPENDIX B1. A CLOSED-FORM FOR $P(Y_{L+1} \mid Y_L)$

The aim of this section is to derive the pdf of Y_{l+1} given Y_l. The classical approach consists of proving that there exists a function $g_{Y_l}(.)$ such that

$$\mathbb{P}(Y_{l+1} \in A \mid Y_l) = \int_A g_{Y_l}(y_{l+1})d\lambda(y_{l+1}) \quad \forall \quad A \in \mathcal{B}\left(\left] -\frac{\pi}{2}, \frac{\pi}{2}\right[\times \mathbb{R}^3\right) \tag{73}$$

where $\mathcal{B}(]-\pi/2,\pi/2[\times\mathbb{R}^3)$ is the σ-algebra of Borel subsets of $]-\pi/2,\pi/2[\times\mathbb{R}^3$ and $\lambda(.)$ is Lebesgue measure. If this property is true then $g_{Y_l}(.)$ is the distribution density function of Y_{l+1} given Y_l. To obtain this result we use the distribution density function of X_{l+1} given X_l. However, computation is not easy because there is no direct bijection between Cartesian and LPC system. We only have (7) and (8). Then we have

$$\mathbb{P}(Y_{l+1} \in A \mid Y_l)$$
$$= \mathbb{P}(f_c^{lp}(X_{l+1}) \in A \mid Y_l)$$
$$= \mathbb{P}(f_c^{lp}(X_{l+1}) \in A \mid Y_l, \{r_y(l) > 0\})\mathbb{P}(\{r_y(l) > 0\} \mid Y_l) \quad (74)$$
$$+ \mathbb{P}(f_c^{lp}(X_{l+1}) \in A \mid Y_l, \{r_y(l) < 0\})\mathbb{P}(\{r_y(l) < 0\} \mid Y_l). \quad (75)$$

Then, using the pdf of X_{l+1} given X_l and the change of variable theorem, we obtain the pdf of Y_{l+1} given Y_l:

$$p(Y_{l+1} \mid Y_l) = r_{l+1}^4 p(X_{l+1} \mid X_l)\alpha(Y_l)$$

with

$$p(X_{l+1} \mid X_l) = \frac{1}{4\pi^2\sqrt{\det(Q)}} e^{-\frac{1}{2}\|X_{l+1}-AX_l-HU_l\|_Q^2},$$
$$\alpha(Y_l) = \mathbf{1}_{\{r_y(l)>0\}}\mathbb{P}(\{r_y(l) > 0\} \mid Y_l) + \mathbf{1}_{\{r_y(l)<0\}}\mathbb{P}(\{r_y(l) < 0\} \mid Y_l). \quad (76)$$

One can remark that the Jacobian term is r_{l+1}^4 where r_{l+1} is the relative range at time $t+1$. Moreover $p(X_{l+1} \mid X_l)$ is the pdf of the diffusion process given by (3). This term can be rewritten as function of Y_l and Y_{l+1} using Cartesian-to-LPC state mapping function given by (7).

APPENDIX B2. LEMMA 2 PROOF

First Relation of Lemma 2

According to (4), the pdf of Z_l given Y_l is

$$p(Z_l \mid Y_l) = \frac{1}{\sqrt{2\pi}\sigma_\beta}\sum_{k\in\mathbb{Z}} e^{-(Z_l-\beta_l-k\pi)^2/2\sigma_\beta^2}\mathbf{1}_{-\pi/2<Z_l<\pi/2}. \quad (77)$$

We can see examples of pdf of Z_l given Y_l in Fig. 1. Using $p(Z_l \mid Y_l)$ given by (77), we can see that the first relation of Lemma 2 is true.

Second Relation of Lemma 2

Looking at (76), we can see that we have just to prove that

$$\lim_{\beta_l\to-\pi/2} p(X_l \mid X_{l-1}) = \lim_{\beta_l\to\pi/2} p(X_l \mid X_{l-1}). \quad (78)$$

Then we need to express X_l as a function which depends on Y_l. Using (7), we obtain

$$\lim_{\beta_l\to-\pi/2} p(X_l \mid X_{l-1}) = \lim_{\beta_l\to-\pi/2} p(f_{lp}^c(Y_l) \mid X_{l-1})\mathbf{1}_{r_y(l)>0} + \lim_{\beta_l\to-\pi/2} p(-f_{lp}^c(Y_l) \mid X_{l-1})\mathbf{1}_{r_y(l)<0} \quad (79)$$

$$\lim_{\beta_l\to\pi/2} p(X_l \mid X_{l-1}) = \lim_{\beta_l\to\pi/2} p(f_{lp}^c(Y_l) \mid X_{l-1})\mathbf{1}_{r_y(l)>0} + \lim_{\beta_l\to\pi/2} p(-f_{lp}^c(Y_l) \mid X_{l-1})\mathbf{1}_{r_y(l)<0}.$$

Now if we note

$$X_l^{\pi/2} = [r_l \;\; 0 \;\; r_l\dot{\rho}_l \;\; -r_l\dot{\beta}_l]^* \quad (80)$$

we finally obtain

$$\lim_{\beta_l\to-\pi/2} p(X_l \mid X_{l-1}) = p(X_l^{\pi/2} \mid X_{l-1}) + p(-X_l^{\pi/2} \mid X_{l-1}) \quad (81)$$
$$\lim_{\beta_l\to\pi/2} p(X_l \mid X_{l-1}) = p(-X_l^{\pi/2} \mid X_{l-1}) + p(X_l^{\pi/2} \mid X_{l-1})$$

so that the second relation of Lemma 2 is true.

Third Relation of Lemma 2

Looking at (76), we can see that we have to prove that

$$\lim_{\beta_l\to-\pi/2} p(X_{l+1} \mid X_l)\alpha(Y_l) = \lim_{\beta_l\to\pi/2} p(X_{l+1} \mid X_l)\alpha(Y_l). \quad (82)$$

The proof is a little bit more difficult because we need to study $\alpha(Y_l)$ limit. First let us remark that $\alpha(Y_t)$ definition given by (76) can rewritten as

$$\alpha(Y_t) = \mathbb{P}(r_y(l) > 0 \mid |r_y(l)|)\mathbf{1}_{\{r_y(l)>0\}} + \mathbb{P}(r_y(l) < 0 \mid |r_y(l)|)\mathbf{1}_{\{r_y(l)<0\}}. \quad (83)$$

Now to study $\alpha(Y_l)$ limit, we need the following lemma.

LEMMA 3 *For X a scalar random variate*

$$\mathbb{P}(X > 0 \mid |X| = x) = \frac{p_X(x)}{p_X(x) + p_X(-x)}$$
$$\mathbb{P}(X < 0 \mid |X| = x) = \frac{p_X(-x)}{p_X(x) + p_X(-x)} \quad (84)$$

where p_X is the pdf *of X.*

PROOF OF LEMMA 3 First let us remark that for a positive ϵ, we can write

$$\mathbb{P}(X > 0 \mid |X| \in [x-\epsilon, x+\epsilon]) = \frac{\int_{x-\epsilon}^{x+\epsilon} p_X(x)dx}{\int_{x-\epsilon}^{x+\epsilon} p_X(x)dx + \int_{-x-\epsilon}^{-x+\epsilon} p_X(x)dx} \quad (85)$$

so that

$$M_\epsilon^- \le \mathbb{P}(X > 0 \mid |X| \in [x-\epsilon, x+\epsilon]) \le M_\epsilon^+$$

with

$$M_\epsilon^- = \frac{\inf_{[x-\epsilon,x+\epsilon]} p_X(x)}{\sup_{[x-\epsilon,x+\epsilon]} p_X(x) + \sup_{[-x-\epsilon,-x+\epsilon]} p_X(x)}$$
$$M_\epsilon^+ = \frac{\sup_{[x-\epsilon,x+\epsilon]} p_X(x)}{\inf_{[x-\epsilon,x+\epsilon]} p_X(x) + \inf_{[-x-\epsilon,-x+\epsilon]} p_X(x)}. \quad (86)$$

Then let ϵ converge to zero so that the first relation of the lemma is proved. The second relation is straightforward.

Applying Lemma 3 with $X = r_y(l)$ and finally remarking that $\lim_{\beta_l \to -\pi/2} r_y(l) = \lim_{\beta_l \to \pi/2} r_y(l) = 0$, we obtain

$$\lim_{\beta_l \to -\pi/2} \alpha(Y_t) = \lim_{\beta_l \to \pi/2} \alpha(Y_t) = \tfrac{1}{2} \tag{87}$$

so that

$$\lim_{\beta_l \to -\pi/2} p(X_{l+1} \mid X_l)\alpha(Y_l) = \tfrac{1}{2} p(X_{l+1} \mid -X_l^{\pi/2}) + \tfrac{1}{2} p(X_{l+1} \mid X_l^{\pi/2}) \tag{88}$$

$$\lim_{\beta_l \to \pi/2} p(X_l \mid X_{l-1})\alpha(Y_l) = \tfrac{1}{2} p(X_{l+1} \mid X_l^{\pi/2}) + \tfrac{1}{2} p(X_{l+1} \mid -X_l^{\pi/2})$$

with $X_l^{\pi/2}$ defined by (80). The third relation of lemma is proven.

APPENDIX C. PROPERTIES OF OPERATORS F AND G

Operators F and G are defined by (33). Before investigating the properties of such operators, let us remark that these operators can be rewritten using direct tensor product. First, let us study F_{X_t} which represents the derivative of the LPC-to-Cartesian mapping w.r.t. state in LPC. Using (7), we have

$$F_{X_t} = \nabla_{Y_t}\{X_t\} = \begin{cases} \nabla_{Y_t} f_{lp}^c(Y_t) & \text{if} \quad r_y(t) > 0 \\ -\nabla_{Y_t} f_{lp}^c(Y_t) & \text{if} \quad r_y(t) < 0. \end{cases} \tag{89}$$

Using now f_{lp}^c definition given by (7), we have

$$\nabla_{Y_t} f_{lp}^c(Y_t) = r_t \begin{bmatrix} \cos\beta_t & \sin\beta_t & 0 & 0 \\ -\sin\beta_t & \cos\beta_t & 0 & 0 \\ \dot{\rho}_t \cos\beta_t - \dot{\beta}_t \sin\beta_t & \dot{\rho}_t \sin\beta_t + \dot{\beta}_t \cos\beta_t & \cos\beta_t & \sin\beta_t \\ -\dot{\rho}_t \sin\beta_t - \dot{\beta}_t \cos\beta_t & \dot{\rho}_t \cos\beta_t - \dot{\beta}_t \sin\beta_t & -\sin\beta_t & \cos\beta_t \end{bmatrix}. \tag{90}$$

We can notice the block structure of $\nabla_{Y_t} f_{lp}^c(Y_t)$. Then using (89) and (90), F_{X_t} can be rewritten using Kronecker products, so that (33) can be rewritten as

$$F_{X_t} = Id_{2\times 2} \otimes R_{X_t} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes V_{X_t}, \qquad G_{X_t} = Id_{2\times 2} \otimes V_{X_t}$$

where

$$R_{X_t} = \begin{bmatrix} r_y(t) & r_x(t) \\ -r_x(t) & r_y(t) \end{bmatrix} \quad \text{and} \quad V_{X_t} = \begin{bmatrix} v_y(t) & v_x(t) \\ -v_x(t) & v_y(t) \end{bmatrix}. \tag{91}$$

Now let us detail the basic properties of $F_.$ and $G_.$ operators.

PROPERTY 1 *$G_.$ and $F_.$ are linear operators, i.e., let X_t and $\tilde{X}_t$ to state vector, then $F_{X_t+\tilde{X}_t} = F_{X_t} + F_{\tilde{X}_t}$ and $G_{X_t+\tilde{X}_t} = G_{X_t} + G_{\tilde{X}_t}$.*

PROPERTY 2 *Reminding that*

$$A = \begin{bmatrix} 1 & \delta_t \\ 0 & 1 \end{bmatrix} \otimes Id_{2\times 2}$$

terms $G_{A^k X_t}$ and $F_{A^k X_t}$ stand as follows:

$$F_{A^k X_t} = F_{X_t} + k\delta_t G_{X_t}, \qquad G_{A^k X_t} = G_{X_t}. \tag{92}$$

Proofs are omitted.

APPENDIX D. CLOSED FORMS FOR D_T^{11}, D_T^{12} AND D_T^{22} AND D_T^{33}

We show in this section that (28) can be rewritten as

$$\begin{aligned} D_t^{11} &= \frac{1}{\sigma_{\max}^2} \mathbb{E}\{F_{X_t}^* A^* Q^{-1} A F_{X_t}\} \\ D_t^{12} &= -\frac{1}{\sigma_{\max}^2} \mathbb{E}\{F_{X_t}^* A^* Q^{-1} F_{AX_t}\} - \Upsilon_t^{12} \\ D_t^{22} &= \frac{1}{\sigma_{\max}^2} \mathbb{E}\{F_{AX_t}^* Q^{-1} F_{AX_t}\} + \mathcal{C} + \Upsilon_t^{22} \\ D_t^{33} &= \begin{bmatrix} \frac{1}{\sigma_\beta^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{93}$$

with

$$\Upsilon_t^{12} = F_{\mathbb{E}X_t}^* A^* Q^{-1} F_{EX_{t+1}} - F_{\mathbb{E}X_t}^* A^* Q^{-1} F_{AEX_t}$$

$$\Upsilon_t^{22} = F_{\mathbb{E}X_{t+1}}^* Q^{-1} F_{\mathbb{E}X_{t+1}} - F_{A\mathbb{E}X_t}^* Q^{-1} F_{A\mathbb{E}X_t}$$

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & \dfrac{2\alpha_3^2}{\alpha_3\alpha_1 - \alpha_2^2} & 0 \\ 0 & 0 & 0 & \dfrac{2\alpha_3^2}{\alpha_3\alpha_1 - \alpha_2^2} \end{pmatrix}.$$

Considering at D_t^{11}, D_t^{12} and D_t^{22} and D_t^{33} formulas given by (28), it is necessary to derive $p(Y_{t+1} \mid Y_t)$ and $p(Z_t \mid Y_t)$. According to Appendix B1:

$$p(Y_{t+1} \mid Y_t) = r_{t+1}^4 p(X_{t+1} \mid X_t)\alpha(Y_t). \qquad (94)$$

More precisely, according to (28), we need $\nabla_{Y_t} \ln p(Y_{t+1} \mid Y_t)$, $\nabla_{Y_{t+1}} \ln p(Y_{t+1} \mid Y_t)$ and $\nabla_{Y_t} \ln p(Z_t \mid Y_t)$. Using $p(Y_{t+1} \mid Y_t)$ as given by (94) and remarking that $\nabla_{Y_t}\alpha(Y_t) = 0$, we obtain

$$\begin{aligned}
&\nabla_{Y_t} p(Y_{t+1} \mid Y_t) \\
&\quad = \frac{1}{\sigma_{\max}^2} r_{t+1}^4 F_{X_t}^* A^* Q^{-1}(X_{t+1} - AX_t - U_t) p(X_{t+1} \mid X_t)\alpha(Y_t) \\
&\nabla_{Y_{t+1}} p(Y_{t+1} \mid Y_t) \\
&\quad = r_{t+1}^4 \left(-\frac{1}{\sigma_{\max}^2} F_{X_{t+1}}^* Q^{-1}(X_{t+1} - AX_t - U_t) + [0\ 4\ 0\ 0]^*\right) \\
&\qquad \times p(X_{t+1} \mid X_t)\alpha(Y_t)
\end{aligned} \qquad (95)$$

where F_{X_t} is defined by (33). Then, using (94) and (95), we obtain

$$\begin{aligned}
&\nabla_{Y_t} \ln p(Y_{t+1} \mid Y_t) \\
&\quad = \frac{1}{\sigma_{\max}^2} F_{X_t}^* A^* Q^{-1}(X_{t+1} - AX_t - U_t) \\
&\nabla_{Y_{t+1}} \ln p(Y_{t+1} \mid Y_t) \\
&\quad = -\frac{1}{\sigma_{\max}^2} F_{X_{t+1}}^* Q^{-1}(X_{t+1} - AX_t - U_t) + [0\ 4\ 0\ 0]^*.
\end{aligned} \qquad (96)$$

Incorporating $\nabla_{Y_t} \ln p(Y_{t+1} \mid Y_t)$, $\nabla_{Y_{t+1}} \ln p(Y_{t+1} \mid Y_t)$ given by (96) in (28), we obtain:

$$\begin{aligned}
D_t^{11} &= \frac{1}{\sigma_{\max}^4}\mathbb{E}\{F_{X_t}^* A^* Q^{-1}(X_{t+1} - AX_t - U_t) \\
&\qquad * (X_{t+1} - AX_t - U_t)^* Q^{-1} A F_{X_t}\}, \\
D_t^{12} &= -\frac{1}{\sigma_{\max}^4}\mathbb{E}\{F_{X_t}^* A^* Q^{-1}(X_{t+1} - AX_t - U_t) \\
&\qquad \times (X_{t+1} - AX_t - U_t)^* Q^{-1} F_{X_{t+1}}\}, \\
D_t^{22} &= \frac{1}{\sigma_{\max}^4}\mathbb{E}\{F_{X_{t+1}}^* Q^{-1}(X_{t+1} - AX_t - U_t) \\
&\qquad * (X_{t+1} - AX_t - U_t)^* Q^{-1} F_{X_{t+1}}\} \\
&\quad - \frac{1}{\sigma_{\max}^2}\mathbb{E}\{F_{X_{t+1}}^* Q^{-1}(X_{t+1} - AX_t - U_t)\}[0\ 4\ 0\ 0] \\
&\quad - \frac{1}{\sigma_{\max}^2}[0\ 4\ 0\ 0]^*\mathbb{E}\{(X_{t+1} - AX_t - U_t)^* Q^{-1} F_{X_{t+1}}\} \\
&\quad + [0\ 4\ 0\ 0]^*[0\ 4\ 0\ 0].
\end{aligned} \qquad (97)$$

Now, we are dealing with the calculation of each elementary term of (97) separately.

D_t^{11} Formula: Let us rewrite D_t^{11} as given by (97), we have

$$\begin{aligned}
D_t^{11} &= \frac{1}{\sigma_{\max}^4}\mathbb{E}\{F_{X_t}^* A^* Q^{-1}(X_{t+1} - AX_t - U_t)(X_{t+1} - AX_t - U_t)^* Q^{-1} A F_{X_t}\} \\
&= \frac{1}{\sigma_{\max}^4}\mathbb{E}\{F_{X_t}^* A^* Q^{-1} \underbrace{\mathbb{E}\{(X_{t+1} - AX_t - U_t)(X_{t+1} - AX_t - U_t)^* \mid X_t\}}_{=\sigma_{\max}^2 Q} Q^{-1} A F_{X_t}\}.
\end{aligned} \qquad (98)$$

Then using the statistical property of X_{t+1} given X_t, i.e., $\mathcal{N}(AX_t + U_t, \sigma_{\max}^2 Q)$ given by (3), we obtain D_t^{11} formula as given by (93).

D_t^{12} Formula: Our aim is now to render explicit D_t^{12} given by (97). Let us first use the linear property of $F_.$:

$$\begin{aligned}
D_t^{12} &= -\frac{1}{\sigma_{\max}^4}\overbrace{\mathbb{E}\{F_{X_t}^* A^* Q^{-1}(X_{t+1} - AX_t - U_t)(X_{t+1} - AX_t - U_t)^* Q^{-1} F_{X_{t+1} - AX_t - U_t}\}}^{=0} \\
&\quad - \frac{1}{\sigma_{\max}^4}\mathbb{E}\{F_{X_t}^* A^* Q^{-1}(X_{t+1} - AX_t - U_t)(X_{t+1} - AX_t - U_t)^* Q^{-1} F_{AX_t + U_t}\}.
\end{aligned} \qquad (99)$$

Using the statistical property of X_{t+1}, i.e., X_{t+1} given X_t is an $\mathcal{N}(AX_t + U_t, Q)$, we obtain

$$D_t^{12} = -\frac{1}{\sigma_{\max}^2}\mathbb{E}\{F_{X_t}^* A^* Q^{-1} F_{AX_t}\} - \frac{1}{\sigma_{\max}^2} F_{\mathbb{E}X_t}^* A^* Q^{-1} F_{U_t}. \tag{100}$$

Now remarking that $U_t = \mathbb{E}X_{t+1} - AX_t$ and the linearity of operator F, we obtain D_t^{12} expression given by (93).

D_t^{22} Formula: Starting from D_t^{22} given by (97) and using again the linearity of F:

where

$$\Omega^{11} = \frac{1}{\sigma_{\max}^2}\begin{pmatrix} \mathbb{E}\{F_{(X_t-AX_{t-1}-U_{t-1})}^* A^* Q^{-1} A F_{(X_t-AX_{t-1}-U_{t-1})}\} \\ \mathbb{E}\{F_{(X_t-AX_{t-1}-U_{t-1})}^* A^* Q^{-1} A G_{(X_t-AX_{t-1}-U_{t-1})}\} \\ \mathbb{E}\{G_{(X_t-AX_{t-1}-U_{t-1})}^* A^* Q^{-1} A F_{(X_t-AX_{t-1}-U_{t-1})}\} \\ \mathbb{E}\{G_{(X_t-AX_{t-1}-U_{t-1})}^* A^* Q^{-1} A G_{(X_t-AX_{t-1}-U_{t-1})}\} \end{pmatrix}. \tag{103}$$

$$D_t^{22} = \frac{1}{\sigma_{\max}^4}\overbrace{\mathbb{E}\{F_{AX_t+U_t}^* Q^{-1}(X_{t+1} - AX_t - U_t)(X_{t+1} - AX_t - U_t)^* Q^{-1} F_{X_{t+1}-AX_t-U_t}\}}^{=0} + \frac{1}{\sigma_{\max}^4}\mathbb{E}\{F_{AX_t+U_t}^* Q^{-1}(X_{t+1} - AX_t - U_t)(X_{t+1} - AX_t - U_t)^* Q^{-1} F_{AX_t+U_t}\} + \mathcal{C} \tag{101}$$

with

$$\mathcal{C} = \frac{1}{\sigma_{\max}^4}\mathbb{E}\{F_{X_{t+1}-AX_t-U_t}^* Q^{-1}(X_{t+1} - AX_t - U_t)(X_{t+1} - AX_t - U_t)^* Q^{-1} F_{X_{t+1}-AX_t-U_t}\} - \frac{1}{\sigma_{\max}^2}\mathbb{E}\{F_{X_{t+1}-AX_t-U_t}^* Q^{-1}(X_{t+1} - AX_t - U_t)\}(0\ 4\ 0\ 0) - \frac{1}{\sigma_{\max}^2}\mathbb{E}\{(0\ 4\ 0\ 0)^* \mathbb{E}(X_{t+1} - AX_t - U_t)^* Q^{-1} F_{X_{t+1}-AX_t-U_t}\} + (0\ 4\ 0\ 0)^*(0\ 4\ 0\ 0).$$

Let us notice that we can show using F definition given by (33) and the statistical property of X_{t+1} (i.e., X_{t+1} given X_t is $\mathcal{N}(AX_t + U_t, \sigma_{\max}^2 Q)$ distributed) that the $\mathcal{C}$ definition given by (102) is equivalent to the $\mathcal{C}$ definition given by (93). Now, using again the statistical property of X_{t+1}, we obtain

$$D_t^{22} = \frac{1}{\sigma_{\max}^4}\mathbb{E}\{F_{AX_t+U_t}^* Q^{-1}(X_{t+1} - AX_t - U_t) \times (X_{t+1} - AX_t - U_t)^* Q^{-1} F_{AX_t+U_t}\} + \mathcal{C}. \tag{102}$$

To end the proof, the linearity of the operator F and the equality $U_t = \mathbb{E}X_{t+1} - X_t$ allow us to infer (93) from (102).

APPENDIX E1. PROOF OF PROPOSITION 5.1

The proof of Proposition 5.1 is based on the properties of F_{X_t} and G_{X_t} investigated in Appendix C. Developing Γ_t^{11} given by (35) and using the linearity of operator F, we obtain

$$\Gamma_t^{11} = \Omega^{11} + \frac{1}{\sigma_{\max}^2}\begin{pmatrix} \mathbb{E}\{F_{(AX_{t-1}+U_{t-1})}^* A^* Q^{-1} A F_{(AX_{t-1}+U_{t-1})}\} \\ \mathbb{E}\{F_{(AX_{t-1}+U_{t-1})}^* A^* Q^{-1} A G_{(AX_{t-1}+U_{t-1})}\} \\ \mathbb{E}\{G_{(AX_{t-1}+U_{t-1})}^* A^* Q^{-1} A F_{(AX_{t-1}+U_{t-1})}\} \\ \mathbb{E}\{G_{(AX_{t-1}+U_{t-1})}^* A^* Q^{-1} A G_{(AX_{t-1}+U_{t-1})}\} \end{pmatrix}$$

Now remarking that $U_{t-1} = \mathbb{E}X_t - A\mathbb{E}X_{t-1}$ and using linear property of operator F, we obtain

$$\Gamma_t^{11} = \Omega^{11} + \frac{1}{\sigma_{\max}^2}\begin{pmatrix} \mathbb{E}\{F_{AX_{t-1}}^* A^* Q^{-1} A F_{AX_{t-1}}\} \\ \mathbb{E}\{F_{AX_{t-1}}^* A^* Q^{-1} A G_{AX_{t-1}}\} \\ \mathbb{E}\{G_{AX_{t-1}}^* A^* Q^{-1} A F_{AX_{t-1}}\} \\ \mathbb{E}\{G_{AX_{t-1}}^* A^* Q^{-1} A G_{AX_{t-1}}\} \end{pmatrix} + \mathbf{\Lambda}_{t-1}^{11} \tag{104}$$

where $\mathbf{\Lambda}_{t-1}^{11}$ is defined by (36). According to Appendix C, $F_{AX_{t-1}} = F_{X_{t-1}} + \delta_t G_{X_{t-1}}$ and $G_{AX_{t-1}} = G_{X_{t-1}}$, so that

$$\Gamma_t^{11} = \Omega^{11} + \Psi \Gamma_{t-1}^{11} + \mathbf{\Lambda}_{t-1}^{11} \tag{105}$$

where Ψ is defined by (36). It remains to show that Ω^{11} has a more simple formula using the following lemma.

LEMMA 4 *For X and Y two state vectors, let us define*

$$\Theta = \begin{pmatrix} \mathbb{E}(F_X^*(\Sigma \otimes Id_{2\times 2})F_Y) \\ \mathbb{E}(F_X^*(\Sigma \otimes Id_{2\times 2})G_Y) \\ \mathbb{E}(G_X^*(\Sigma \otimes Id_{2\times 2})F_Y) \\ \mathbb{E}(G_X^*(\Sigma \otimes Id_{2\times 2})G_Y) \end{pmatrix} \tag{106}$$

where operators F and G are defined by (33). *Then*

$$\Theta = \begin{pmatrix} \Sigma \otimes \mathbb{E}\{R_X^* R_Y\} + \Sigma_{\nwarrow} \otimes \mathbb{E}\{V_X^* V_Y\} + \Sigma_{\downarrow} \otimes \mathbb{E}\{V_X^* R_Y\} + \Sigma_{\uparrow} \otimes \mathbb{E}\{R_X^* V_Y\} \\ \Sigma_{\uparrow} \otimes \mathbb{E}\{V_X^* V_Y\} + \Sigma \otimes \mathbb{E}\{R_X^* V_Y\} \\ \Sigma_{\leftarrow} \otimes \mathbb{E}\{V_X^* V_Y\} + \Sigma \otimes \mathbb{E}\{V_X^* R_Y\} \\ \Sigma \otimes \mathbb{E}\{V_X^* V_Y\} \end{pmatrix}$$

where

$$\Sigma_{\uparrow} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Sigma, \qquad \Sigma_{\leftarrow} = \Sigma \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$
$$\Sigma_{\nwarrow} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Sigma \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{107}$$

PROOF OF LEMMA 4 We just have to rewrite (106) using F and G formulas given by (33). We prove Lemma 4 using direct tensor product properties.

To end the proof, Lemma 4 is applied with

$$\begin{aligned} X &= X_t - AX_{t-1} - U_{t-1} \\ Y &= X_t - AX_{t-1} - U_{t-1} \\ \Sigma \otimes Id_{2\times 2} &= \frac{1}{\sigma_{\max}^2} A^* Q^{-1} A. \end{aligned} \tag{108}$$

Then, using the statistical property of X_t, i.e., X_t given X_{t-1} is $\mathcal{N}(AX_{t-1} + U_{t-1}, \sigma_{\max}^2 Q)$-distributed, we obtain

$$\begin{aligned} \mathbb{E}\{R_X^* R_Y\} &= 2\sigma_{\max}^2 \alpha_3 Id_{2\times 2} \\ \mathbb{E}\{R_X^* V_Y\} &= 2\sigma_{\max}^2 \alpha_2 Id_{2\times 2} \\ \mathbb{E}\{V_X^* R_Y\} &= 2\sigma_{\max}^2 \alpha_2 Id_{2\times 2} \\ \mathbb{E}\{V_X^* V_Y\} &= 2\sigma_{\max}^2 \alpha_1 Id_{2\times 2} \end{aligned} \tag{109}$$

so that Ω^{11} is given by (101).

APPENDIX E2. PROOF OF PROPOSITION 5.2

Using the same approach as in Proposition 5.1 proof, we have

$$\Gamma_t^{12} = \Omega^{12} + \Psi \Gamma^{12}(t-1) + \mathbf{\Lambda}_{t-1}^{12}$$

where Ψ and $\mathbf{\Lambda}_{t-1}^{12}$ are given by (36) and (42) and

$$\Omega^{12} = \frac{1}{\sigma_{\max}^2} \begin{pmatrix} \mathbb{E}\{F^*_{(X_t-AX_{t-1}-U_{t-1})} A^* Q^{-1} F_{A(X_t-AX_{t-1}-U_{t-1})}\} \\ \mathbb{E}\{F^*_{(X_t-AX_{t-1}-U_{t-1})} A^* Q^{-1} G_{A(X_t-AX_{t-1}-U_{t-1})}\} \\ \mathbb{E}\{G^*_{(X_t-AX_{t-1}-U_{t-1})} A^* Q^{-1} F_{A(X_t-AX_{t-1}-U_{t-1})}\} \\ \mathbb{E}\{G^*_{(X_t-AX_{t-1}-U_{t-1})} A^* Q^{-1} G_{A(X_t-AX_{t-1}-U_{t-1})}\} \end{pmatrix}. \tag{110}$$

Lemma 4 is again the key for simplifying Ω^{12}, and is used with

$$\begin{aligned} X &= X_t - AX_{t-1} - U_{t-1} \\ Y &= A(X_t - AX_{t-1} - U_{t-1}) \\ \Sigma \otimes Id_{2\times 2} &= \frac{1}{\sigma_{\max}^2} A^* Q^{-1}. \end{aligned} \tag{111}$$

Now, using the statistical property of X_t, i.e., X_t given X_{t-1} is $\mathcal{N}(AX_{t-1} + U_{t-1}, \sigma_{\max}^2 Q)$-distributed, we obtain for Ω^{12} the simple formula given by (39).

APPENDIX E3. PROOF OF PROPOSITION 5.3

The proof again mimics that of Proposition 5.1. Thus, we first obtain

$$\Gamma_t^{22} = \Omega^{22} + \Psi \Gamma_{t-1}^{22} + \mathbf{\Lambda}_t^{22}$$

where Ψ and $\mathbf{\Lambda}_{t-1}^{22}$ given by (36) and (42), and

$$\Omega^{22} = \frac{1}{\sigma_{\max}^2} \begin{pmatrix} \mathbb{E}\{F^*_{A(X_t-AX_{t-1}-U_{t-1})} Q^{-1} F_{A(X_t-AX_{t-1}-U_{t-1})}\} \\ \mathbb{E}\{F^*_{A(X_t-AX_{t-1}-U_{t-1})} Q^{-1} G_{A(X_t-AX_{t-1}-U_{t-1})}\} \\ \mathbb{E}\{G^*_{A(X_t-AX_{t-1}-U_{t-1})} Q^{-1} F_{A(X_t-AX_{t-1}-U_{t-1})}\} \\ \mathbb{E}\{G^*_{A(X_t-AX_{t-1}-U_{t-1})} Q^{-1} G_{A(X_t-AX_{t-1}-U_{t-1})}\} \end{pmatrix}. \tag{112}$$

We prove now that Ω^{22} has a more simple formula using Lemma 4 with

$$\begin{aligned} X &= A(X_t - AX_{t-1} - U_{t-1}) \\ Y &= A(X_t - AX_{t-1} - U_{t-1}) \\ \Sigma \otimes Id_{2\times 2} &= \frac{1}{\sigma_{\max}^2} Q^{-1}. \end{aligned} \tag{113}$$

Then, using the statistical property of X_t, i.e., X_t given X_{t-1} is $\mathcal{N}(AX_{t-1} + U_{t-1}, \sigma_{\max}^2 Q)$-distributed, we obtain for Ω^{22} the formula given by (42).

REFERENCES

[1] Rong Li, X., and Jilkov, V.
A survey of maneuvering target tracking. Part I: Dynamics models.
IEEE Transactions on Aerospace and Electronic Systems, **39**, 4 (Oct. 2003), 1333–1364.

[2] Lindgren, A. G., and Gong, K. F.
Position and velocity estimation via bearings observations.
IEEE Transactions on Aerospace and Electronic Systems, **AES-14**, 4 (July 1978), 564–577.

[3] Song, T. L., and Speyer, J. L.
A stochastic analysis of a modified gain extended Kalman filter with applications to estimation with bearings only measurements.
IEEE Transactions on Autmatic Control, **30**, 10 (Oct. 1985), 940–949.

[4] Aidala, V. J., and Nardone, S. C.
Biased estimation properties of the pseudolinear tracking filter.
IEEE Transactions on Aerospace and Electronic Systems, **AES-18**, 4 (July 1982), 432–441.

[5] Lerro, D., and Bar-Shalom, Y.
Bias compensation for improved recursive bearings-only target state estimation.
In *American Control Conference, Seattle, WA*, June 1995, 648–652.

[6] Aidala, V. J.
Kalman filter behaviour in bearings-only tracking applications.
IEEE Transactions on Aerospace and Electronic Systems, **AES-15**, 1 (Jan. 1979), 29–39.

[7] Nardone, S. C., and Aidala, V. J.
Observability criteria for bearings-only target motion analysis.
IEEE Transactions on Aerospace and Electronic Systems, **17**, 2 (Mar. 1981), 161–166.

[8] Mohler, R. R., and Hwang, S.
Nonlinear data observability and information.
The Franklin Institute, **325**, 4 (1988), 443–464.

[9] Aidala, V. J., and Hammel, S. E.
Utilization of modified polar coordinates for bearing-only tracking.
IEEE Transactions on Automatic Control, **28**, 3 (Mar. 1983), 283–294.

[10] Peach, N.
Bearing-only tracking using a set of range-parametrised extented Kalman filters.
IEEE Proceedings on Control Theory Application, **142**, 1 (Jan. 1995), 73–80.

[11] Gordon, N., Salmond, D., and Smith, A.
Novel approach to non-linear/non-Gaussian Bayesian state estimation.
Proceedings of the IEE, **140**, 2 (Apr. 1993), 107–113.

[12] Pitt, M. K., and Shephard, N.
Filtering via simulation: Auxiliary particle filters.
Journal of the American Statistical Association, **94** (1999), 590–599.

[13] Hue, C., Le Cadre, J.-P., and Pérez, P.
Sequential Monte Carlo methods for multiple target tracking and data fusion.
IEEE Transactions on Signal Processing, **50**, 2 (Feb. 2002), 309–325.

[14] Arulampalam ,S., and Ristic, B.
Comparaison of the particule filter with range-parameterised and modified polar EKFs for angle-only tracking.
In *Conference on Signal and Data Processing of Small Targets*, vol. 4048, (SPIE Annual International Symposium on Aerosense), 2000, 288–299.

[15] Ristic, B., Arulampalam, S., and Gordon, N.
Beyond the Kalman Filter, Particle Filters for Tracking Applications.
Norwood, MA: Artech House, 2004.

[16] Van Trees, H. L.
Detection, Estimation and Modulation Theory.
New York: Wiley, 1968.

[17] Bergman, N., Doucet, A., and Gordon, N.
Optimal estimation and Cramér-Rao bounds for partial non-Gaussian state space models.
Ann. Ins. Statist. Math, **53**, 1 (1998), 97–112.

[18] Zhang, X., Willett, P., and Bar-Shalom, Y.
The Cramér-Rao bound for dynamic target tracking with measurement origin uncertainty.
In *The 41st IEEE Conference on Decision and Control*, 2002.

[19] Hernandez, M. L., Marrs, A. D., Gordon, N. J., Marskell, S. R., and Reed, C. M.
Cramér-Rao bounds for non-linear filtering with measurement origin uncertainty.
In *Proceedings of the 5th International Conference on Information Fusion*, Annapolis, MD, July 2002.

[20] Bessel, A., Ristic, B., Farina, A., Wang, X., and Arulampalam, M. S.
Error performance bounds for tracking a manoeuvring target.
In *Proceedings of the 6th International Conference on Information Fusion*, Cairns, Queensland, Australia, 2003.

[21] Ristic, B., Zolllo, S., and Arulampalam, S.
Performance bounds for manoeuvring target tracking using asynchronous multi-platform angle-only measurements.
In *Proceedings of the 4th International Conference on Information Fusion*, Montréal, Quebec, Canada, July 2001.

[22] Hue, C., Le Cadre, J.-P., and Pérez, P.
Performance analysis of two sequential Monte Carlo methods and posterior Cramér-Rao bounds for multi-target tracking.
Technical report, IRISA, 2002.

[23] Tichavský, P., Muravchik, C., and Nehorai, A.
Posterior Cramér-Rao bounds for discrete-time nonlinear filtering.
IEEE Transactions on Signal Processing, **46**, 5 (May 1998), 1386–1396.

[24] Hernandez, M., Kirubarajan, T., and Bar-Shalom, Y.
Multisensor resource deployment using posterior Cramér-Rao bounds.
IEEE Transactions on Aerospace and Electronic Systems, **40**, 2 (Apr. 2004), 399–416.

[25] Hernandez, M.
Optimal sensor trajectories in bearings-only tracking.
In *Proceedings of the 7th International Conference on Information Fusion*, Stockholm, Sweden, July 2004.

[26] Doucet, A., De Freitas, N., and Gordon, N.
Sequential Monte Carlo Methods in Practice. New York: Springer, 2001.

[27] Arulampalam, M. S., Maskell, S., Gordon, N., and Clapp, T.
A tutorial on particule filters for online non-linear/non-Gaussian Bayesian tracking.
IEEE Transactions on Signal Processing, **50**, 2 (Feb. 2002).

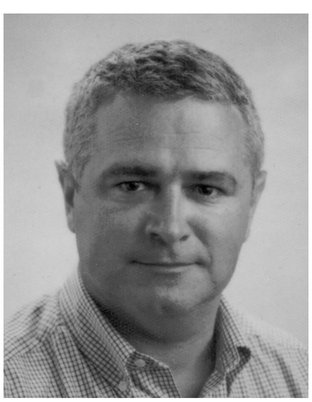
[28] Horn, R. A., and Johnson, C. R.
Matrix Analysis.
New York: Cambridge Univ. Press, 1985.

[29] Avitzour, D., and Rogers, S. R.
Optimal measurements scheduling for prediction and estimation.
IEEE Transactions on Acoustic, Speech and Signal Processing, **38**, 10 (Oct. 1990), 1733–1739.

[30] Shakeri, M., Pattipati, K. R., and Kleinman, D. L.
Optimal measurements scheduling for state estimation.
IEEE Transactions on Aerospace and Electronic Systems, **31**, 2 (Apr. 1995), 716–729.

[31] Le Cadre, J.-P.
Scheduling active and passive measurements.
In *Proceedings of the 3rd International Conference on Information Fusion*, Paris, France, July 2000.

Thomas Bréhard was born in 1978. He graduated from the Superior Engineering School of Statistics and Information Sciences in 2002. He received the M.Sc. degree in statistics in 2002 and the Ph.D. degree in applied mathematics in 2005, both from the University of Rennes, France. His research interests include statistical methods for target tracking, performance analysis and sensor management.



Jean-Pierre Le Cadre (M'93) received the M.S. degree in mathematics in 1977, the "Doctorat de 3^{-eme} cycle" in 1982, and the "Doctorat d'Etat" in 1987, both from INPG, Grenoble.

From 1980 to 1989, he worked at the GERDSM (Groupe d'Etudes et de Recherche en Detection Sous-Marine), a laboratory of the DCN (Direction des Constructions Navales), mainly on array processing. Since 1989, he is with IRISA/CNRS, where he is "Directeur de Recherche" at CNRS. His interests are now topics like system analysis, detection, multitarget tracking, data association, and operations research.

Dr. Le Cadre has received (with O. Zugmeyer) the Eurasip Signal Processing best paper award (1993).