2005 7th International Conference on Information Fusion (FUSION)

Performance Analysis of Optimal Data Association within a Linear Regression Framework

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Abstract – *This paper is concerned with performance prediction of multiple target tracking system and especially with the analysis of track purity. Effects of misassociation is considered so as to provide closed-form expressions of the probability of correct association. The linear regression framework is used throughout and provide powerful tools.*

1 Introduction

One important problem in multiple-target tracking is to evaluate the performance of a multiple-target tracking system, in which we would like to relate system environment parameters to key performance measures. In particular, two performance measures are relevant: track purity (i.e. percentage of data originating form the track to be tracked), and track accuracy. In this area, seminal contributions certainly include those of K.C. Chang, C.Y. Chong and S. Mori [2], [3], [4]. Following the general guidelines of these works, our general aim is to use the linear regression [1] framework for calculating the probability of correctly associating a measure to a given track in a multi-target environment.

Using basic results of linear regression [1], a general expression of the probability of correct association is provided. This expression is valid in a general N-scan environment and involve only elementary parameters like the inter-track distance and the variance of measurement. Actually, it is simply the integral of a non-linear function (*erfc*) of the quotient of two quadratic forms. For easing interpretations, approximations of the integrand are considered and provide meaningful results. This approach can be extended to a variety of situations (multiple outliers, crossing tracks, etc.).

The linear regression framework is used throughout and provide a simple and feasible way for analyzing the effect of misassociation. In the non-linear case, having a similar comfort is certainly hopeless but our approach is still relevant via local *linear* approximations. Practically, linear regression is relevant for completely (or almost completely) observed systems, like for radar applications.

2 Problem formulation

A target is moving with rectilinear and uniform motion. Noisy measurements consisting of Cartesian positions represented by the points $P_1 = (x_1, y_1), P_2 = (x_2, y_2), \cdots, P_N = (x_N, y_N)$ are observed by an observer at time periods t_1, t_2,$\cdots$,t_N and called the "scans". The position measurements are the exact Cartesian positions $P_1^* = (x_1^*, y_1^*), \cdots, P_N^* = (x_N^*, y_N^*)$ corrupted by a sequence of independent and identically normally distributed noises.

When a target is (*sufficiently*) isolated from others, there is no ambiguity about the measurement origin. It is not true any more if it happens that a second target comes to stand in the vicinity of the first target. In this case, it becomes possible to make a mistake about the origin of an observation by associating it to the wrong target; thus corrupting target trajectory estimation. But the question is to give a more precise meaning to the term "sufficiently isolated".

Thus, the aim of this article is to give a closed-form expression for the probability of correct association of a measurement to a target track as a function of the number of scans and the distance of the outlier observation. In order to simplify the scenario, we consider that one outlier measurement $P_E = (x_E, y_E)$ is located close to the true last target position $P_N^* = (x_N^*, y_N^*)$ at time t_N with a distance of λ. The general problem setting and definitions are depicted in fig. 1.

Let us denote $\delta_i = t_{i+1} - t_i \quad , \; i = 1 \cdots N-1$ the duration between the times t_{i+1} and t_i, and:

$$\vec{v} = (v_x^*, v_y^*)$$

the two components of the constant target velocity on the Cartesian axis. The target trajectory is then exactly defined by the true parameters state vector $(x_1^*, y_1^*, v_x^*, v_y^*)$. For the rest of the paper, an uppercase * will correspond to the exact (or ideal) model.

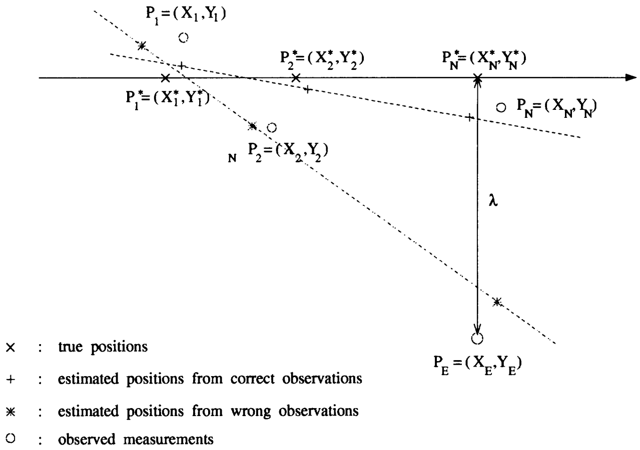

Figure 1: correct association (P_1, P_2, P_3) and wrong association (P_1, P_2, P_6).

3 Problem analysis

Under the assumption of correct association, the measurements depicted in fig. 1 can be written:

$$\left\{\begin{array}{lcl} x_1 & = & x_1^* + \varepsilon_{x_1}^* \, , \\ y_1 & = & y_1^* + \varepsilon_{y_1}^* \, , \\ x_2 & = & x_2^* + \varepsilon_{x_2}^* \, , \\ y_2 & = & y_2^* + \varepsilon_{y_2}^* \, , \\ \vdots & \vdots & \quad \vdots \\ x_N & = & x_N^* + \varepsilon_{x_N}^* \, , \\ y_N & = & y_N^* + \varepsilon_{y_N}^* \, . \end{array}\right.$$

where $(\varepsilon_{x_1}^*, \varepsilon_{y_1}^*, \varepsilon_{x_2}^*, \varepsilon_{y_2}^*, \cdots, \varepsilon_{x_N}^*, \varepsilon_{y_N}^*)^T$ is the vector of true[1] noise measurements, normally distributed,i.e. $\varepsilon_{x_i}^* \sim \mathcal{N}(0, \sigma_{x_i})$ and $\varepsilon_{y_i}^* \sim \mathcal{N}(0, \sigma_{y_i})$. To simplify the problem, we assume that $\sigma_{x_i} = \sigma_{y_i} = 1$.

Taking into account the uniform rectilinear target motion, the correct association model is:

$$\left\{\begin{array}{lcl} x_1 & = & x_1^* + \varepsilon_{x_1}^* \\ y_1 & = & y_1^* + \varepsilon_{y_1}^* \\ x_2 & = & x_1^* + v_x^* \cdot \delta_1 + \varepsilon_{x_2}^* \\ y_2 & = & y_1^* + v_y^* \cdot \delta_1 + \varepsilon_{y_2}^* \\ x_3 & = & x_1^* + v_x^* \cdot (\delta_1 + \delta_2) + \varepsilon_{x_2}^* \\ y_3 & = & y_1^* + v_y^* \cdot (\delta_1 + \delta_2) + \varepsilon_{y_2}^* \\ \vdots & \vdots & \vdots \\ x_N & = & x_1^* + v_x^* \cdot (\delta_1 + \delta_2 + \cdots + \delta_{N-1}) + \varepsilon_{x_N}^* \\ y_N & = & y_1^* + v_y^* \cdot (\delta_1 + \delta_2 + \cdots + \delta N - 1) + \varepsilon_{y_N}^* \end{array}\right. \tag{1}$$

Denoting $\tau_i \triangleq \delta_1 + \delta_2 + \cdots + \delta_i \quad , \quad i = 1 \cdots N-1$, eq. 1 has the equivalent matrix formulation:

$$\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ \vdots \\ x_N \\ y_N \end{pmatrix} = \underbrace{\begin{pmatrix} I_2 & 0_2 \\ I_2 & \tau_1 I_2 \\ I_2 & \tau_2 I_2 \\ \vdots & \vdots \\ I_2 & \tau_{N-1} I_2 \end{pmatrix}}_{\mathcal{X}} \begin{pmatrix} x_1^* \\ y_1^* \\ v_x^* \\ v_y^* \end{pmatrix} + \begin{pmatrix} \varepsilon_{x_1}^* \\ \varepsilon_{y_1}^* \\ \varepsilon_{x_2}^* \\ \varepsilon_{y_2}^* \\ \varepsilon_{x_3}^* \\ \varepsilon_{y_3}^* \\ \vdots \\ \varepsilon_{x_N}^* \\ \varepsilon_{y_N}^* \end{pmatrix} \tag{2}$$

Thus, let us denote:
$Y^* \stackrel{\delta}{=} (x_1^*, y_1^*, x_2^*, y_2^*, \cdots, x_N^*, y_N^*)^T$
the vector made of the exact positions, while $Y_C \stackrel{\delta}{=} (x_1, y_1, x_2, y_2, \cdots, x_N, y_N)^T$ stands for the vector of the (complete) observations associated with a correct association hypothesis. To ease further calculations, the following notations are defined:

- $\mathcal{X} \triangleq \begin{pmatrix} I_2 & 0_2 \\ I_2 & \tau_1 I_2 \\ I_2 & \tau_2 I_2 \\ \vdots & \vdots \\ I_2 & \tau_{N-1} I_2 \end{pmatrix}$, a $(2N) \times 4$ matrix,
- $\beta^* = (x_1^*, y_1^*, v_x^*, v_y^*)^T$ the true state vector,
- $\varepsilon_C^* = (\varepsilon_{x_1}^*, \varepsilon_{y_1}^*, \varepsilon_{x_2}^*, \varepsilon_{y_2}^*, \cdots, \varepsilon_{x_N}^*, \varepsilon_{y_N}^*)^T$, measurement noise vector under the correct association assumption.

With these definitions and under the correct association hypothesis, the model simply stands as follows:

$$\begin{array}{lcl} Y_C & = & Y^* + \varepsilon_C^* \\ & = & \mathcal{X}\,\beta^* + \varepsilon_C^* \end{array}$$

3.1 The regression model

Consider the following general regression model:

$$Y = \mathcal{X}\,\beta + \varepsilon \, , \tag{3}$$

where Y are the data, $\mathcal{X}$ are the regressors and β is the vector of parameters, to be estimated. Under the perfect assumption hypothesis (denoted by * uppercases), we thus have:

$$Y = \mathcal{X}\,\beta^* + \varepsilon^* \, . \tag{4}$$

The observed value of random vector Y is then the sum of a deterministic component $\mathcal{X} \cdot \beta$ and a random component ε^* which represents observation noise. From this model, β is to be estimated by minimizing the norm of the residual vector $\varepsilon = Y - \mathcal{X}\,\beta$. Generally, the estimation of β is made by using the quadratic loss function:

$$\mathcal{L}_2(\beta) = (Y - \mathcal{X}\,\beta)^T (Y - \mathcal{X}\,\beta) = \|Y - \mathcal{X}\,\beta\|^2 \, . \tag{5}$$

[1] *True* means here correct association.

If the matrix $\mathcal{X}^T\mathcal{X}$ is non-singular, then $\mathcal{L}_2(\beta)$ is minimum for the unique value $\hat{\beta}$ of β such that:

$$\hat{\beta} = (\mathcal{X}^T\,\mathcal{X})^{-1}\mathcal{X}^T Y\,. \tag{6}$$

From the estimation $\hat{\beta}$ of β, let be $\hat{Y}$ the estimator of the mean $\mathcal{X}\,\beta$ of the random vector Y defined by:

$$\begin{aligned} \widehat{Y} &= \mathcal{X}\,\hat{\beta}\,, \\ &= \mathcal{X}(\mathcal{X}^T\mathcal{X})^{-1}\,\mathcal{X}^T\,Y\,, \\ &= \mathcal{H}\,Y\,, \end{aligned} \tag{7}$$

with $\mathcal{H} \triangleq \mathcal{X}(\mathcal{X}^T\mathcal{X})^{-1}\,\mathcal{X}^T$.

The vector of the residuals $\hat{\varepsilon} \triangleq Y - \mathcal{X}\,\hat{\beta}$ is given by:

$$\begin{aligned} \hat{\varepsilon} &= Y - \widehat{Y}\,, \\ &= Y - \mathcal{H}\,Y \\ &= (I - \mathcal{H})\,Y\,, \\ &= M\,Y \end{aligned} \tag{8}$$

with $M \triangleq I - \mathcal{H}$, where I is the identity matrix.

It is easy to check that M is a projection matrix, i.e :

$$\begin{cases} M^T &= M\,, \\ M^2 &= M\,. \end{cases}$$

We also recall the following classical identities, which will be used subsequently:

$$\begin{aligned} M\,\mathcal{X} &= (I - \mathcal{H})\,\mathcal{X}\,, \\ &= \mathcal{X} - \mathcal{H}\,\mathcal{X}\,, \\ &= \mathcal{X} - \mathcal{X}(\mathcal{X}^T\mathcal{X})^{-1}\,\mathcal{X}^T\mathcal{X}\,, \\ &= 0\,, \end{aligned}$$

and:

$$\begin{aligned} \hat{\varepsilon} &= M\,Y\,, \\ &= M(\mathcal{X}\beta^* + \varepsilon^*)\,, \\ &= M\varepsilon^*\,. \end{aligned} \tag{9}$$

3.2 Evaluation of the correct association probability

Assume that the outlier measurement $P_E = (x_E, y_E)$ is located at the point (see fig. 1):

$$\begin{cases} x_E &= x_N^*\,, \\ y_E &= y_N^* - \lambda\,. \end{cases}$$

The correct association is then defined by the points $P_1, P_2, \cdots, P_{N-1}, P_N$ whereas the wrong association is defined by $P_1, P_2, \cdots, P_{N-1}, P_E$ (the lowercase E stands for erroneous association).

Now, let us define:

$$Y_F \triangleq (x_1, y_1, x_2, y_2, \cdots, x_E, y_E)^T\,, \tag{10}$$

the vector of erroneous association. So that we have:

$$\begin{cases} Y_C = Y^* + \varepsilon_C^* \\ \text{and:} \\ Y_F = Y^* + \varepsilon_F^*\,, \end{cases}$$

with:

$$\varepsilon_C^* = \begin{pmatrix} \varepsilon_{x_1}^* \\ \varepsilon_{y_1}^* \\ \varepsilon_{x_2}^* \\ \varepsilon_{y_2}^* \\ \vdots \\ \varepsilon_{x_N}^* \\ \varepsilon_{y_N}^* \end{pmatrix} \quad \text{and} \quad \varepsilon_F^* = \begin{pmatrix} \varepsilon_{x_1}^* \\ \varepsilon_{y_1}^* \\ \varepsilon_{x_2}^* \\ \varepsilon_{y_2}^* \\ \vdots \\ 0 \\ -\lambda \end{pmatrix}$$

In eq. 11, ε_C^* is the vector made of differences between the true positions $P_1^*, P_2^*, \cdots, P_N^*$ and "correct" observations $P_1, P_2, \cdots, P_N$, whereas ε_F^* is the vector of differences between the true positions $P_1^*, P_2^*, \cdots, P_N^*$ and "incorrect" observations $P_1, P_2, \cdots, P_{N-1}, P_E$ (erroneous association on scan N).

The vectors of residuals $\varepsilon_C^* = Y^* - Y_C$ and $\varepsilon_F^* = Y^* - Y_F$ being unknown are estimated from a linear regression on the available observations vectors Y_C and Y_F, leading to $\hat{\varepsilon}_C$ and $\hat{\varepsilon}_F$, defined as:

$$\begin{cases} \hat{\varepsilon}_C = Y_C - \hat{Y}_C\,, \\ \text{and:} \\ \hat{\varepsilon}_E = Y_F - \hat{Y}_F\,. \end{cases}$$

The costs of correct ($\mathcal{C}_C$) and erroneous ($\mathcal{C}_F$) associations are defined by the square residuals norms and developed by using the relation 9 :

$$\begin{aligned} \mathcal{C}_C &= (Y_C - \widehat{Y_C})^T\,(Y_C - \widehat{Y_C}) \\ &= \hat{\varepsilon}_C^T\,\hat{\varepsilon}_C \\ &= (M\,\varepsilon_C^*)^T(M\,\varepsilon_C^*) \\ &= (\varepsilon_C^*)^T\,M^TM\,(\varepsilon_C^*) \\ &= (\varepsilon_C^*)^T\,M\,(\varepsilon_C^*)\,. \end{aligned} \tag{11}$$

In the same way, we have also:

$$\mathcal{C}_F = (\epsilon_F^*)^T\,M\,(\epsilon_F^*) \tag{12}$$

Let us define now Δ_c the difference between the correct and wrong costs, i.e. :

$$\Delta_c \triangleq \mathcal{C}_C - \mathcal{C}_F\,. \tag{13}$$

Then, the probability of correct association is defined by the probability that $\Delta_c \geq 0$. **The aim of this article is to give closed-form expressions for this probability**.

Let be ε_{COM}^* the vector of components, that vectors ε_C^*

and ε_F^* have in common, i.e.:

$$\varepsilon_{COM}^* = \begin{pmatrix} \varepsilon_{x_1}^* \\ \varepsilon_{y_1}^* \\ \varepsilon_{x_2}^* \\ \varepsilon_{y_2}^* \\ \vdots \\ \varepsilon_{x_{N-1}}^* \\ \varepsilon_{y_{N-1}}^* \\ 0 \\ 0 \end{pmatrix}$$

and define ε_P^* and L as the complementary vectors:

$$\varepsilon_P^* = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ x_N^* \\ y_N^* \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -\lambda \end{pmatrix} \tag{14}$$

then:

$$\varepsilon_C^* = \varepsilon_{COM}^* + \varepsilon_P^* \quad , \quad \varepsilon_F^* = \varepsilon_{COM}^* + L$$

With these notations, the difference between the correct and wrong costs Δ_c (see eq. 13) can be written:

$$\begin{aligned} &\Delta_c \\ &= c_{MA} - c_{BA} \,, \\ &= (\varepsilon_F^*)^T M \varepsilon_F^* - (\varepsilon_C^*)^T M \varepsilon_C^* \,, \\ &= (\varepsilon_{COM}^* + L)^T M (\varepsilon_{COM}^* + L) \,, \\ &-(\varepsilon_{COM}^* + \varepsilon_P^*)^T M (\varepsilon_{COM}^* + \varepsilon_P^*) \\ &= L^T M L - (\varepsilon_P^*)^T M \varepsilon_P^* - 2(\varepsilon_P^* - L)^T M \varepsilon_{COM}^* \,. \end{aligned}$$

Since the components of the vector ε_{COM}^* are normally distributed and supposed independent, this vector is Gaussian and distributed as:

$$\varepsilon_{COM}^* \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & & & & & & (0) \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & (0) & & & & 0 & \\ & & & & & & 0 \end{pmatrix}\right)$$

Let us define:

$$\sum\nolimits_{COM} = \begin{pmatrix} 1 & & & & & & (0) \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & (0) & & & & 0 & \\ & & & & & & 0 \end{pmatrix} \tag{15}$$

For the same reasons, we have also:

$$\varepsilon_P^* \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & & & & & & (0) \\ & 0 & & & & & \\ & & 0 & & & & \\ & & & 0 & & & \\ & & & & \ddots & & \\ & (0) & & & & 1 & \\ & & & & & & 1 \end{pmatrix}\right)$$

Assuming that the vector ε_P^* is set to a *fixed* value e_P, the law of the difference of costs $f(\Delta_{coûts} | \varepsilon_P^* = e_P)$ is Gaussian with characteristics :

$$\begin{aligned} &f(\Delta_c | \varepsilon_P^* = e_P) \\ &= \mathcal{N}\left(L^T M L - (e_P)^T M e_P \,, 4(e_P - L)^T M \textstyle\sum_{COM} M (e_P - L)\right) \end{aligned}$$

Defining the Φ projection matrix :

$$\Phi = M \sum\nolimits_{COM} M \tag{16}$$

the precedent density becomes:

$$\begin{aligned} &f(\Delta_c | \varepsilon_P^* = e_P) \\ &= \mathcal{N}\left(L^T M L - (e_P)^T M e_P \,, 4(e_P - L)^T \Phi \, (e_P - L) \,.\right) \end{aligned} \tag{17}$$

Therefore the conditional probability of correct association given $\varepsilon_P^* = e_P$ is:

$$P(\Delta_c \geq 0 | \varepsilon_P^* = e_P) = erfc\left(\frac{(e_P)^T M e_P - L^T M L}{2\sqrt{(e_P - L)^T \Phi \, (e_P - L)}} \,.\right) \tag{18}$$

where *erfc* refers to the complementary function of the normal cumulative density function.

The probability of correct association is then obtained by integrating the conditional density relating to the Gaussian vector ε_P^* :

$$\begin{aligned} &P(\Delta_c \geq 0) \\ &,= \iint P(\Delta_c \geq 0 | \varepsilon_P^* = e_P) \; f_{\varepsilon_P}(e_P) \; de_P \,, \\ &= \iint erfc\left(\frac{(e_P)^T M e_P - L^T M L}{2\sqrt{(e_P - L)^T \Phi \, (e_P - L)}}\right) \; f_{\varepsilon_P}(e_P) \; de_P \,, \\ &= \iint erfc(\Psi) \; f_{\varepsilon_P}(e_P) \; de_P \,, \end{aligned} \tag{19}$$

where we have defined the *functional* Ψ by:

$$\Psi = \frac{(e_P)^T M e_P - L^T M L}{2\sqrt{(e_P - L)^T \Phi \, (e_P - L)}} \,. \tag{20}$$

Considering eq. 19, it is not surprising that it is the functional Ψ as defined by eq. 20 which will play the fundamental role for analyzing the probability of correct association.

3.3 Closed-form approximations of the probability of correct association

Our aim is now to obtain a closed-form approximation of the probability of correct association. To obtain it, it is possible to use a (limited) expansion of the Ψ functional by considering that e_P is very small in regard of L, more precisely by assuming :

$$\|M \cdot e_p\| \ll \|M \cdot L\| \,. \tag{21}$$

Then let us define the factor ρ by the following equation:

$$e_p^T M e_p = \rho^2 \left(L^T M L\right) \,. \tag{22}$$

In order to express the denominator of Ψ, it is necessary to compute : $e_p^T \Phi\, e_p$, $e_p^T \Phi\, L$ et $L \Phi L$. Referring to (16)), we have:

$$\begin{aligned} e_p^T \Phi\, e_p &= e_P^T M \sum\nolimits_{COM} M e_p \,, \\ &= e_P^T M \left(\sum\nolimits_{COM} \cdot \sum\nolimits_{COM}\right) M e_p \,, \\ &= \left(\sum\nolimits_{COM} M e_p\right)^T \left(\sum\nolimits_{COM} M e_p\right) \,, \\ &= \left\|\sum\nolimits_{COM} M e_p\right\|^2 \,. \end{aligned} \tag{23}$$

Now, it is trivially shown that $\sum_{COM}$ (see eq. 15) is a projection matrix, therefore:

$$\left\|\sum\nolimits_{COM} M e_p\right\|^2 \leq \|M e_p\|^2$$

so, that the following inequality holds true (see eq.22) :

$$\left\|\sum\nolimits_{COM} M e_p\right\|^2 \leq \rho^2 \cdot \|M \cdot L\|^2 \tag{24}$$

This inequality allows to define $\alpha \in [0, 1]$ such that :

$$\begin{aligned} e_p^T \Phi e_P &= \alpha \|M \cdot L\|^2 \rho^2 \\ &= \alpha \left(L^T M L\right) \rho^2 \end{aligned} \tag{25}$$

In the same way :

$$\begin{aligned} e_p^T \Phi L &= \left(e_p^T M \sum\nolimits_{COM}\right) \left(\sum\nolimits_{COM} M L\right) \\ &= \left\langle \sum\nolimits_{COM} M e_p, \sum\nolimits_{COM} M L \right\rangle \end{aligned}$$

(this notation refers to the scalar product).
From the Schwarz inequality, we then have:

$$e_p^T \Phi L \leq \left\|\sum\nolimits_{COM} M e_p\right\| \cdot \left\|\sum\nolimits_{COM} M L\right\|$$

$\sum_{COM}$ being a projection matrix :

$$\left\|\sum\nolimits_{COM} M e_p\right\| \cdot \left\|\sum\nolimits_{COM} M L\right\| \leq \|M e_p\| \cdot \|M L\|$$

by using (22) :

$$\begin{aligned} \|M e_P\| &= \sqrt{e_p^T M e_p} \\ &= \sqrt{L^T M L} \cdot \rho \end{aligned}$$

then:

$$\begin{aligned} \|M e_p\| \cdot \|M L\| &= \left(\sqrt{L^T M L}\, \rho\right) \times \sqrt{L^T M L} \\ &= L^T M L \cdot \rho \end{aligned}$$

so :

$$e_p^T \Phi L \leq L^T M L \cdot \rho$$

It is then possible to define $\beta \in [0, 1]$ by :

$$e_p^T \Phi L = \beta \left(L^T M L\right) \rho \tag{26}$$

The last term is expressed in a similar manner :

$$\begin{aligned} L^T \Phi L &= \left(L^T M \sum\nolimits_{COM}\right) \left(\sum\nolimits_{COM} M L\right) \\ &= \left\|\sum\nolimits_{COM} M L\right\|^2 \\ &\leq \|M L\|^2 \end{aligned}$$

Let $\gamma \in [0, 1]$ such that :

$$L^T \Phi L = \gamma \left(L^T M L\right) \tag{27}$$

From the expressions (25), (26) et (27), we have finally:

$$\begin{aligned} &(e_p - L)^T \Phi (e_p - L) \\ &= e_p^T \Phi e_p - 2 e_p^T \Phi L + L^T \Phi L \\ &= \alpha \left(L^T M L\right) \rho^2 - 2\beta \left(L^T M L\right) \rho + \gamma \left(L^T M L\right) \end{aligned}$$

This formula allows to express the functional Ψ under a new manner :

$$\begin{aligned} \Psi &= \frac{\frac{e_p^T M e_p}{L^T M L} - 1}{\frac{2}{L^T M L} \sqrt{\alpha \left(L^T M L\right) \rho^2 - 2\beta \left(L^T M L\right) \rho + \gamma \left(L^T M L\right)}} \\ &= \frac{\rho^2 - 1}{\frac{2}{L^T M L} \sqrt{\gamma \left(L^T M L\right)} \times \sqrt{1 + \frac{\alpha}{\gamma} \rho^2 - 2\frac{\beta}{\gamma} \rho}} \,, \\ &= \frac{\sqrt{L^T M L}}{2\sqrt{\gamma}} \times \frac{\rho^2 - 1}{\sqrt{1 + a\rho^2 - b\rho}} \,, \end{aligned} \tag{28}$$

with :

$$a = \frac{\alpha}{\gamma} \quad \text{and} \quad b = \frac{2\beta}{\gamma}$$

Considering the classical expansion (u small vs 1):

$$\frac{1}{\sqrt{1+u}} = 1 - \frac{u}{2} + \frac{3}{8} u^2 - \frac{15}{48} u^3 + o(u^3)$$

we have:

$$\begin{aligned} \frac{1}{\sqrt{1 + a\,\rho^2 - b\,\rho}} &= 1 + \frac{b}{2}\rho + \left(-\frac{a}{2} + \frac{3}{8} b^2\right) \rho^2 + \cdots \\ &\quad + \left(-\tfrac{3}{4}\, a\, b + \tfrac{15}{48}\, b^3\right) \rho^3 + o(\rho^3) \,, \end{aligned}$$

i.e :

$$\begin{aligned} \frac{1}{\sqrt{1 + \frac{\alpha}{\gamma} \rho^2 - 2\frac{\beta}{\gamma} \rho}} &= 1 + \frac{\beta}{\gamma}\rho + \left(-\frac{\alpha}{2\gamma} + \frac{3}{2}\frac{\beta^2}{\gamma^2}\right) \rho^2 + \cdots \\ &\quad + \left(-\frac{3}{2}\frac{\alpha\beta}{\gamma^2} + \frac{15}{6}\frac{\beta^3}{\gamma^3}\right) \rho^3 + o(\rho^3) \,. \end{aligned}$$

Therefore, by using (28), the functional Ψ has the following expansion:

$$\begin{aligned}\Psi &= \frac{\sqrt{L^TML}}{2\sqrt{\gamma}} \times \frac{\rho^2-1}{\sqrt{1+a\rho^2-b\rho}} \\ &= -\frac{\sqrt{L^TML}}{2\sqrt{\gamma}}\left[1+\left(\frac{\beta}{\gamma}\right)\rho+\left(-\frac{\alpha}{2\gamma}+\frac{3}{2}\frac{\beta^2}{\gamma^2}-1\right)\rho^2+\cdots \right.\\ &\left. +\left(-\frac{3}{2}\frac{\alpha\beta}{\gamma^2}+\frac{15}{6}\frac{\beta^3}{\gamma^3}-\frac{\beta}{\gamma}\right)\rho^3+o(\rho^3)\right].\end{aligned}$$

Taking the 0-th order expansion, Ψ is approximated by :

$$\begin{aligned}\Psi &\cong -\frac{\sqrt{L^TML}}{2\sqrt{\gamma}}, \\ &\cong -\frac{\sqrt{L^TML}}{2\sqrt{\frac{L^T\Phi L}{L^TML}}}, \\ &\cong -\frac{L^TML}{2\sqrt{L^T\Phi L}}. \end{aligned} \tag{29}$$

Then, the 0-th order expansion of the probability of correct association as given in (19) is *simply*:

$$\begin{aligned}P(\Delta_{coûts}\geq 0) &\cong \iint erfc(\Psi)\ f_{\varepsilon_P}(e_P)\,de_P \\ &\cong \iint erfc\left(-\frac{L^TML}{2\sqrt{L^T\Phi L}}\right)\ f_{\varepsilon_P}(e_P)\,de_P \\ &\cong erfc\left(-\frac{L^TML}{2\sqrt{L^T\Phi L}}\right)\iint f_{\varepsilon_P}(e_P)\,de_P \\ &\cong erfc\left(-\frac{L^TML}{2\sqrt{L^T\Phi L}}\right) \\ &\cong erf\left(\frac{L^TML}{2\sqrt{L^T\Phi L}}\right) \end{aligned} \tag{30}$$

where "erf" refers to the normal cumulative density function.

3.4 Expliciting the quadratic forms

We shall now explicit the two quadratic forms L^TML and $L^T\Phi L$ (see eq. 30). More precisely, it can be shown easily that the M matrix can be expressed by :

$\forall i = 1\cdots N,\ \forall j = 1\cdots N$:

- If $i=j$:

$$\begin{aligned}M_{2i-1,2j-1} &= M_{2i,2j}\ ;, \\ &= 1-\frac{(S^2-s\,\tau_{i-1})+\tau_{j-1}\,(-s+N\,\tau_{i-1})}{N\,S^2-s^2}\end{aligned}$$

- If $i\neq j$:

$$\begin{aligned}M_{2i-1,2j-1} &= M_{2i,2j}\ , \\ &= -\frac{(S^2-s\,\tau_{i-1})+\tau_{j-1}\,(-s+N\,\tau_{i-1})}{N\,S^2-s^2}\end{aligned}$$

- If k and l have the same parity : $M_{k,l}=0$

with the following definitions:

$$\begin{cases}\tau_0=0\,, \\ s=\tau_1+\tau_2+\cdots+\tau_{N-1}\,, \\ \text{and: }, S^2=\tau_1^2+\tau_2^2+\cdots+\tau_{N-1}^2\,. \end{cases} \tag{31}$$

Therefore, with the definition of L and Φ given in (14) and (16) :

$$L^T\Phi L = \left[\sum_{k=1\cdots N-1} M^2_{2N,2k}\right]\lambda^2$$

Since k always differs from N, we have finally:

$$L^T\Phi L = \sum_{k=1\cdots N-1}\left(\frac{(S^2-s\,\tau_{N-1})+\tau_{k-1}\,(-s+N\,\tau_{N-1})}{N\,S^2-s^2}\right)^2\lambda^2 .$$

To conclude the 0-th order approximation of Ψ, we have finally obtain:

$$\Psi \overset{0}{\cong} -\lambda\,\frac{N\,S^2-s^2-(S^2-s\,\tau_{N-1})-\tau_{N-1}\,(-s+N\,\tau_{N-1})}{2\sqrt{\sum_{k=1\cdots N-1}\left[(S^2-s\,\tau_{N-1})+\tau_{k-1}\,(-s+N\,\tau_{N-1})\right]^2}} . \tag{32}$$

Concerning the probability of correct association, a 0-th order approximation is straightforwardly deduced from (30), yielding:

$$\begin{aligned}P(\Delta_c) &\overset{0}{\cong} erf\left(\frac{L^TML}{2\sqrt{L^T\Phi L}}\right), \\ &\overset{0}{\cong} erf\left(\frac{N\,S^2-s^2-(S^2-s\,\tau_{N-1})-\cdots}{2\sqrt{\sum_{k=1\cdots N-1}\left[(S^2-s\,\tau_{N-1})+\cdots\right.}}\right. \\ &\qquad \left.\frac{\cdots-\tau_{N-1}\,(-s+N\,\tau_{N-1})}{\left.\cdots+\tau_{k-1}\,(-s+N\,\tau_{N-1})\right]^2}\cdot\lambda\right)\end{aligned} \tag{33}$$

3.5 Particular case of regularly spaced times measurements

Considering the previous calculations, important simplifications can be made. Indeed, it is usual to consider that the duration δ between each measurement is constant, i.e. assuming that:

$$\begin{aligned}\tau_1 &= \delta \\ \tau_2 &= 2\,\delta \\ \tau_3 &= 3\,\delta \\ &\vdots \\ \tau_{N-1} &= (N-1)\,\delta\end{aligned}$$

The expressions of s and S^2 simply reduce to:

$$\begin{aligned} s &= \tau_1 + \tau_2 + \cdots + \tau_{N-1} \\ &= \frac{N(N-1)}{2}\,\delta \end{aligned}$$

and :

$$\begin{aligned} S^2 &= \tau_1^2 + \tau_2^2 + \cdots + \tau_{N-1}^2 \\ &= [1^2 + 2^2 + 3^2 + \cdots + (N-1)^2]\,\delta^2 \\ &= \frac{(N-1)N(2N-1)}{6}\,\delta^2 \end{aligned}$$

Finally, the correct association probability approximation can be computed and is given by :

$$P(\Delta_{coûts} \geq 0) \quad 0 \quad erf\left(\sqrt{\frac{1}{8}\cdot\frac{(N-2)(N-1)}{2N-1}}\cdot\lambda\right) \tag{34}$$

It is useful to notice that the probability doesn't depend of the constant duration δ between each measurement but depend just of the number of measurements N and naturally of the distance λ from the correct last target position P_N and the outlier observation P_E.

4 Results

The following results are given within the framework of times measurements regularly spaced, as previously seen. In the figure 2, we represent the N scans problem (here $N = 5$) : the correct target positions (red crosses), the correct noisy positions (magenta circles) and the outlier measurement (black rhombus) located at a distance $\lambda = 5$ under the correct last target position. The target speed is $v = 5$ and the duration between two consecutive measurements is fixed to 2.

The left figure presents the estimated positions (blue cross) belonging to the straight line regression from the correct noisy measurements, whereas the right one concerns the regression made from the wrong observations, i.e by replacing the right last observation by the outlier one.

On the five following figures is represented the probability of correct association (red curve) as a function of the number of scans, obtained by a Monte-Carlo method (5 000 runs) from eq. (19), for different values of λ : 1, 2, 3, 5 and 7. The blue dash line represents the approximation of probability of correct association given by the formula (34).

The fig. 3 below confirms that the hypothesis made to get the order 0 approximation (eq. (21)) is not valid. Indeed λ is of the same order that the noise assigned to true positions. The outlier observation P_E is too close to the correct observation P_5 for implying a significant difference on the costs. The approximation formula cannot be used is this case.

However, the four following figures show that the approximation becomes far better as the outlier observation departs from the correct position. From $\lambda = 3$, it can be considered that the 0-th order approximation is useful giving a realistic value of the true association probability. The approximation for $\lambda = 7$ is almost perfect.

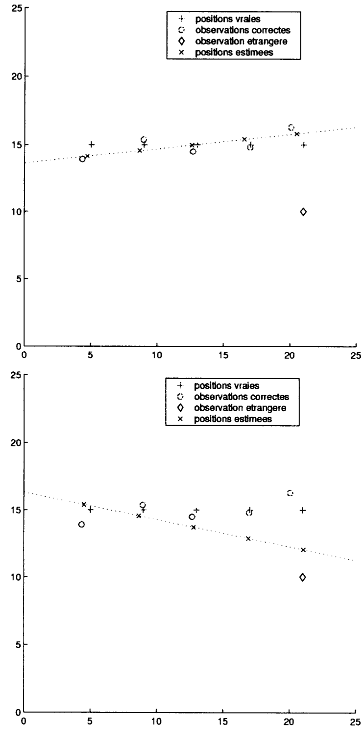

Figure 2: Left : Regression on correct observations (P_1, P_2, P_3, P_4, P_5). Right : Regression on wrong observations (P_1, P_2, P_3, P_4, P_E).

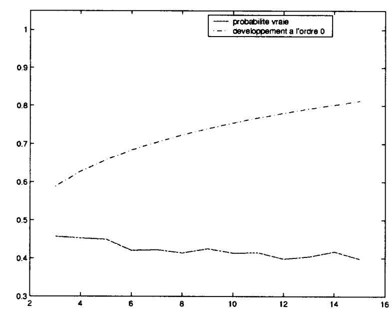

Figure 3: Right association probability (red) and approximated (blue dash curve) versus number of scans for $\lambda = 1$.

5 Conclusion

It has been shown that classical results of linear regression allow us to derive closed-form expressions of the probability of correct association, in the general N-scan case and for one outlier. These results can be easily extended to more general case studies: arbitrary number of "outliers", two (crossing) tracks, etc. In the non-linear case, the situation becomes more difficult, but linear local approximations are still relevant thus showing the pertinence of this approach.

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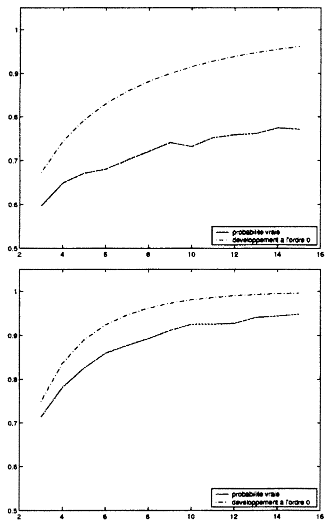

Figure 4: Right association probability (red) and approximated (blue dash curve) versus number of scans. Left : $\lambda = 2$. Right : $\lambda = 3$.

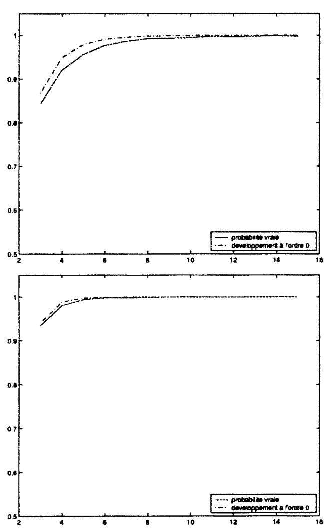

Figure 5: Right association probability (red) and approximated (blue dash curve) versus number of scans. Left : $\lambda = 5$. Right : $\lambda = 7$.

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