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Closed-form Posterior Cramér-Rao Bound for Bearings-Only Tracking

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Abstract: We here address the classical bearings-only tracking problem (BOT) for a single object, which belongs to the general class of nonlinear filtering problems. Recently, algorithms based on sequential Monte Carlo methods (particle filtering) have been proposed. As far as performance analysis is concerned, the Posterior Cramér-Rao Bound (PCRB) provides a lower bound on the mean square error. Classically, under a technical assumption named ”*asymptotic unbiasedness assumption*”, the PCRB is given by the inverse Fisher Information Matrix (FIM). The latter is computed using Tichavský's recursive formula via Monte Carlo methods. In this paper, two major problems are studied. First, we show that the ”*asymptotic unbiasedness assumption*” can be replaced by an assumption which is more meaningful. Second, an exact algorithm to compute the PCRB is derived via Tichavský's recursive formula without using Monte-Carlo methods. This result is based on a new coordinates system named Logarithmic Polar Coordinates (LPC) system. Simulation results illustrate that PCRB can now be computed accurately and quickly, making it suitable for sensor management applications.

Key-words: bearings-only tracking, sequential Monte Carlo methods, posterior Cramér-Rao bound, performance analysis, sensor management.

(Résumé : tsvp)

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Formule pour la borne de Cramér-Rao *a posteriori* dans le cadre du suivi par mesure d'angles.

Résumé : Le suivi mono-cible par mesure d'angles appartient à la classe des problèmes de filtrage non linéaires. Ce problème a été résolu récemment à l'aide de méthodes d'estimation séquentielles de type Monte Carlo. Concernant l'analyse de performances, la borne de Cramér-Rao *a posteriori* (PCRB) fournit une borne inférieure pour la covariance de l'erreur d'estimation. Classiquement, sous l'*hypothèse de biais asymptotique*, la PCRB est l'inverse de la matrice d'information de Fisher. Cette dernière est estimée l'aide de la formule Tichavský, les termes inclus dans la formule étant estimés à l'aide de simulations de type Monte Carlo. Dans cet article, deux problèmes sont étudiés. Tout d'abord, nous montrons que *l'hypothèse de biais asymptotique* peut-être remplacée par une hypothèse ayant un sens plus "physique". Puis, nous montrons que la PCRB peut-être calculée de manière exacte via la formule de Tichavský sans avoir recours à des simulations. Ces deux résultats sont obtenus grâce à un nouveau système de coordonnées nommé coordonnées logarithmiques polaires. Des simulations illustrent le fait la PCRB peut désormais être calculée rapidement et de manière exacte, la rendant ainsi utilisable pour des applications de type gestion de capteurs.

Mots clés : suivi par mesure d'angles, méthodes séquentielles de Monte Carlo , borne de Cramér-Rao *a posteriori*, analyse de performances, gestion de capteurs.

Notation

LP(C): Logarithmic Polar (Coordinates),

MP(C): Modified Polar (Coordinates),

BOT: Bearings-Only Tracking,

X_t: is the target state in th Cartesian coordinates system,

Y_t: is the target state in the LPC system,

n_y: size of the target state ($n_y = 4$),

$\succcurlyeq$: inequality $R \succcurlyeq S$ means that $R - S$ is a positive semi-definite matrix,

Id_n: $n \times n$ identity matrix,

$0_{n \times m}$: $n \times m$ matrix composed of zero element,

$\otimes$: Kronecker product,

X^*: denotes the transpose of matrix X.

$||X||_Q^2$: , $||X||_Q^2 = \mathbb{E}\left\{X^* Q^{-1} X\right\}$ where X is a column vector,

δ: Dirac delta function,

Δ: Laplacian operator,

∇: gradient operator,

$det(X)$: the determinant of matrix X,

PDF: Probability Density Function,

A: $A = Id_4 + \delta_t B$ with $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes Id_2$,

$H : H = \begin{pmatrix} \delta_t \\ 1 \end{pmatrix} \otimes Id_2$,

$Q : Q = \Sigma \otimes Id_2$ with $\Sigma = \begin{pmatrix} \alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_1 \end{pmatrix}$.

1 Introduction

In many applications (submarine tracking, aircraft surveillance), a bearings-only sensor is used to collect observations about target trajectory. This problem of tracking has been of interest for the past thirty years. The aim of Bearings-Only Tracking (BOT) is to determine the target trajectory using noise-corrupted bearing measurements from a single observer. Target motion is classically described by a diffusion model [1] so that the filtering problem is composed of two stochastic equations. The first one represents the temporal evolution of the target state (position and velocity) called state equation. The second one links the bearing measurement to the target state at time t (measurement equation).

[1] see [1] for an exhaustive review on dynamic models

One of the characteristics of the problem is the nonlinearity of the measurement equation so that the classical Kalman filter is not convenient in this case. We can find in literature two kinds of solutions to this problem. The first one, proposed by Lindgren and Gong in [2], consists of deriving a pseudolinear measurement equation. Then, a Kalman filter can be used to solve the problem. The stochastic stability analysis of the estimates had been addressed by Song and Speyer in [3]. However, Aidala and Nardone show in [4] that this approach produces bias range estimate which can be reduced if the observer executes a maneuver. Consequently, bias range can be estimated as soon as it becomes observable [5]. A second idea consists of using the Extended Kalman Filter (EKF) in Cartesian coordinates system to solve the problem. However, simulations show that this algorithm is often divergent due to the weak observability of range ([6, 7, 8]). To remedy this problem, Aidala and Hammel in [9] proposed an EKF using another system named Modified Polar Coordinates (MPC) system whose one salient feature is that range is not coupled with the observable components. This constitutes a neat improvement. Another solution proposed by Peach in [10] is a range-parametrized EKF, in which a number of EKF trackers parametrized by range run in parallel. Recently, particle filtering algorithms have been proposed in this context ([11, 12, 13]). In [14], Arulampalam and Ristic compare the particle filter with the range-parametrized and EKF in MPC system; while a comprehensive overview of the state of art can be found in [15].

As far as performance analysis is concerned, the Posterior Cramér-Rao Bound (PCRB) proposed in [16] is widely used to assess the performance of filtering algorithms, by the tracking community ([17, 18, 19, 20]) and in particular in the bearings-only context ([15, 21, 22]). The PCRB gives a lower bound for the Error Covariance Matrix (ECM). More precisely, under a technical assumption, the PCRB is the inverse of the Fisher Information Matrix (FIM). A seminal contribution on performance analysis is the paper from Tichavský et al. [23]. Here, the authors noticed that only the right lower block of the FIM inverse was of interest for investigating tracking performance. This was the key idea for deriving a practical updating formula for the PCRB. Recently, PCRB has been used for various sensor management problems like automating the deployment of sensors in [24] or determining the optimal sensor trajectory in the bearings-only context in [25]. Moreover, PCRB can be used to schedule active measurements in a system involving active and passive subsystems. This application will be addressed in the simulation section.

However, some problems remain to be solved. In this paper, two major issues of the PCRB are addressed. First, under a technical assumption named ”*asymptotic unbiasedness assumption*”, the PCRB is the FIM inverse. However, the validity of this assumption has not been thoroughly investigated in the BOT context yet. Here, our approach consists of deriving the PCRB in an original coordinates system named Logarithmic Polar Coordinates (LPC) system . Using this coordinates system, it is shown that the ”*asymptotic unbiasedness assumption*” can be replaced with another one, more meaningful in the BOT context. Second, Tichavský’s recursive formula is a powerful result to compute the right lower block of the FIM inverse. However, complex integrals without any closed-forms are involved in this recursion. So, these complex integrals must be approximated via Monte Carlo methods. This approach is quite feasible but induces high computation requirements which highly reduces its suitability for complex problems like sensor management.

For instance, measurement scheduling would imply to consider a large number of active measurement sequences and to perform Monte-Carlo evaluations of the PCRB for each sequence, which would rapidly become infeasible.

Another approach proposed by Ristic et al. in [15] consists of assuming that the target process noise is sufficiently small for drastically PCRB computation. In the general case, we show that the complex integrals required for calculating the PCRB admit closed-form expressions if the PCRB is derived in an original coordinates system named Logarithmic Polar Coordinates (LPC) system . Remarkably, though this coordinate system is only a slight modification of the MPC [9], it allows instrumental simplifications in the calculation of the elementary terms of the PCRB recursion. Applications to active measurement scheduling is briefly considered in a simulation framework.

In section II, the BOT problem is presented in the Cartesian coordinates system and then in the LPC system. This original coordinates system is the key point to derive a closed-form for the PCRB. In Section III, the classical PCRB is presented. A close examination of the ”*asymptotic unbiasedness assumption*” is achieved so as to prove the validity of the ”usual” PCRB, as given by the FIM inverse. We study this assumption and derive a more meaningful condition. In particular, conditions ensuring its validity are examined in the BOT context. Calculation of closed-form expressions of the right lower block of the FIM inverse via Tichavský’s recursive formula is addressed in section IV, in the LPC setting. Then, the closed-form PCRB is investigated for scheduling active measurements in section V. In section VI, simulation results present a comparison between the closed-form PCRB and the classical one (i.e. where the terms involved in Tichavský’s formula are approximated by Monte Carlo methods). Finally, the closed-form PCRB is used for investigating scheduling of passive and active measurements.

2 From Cartesian to LPC system

2.1 Cartesian framework for BOT

Historically, BOT is presented in the Cartesian system. Let us define target state at time t:

$$X_t = \begin{bmatrix} r_x(t) & r_y(t) & v_x(t) & v_y(t) \end{bmatrix}^* , \tag{1}$$

made of target relative velocity and position in the $x - y$ plane. It is assumed that the target follows a nearly constant-velocity model. The discretized state equation[2] is given by:

$$X_{t+1} = AX_t + HU_t + \sigma W_t \ , \tag{2}$$

[2]For a general review of dynamic models for target tracking see [1].

where:

$$\left| \begin{array}{l} W_t \sim \mathcal{N}(0, Q) \ , \\ A = Id_4 + \delta_t B \text{ with } B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes Id_2, \\ H = \begin{bmatrix} \delta_t \\ 1 \end{bmatrix} \otimes Id_2 \ , \\ Q = \Sigma \otimes Id_2 \text{ with } \Sigma = \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_1 \end{bmatrix} \ . \end{array} \right.$$

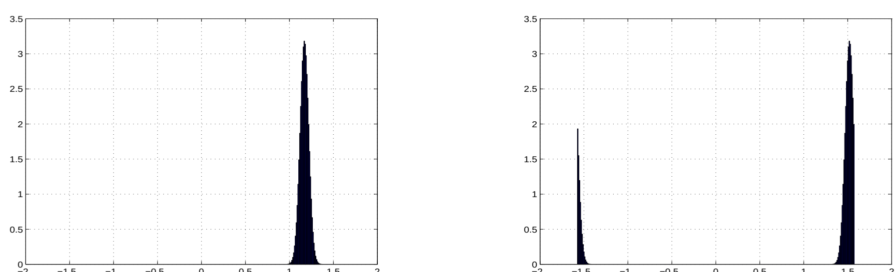
δ_t is the elementary time period and U_t is the known difference between observer velocity at time $t+1$ and t. The state covariance σ is unknown. However we assume classically that $\sigma < \sigma_{max}$, so that we use in practice the following equation:

$$X_{t+1} = AX_t + HU_t + \sigma_{max} W_t \ . \tag{3}$$

Otherwise, we note Z_t the bearing measurement received at time t. The target state is related to this measurement through the following equation:

$$Z_t = \arctan\left(\frac{r_x(t)}{r_y(t)}\right) + V_t \underbrace{- \pi 1\!\!1_{\arctan\left(\frac{r_x(t)}{r_y(t)}\right) + V_t > \frac{\pi}{2}} + \pi 1\!\!1_{\arctan\left(\frac{r_x(t)}{r_y(t)}\right) + V_t < -\frac{\pi}{2}}}_{(\star)} \ , \tag{4}$$

where $V_t \sim \mathcal{N}(0, \sigma_\beta^2)$ and σ_β^2 is known. Let us notice that the term $(\star)$ is classically avoided. However, it is frequent to consider that measurement Z_t is restricted to a part of the space. This is the case if symmetry of the receiver (e.g. linear array) leads to consider measurements belonging in the interval $]-\frac{\pi}{2}, \frac{\pi}{2}[$, so that the additional term $\star$ in eq.(4) is necessary. Two examples of Probability Density Function (PDF) of Z_t given X_t are presented in fig.1 to enlighten the importance of the additional term $(\star)$. In fig.1.(b), the bearing measurement is close to $\frac{\pi}{2}$ so that there is an overlapping phenomena.



(a) (b)

Figure 1: *two examples of PDF of Z_t given X_t (a) if Z_t is far from the bounds (b) if Z_t is close to $\frac{\pi}{2}$.*

The system (3-4) has two components : a linear state equation (3) and a nonlinear measurement equation (4). Particle filter techniques (see [26],[27]) are, thus, particularly appealing. Otherwise, practical implementations of

EKF-based algorithms ([9] and [10]) use a specific coordinates system, namely Modified Polar Coordinates (MPC). Indeed, if the target follows a deterministic trajectory (i.e. $W_t = 0 \;\; \forall t \in \{0, \ldots, T\}$ in eq.(3)), Nardone and Aidala have demonstrated in [7] that no information on range exists as long as the observer is not maneuvering. So the idea consists of using a coordinates system for which unobservable component (range) is not coupled with the observable part. This is also the motivation of Aidala and Hammel [9] for defining the MPC system:

$$\left[\begin{array}{cccc} \beta_t & \frac{1}{r_t} & \dot{\beta}_t & \frac{\dot{r}_t}{r_t} \end{array} \right]^* . \tag{5}$$

Thus, the target state at time t is defined by eq.(5), where β_t and r_t are the relative bearing and target range. We propose in the following section a slight modification of the MPC system, named **Logarithmic** Polar Coordinates (LPC) system. The only difference is that the second component is not $\frac{1}{r_t}$ but $\mathbf{ln}(r_t)$. Even if this tiny difference appears very minor, it will be shown that it is instrumental for deriving a closed-form of the PCRB. Let us now derive BOT equations given by eqs.(3,4) in the LPC framework.

2.2 LPC framework for BOT

We consider now that the system state Y_t is expressed in the Logarithmic Polar Coordinates (LPC) system, i.e. :

$$Y_t = \left[\begin{array}{cccc} \beta_t & \ln r_t & \dot{\beta}_t & \frac{\dot{r}_t}{r_t} \end{array} \right]^* . \tag{6}$$

As between Cartesian and MP system, we do not have a direct bijection between Cartesian and LPC system due to arctan function definition. We just have f_{lp}^c and f_c^{lp} respectively LPC-to-Cartesian and Cartesian-to-LPC state mapping functions such that:

$$X_t = \left\{ \begin{array}{l} f_{lp}^c(Y_t) \text{ if } r_y(t) > 0 \\ -f_{lp}^c(Y_t) \text{ if } r_y(t) < 0 \end{array} \right. \quad \text{with } f_{lp}^c(Y_t) = r_t \left[\begin{array}{c} \sin \beta_t \\ \cos \beta_t \\ \dot{\beta}_t \cos \beta_t + \frac{\dot{r}_t}{r_t} \sin \beta_t \\ -\dot{\beta}_t \sin \beta_t + \frac{\dot{r}_t}{r_t} \cos \beta_t \end{array} \right] \tag{7}$$

and

$$Y_t = f_c^{lp}(X_t) = \left[\begin{array}{c} \arctan\left(\frac{r_x(t)}{r_y(t)}\right) \\ \ln\left(\sqrt{r_x^2(t) + r_y^2(t)}\right) \\ \frac{v_x(t) r_y(t) - v_y(t) r_x(t)}{r_x^2(t) + r_y^2(t)} \\ \frac{v_x(t) r_x(t)) + v_y(t) r_y(t)}{r_x^2(t) + r_y^2(t)} \end{array} \right] . \tag{8}$$

Thus, using eqs.(7,8), the stochastic system given by eqs.(3,4) becomes:

$$\left\{ \begin{array}{l} Y_{t+1} = \left\{ \begin{array}{l} f_c^{lp}\left(A f_{lp}^c(Y_t) + H U_t + \sigma_{max} W_t\right) \text{ if } r_y(t) > 0, \\ f_c^{lp}\left(-A f_{lp}^c(Y_t) + H U_t + \sigma_{max} W_t\right) \text{ if } r_y(t) < 0. \end{array} \right. \\ Z_t = \beta_t + V_t - \pi \mathbb{1}_{\beta_t + V_t > \frac{\pi}{2}} + \pi \mathbb{1}_{\beta_t + V_t < -\frac{\pi}{2}} . \end{array} \right. \tag{9}$$

Though, it seems that the LPC increases the complexity of the BOT problem, it has also the advantage to highlight the multi-modality associated with the two solutions corresponding to $r_y(t) > 0$ and $r_y(t) < 0$ respectively.

3 PCRB for state estimation

In this section, ”usual” PCRB given by the inverse Fisher Information Matrix (FIM) is presented. Notably, we present in sub-section A, the proof of this classical result. The role of a technical hypothesis named ”*asymptotic unbiasedness assumption*” is thus highlighted, especially in the LPC system. Then, we show in sub-section B that this hypothesis is not always satisfied in the BOT context and we propose to replace it by an original extension. Finally, it is shown that the ”usual” PCRB as given by FIM inverse is valid if bearing measurements are ”sufficiently” far from $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Let us remark that the PCRB is not derived in the Cartesian framework for two reasons. First, the ”*asymptotic unbiasedness assumption*” seems rather difficult to address in this setting. Second, it is shown that a closed-form exists in LPC but not in the classical coordinates systems (Cartesian or MPC).

3.1 Classical PCRB

Let $Y_{0:t}$ and $Z_{1:t}$ be the trajectory and the set of bearing measurements up to time t, they are random vectors of size $n_y(t+1)$ and t, respectively. Let $\hat{Y}_{0:t}$ be an estimator of $Y_{0:t}$ which is a function of $Z_{1:t}$. We focus here on the Error Covariance Matrix (ECM) at time t which is $n_y(t+1) \times n_y(t+1)$-matrix, defined by:

$$\mathrm{ECM}_{0:t} = \|\hat{Y}_{0:t} - Y_{0:t}\|^2 \ . \tag{10}$$

First, let us recall the Fisher Information Matrix (FIM) and bias definitions.

Definition 1 (FIM) *For the filtering problem given by eq.(9); the FIM ,at time t, is denoted $J_{0:t}$ and defined as:*

$$J_{0:t} = \mathbb{E}\left\{\nabla_{Y_{0:t}} \ln p(Z_{1:t}, Y_{0:t}) \nabla^*_{Y_{0:t}} \ln p(Z_{1:t}, Y_{0:t})\right\} \ , \tag{11}$$

where $p(Z_{1:t}, Y_{0:t})$ is the joint Probability Density Function (PDF) of $Z_{1:t}$ and $Y_{0:t}$.

Definition 2 (Bias) *For the filtering problem described by eq.(9), estimation bias related to the observation sequence $\hat{Y}_{0:t}$ is defined as:*

$$B(Y_{0:t}) = \mathbb{E}\left\{ \hat{Y}_{0:t} - Y_{0:t} \Big| Y_{0:t} \right\} \ . \tag{12}$$

$Y_{0:t}$ is a $n_y(t+1)$ vector so that $B(Y_{0:t})$ is a $n_y(t+1)$ vector too. The estimator of the trajectory $\hat{Y}_{0:t}$ is unbiased if vector $B(Y_{0:t})$ is almost surely equal to zero. This choice of the bias definition is justified in Appendix A. Proposition 1 ensures that the FIM gives a lower bound for the ECM under a specific assumption called “*asymptotic unbiasedness assumption*”. Before introducing this technical assumption let us introduce a notation to simplify the presentation:

Notation 1 *For a function $F : \mathbb{R}^d \to \mathbb{R}^n$, U and $\mathcal{U}$ two $\mathbb{R}^d$-vectors such that $U = \Big[U_1, \dots, U_d\Big]^*$ and $\mathcal{U} = \Big[\mathcal{U}_1, \dots, \mathcal{U}_d\Big]^*$, we define:*

$$\lim_{U \to \mathcal{U}} F(U) = \begin{bmatrix} \lim_{U_1 \to \mathcal{U}_1} (F(U))_1 & \dots & \lim_{U_d \to \mathcal{U}_d} (F(U))_1 \\ \vdots & & \vdots \\ \lim_{U_1 \to \mathcal{U}_1} (F(U))_n & \dots & \lim_{U_d \to \mathcal{U}_d} (F(U))_n \end{bmatrix} \tag{13}$$

where $(F(U))_i$ is the ith component of vector $F(U)$.

Let us notice that $\lim_{U_1 \to \mathcal{U}_1} (F(U))_1$ is a function which depends on variables $\mathcal{U}_1$ **and** $\{U_2, \ldots, U_d\}$ so that $\lim_{U \to \mathcal{U}} F(U)$ depends on variables $\mathcal{U}$ **and** U. We will see that notation 1 is defined unambiguously in proposition 1 proof and will be helpful to present the following assumption.

Assumption 1 (Asymptotic unbiasedness) *For the filtering problem given by eq.(9), the asymptotic unbiasedness assumption is defined as:*

$$\forall k \in \{1, \ldots, t\}, \lim_{Y_k \to \mathcal{Y}_k^+} B(Y_{0:t})p(Y_{0:t}) = \lim_{Y_k \to \mathcal{Y}_k^-} B(Y_{0:t})p(Y_{0:t}) \tag{14}$$

where $\mathcal{Y}_k$ is the (connected) domain of Y_k , $k \in \{1, \ldots, t\}$, while $\left\{\mathcal{Y}_k^- \ , \ \mathcal{Y}_k^+\right\}$ are its bounds.

Looking at LPC's definition given by eq.(6), we have $\mathcal{Y}_l^- = \left[-\frac{\pi}{2}, -\infty, -\infty, -\infty\right]^*$ and $\mathcal{Y}_l^+ = \left[\frac{\pi}{2}, +\infty, +\infty, +\infty\right]^*$. Moreover, $B(Y_{0:t})p(Y_{0:t})$ is a $n_y(t+1)$ vector following notation 1, $\lim_{Y_k \to \mathcal{Y}_k^+} B(Y_{0:t})p(Y_{0:t})$ is a $n_y(t+1) \times n_y$ matrix. After introducing assumption 1, we can now present the classical result on the PCRB.

Proposition 1 (PCRB) *Under assumption 1,*

$$ECM_{0:t} \succcurlyeq J_{0:t}^{-1} \ . \tag{15}$$

Proposition 1 ensures that the FIM inverse gives a lower bound for the ECM conditionally to the validity of the technical Assumption 1 named "*asymptotic unbiasedness assumption*". Classically, Assumption 1 is true if the estimator $\hat{Y}_{0:t}$ is unbiased when $Y_k \approx \mathcal{Y}_k^-$ and $Y_k \approx \mathcal{Y}_k^+$. However, this point is relatively complex to verify in the bearings-only context. We propose to study assumption 1 to find a more concrete one. First, let us present a proof of the rather classical Prop. 1. For the sake of completeness, the following lemma is reminded.

Lemma 1 *Let S be a symmetric matrix defined as:*

$$S = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix} , \tag{16}$$

where

- *A is a non negative real symmetric matrix,*
- *B is a positive real symmetric matrix,*
- *C is a real matrix.*

then $S \succcurlyeq 0$ implies $A - CB^{-1}C^* \succcurlyeq 0$.

Proof of lemma 1 This lemma is a classical algebraic result given in [28].□

Proof of proposition 1 Using lemma 1, we build the S matrix such that:

$$S = \begin{bmatrix} A_{0:t} & C_{0:t} \\ C_{0:t}^* & B_{0:t} \end{bmatrix} ,$$

where:

$$\begin{cases} A_{0:t} = \text{ECM}_{0:t} \ , \\ B_{0:t} = J_{0:t} \ , \\ C_{0:t} = \mathbb{E}\left\{(\hat{Y}_{0:t} - Y_{0:t})\nabla^*_{Y_{0:t}} \ln p(Z_{1:t}, Y_{0:t})\right\} \ . \end{cases} \tag{17}$$

From this definition, S is a non negative matrix. Using lemma 1, one remarks that we just have to prove that $C_{0:t}$ is equal to the identity matrix. The "*asymptotic unbiasedness assumption*" will be used to do so. First, let us notice that $C_{0:t}$ can be rewritten as:

$$C_{0:t} = \int (\hat{Y}_{0:t} - Y_{0:t})\nabla^*_{Y_{0:t}} p(Z_{1:t}, Y_{0:t}) d(Z_{1:t}, Y_{0:t}) \ . \tag{18}$$

$C_{0:t}$ is a $n_y(t+1) \times n_y(t+1)$ matrix made of $(t+1) \times (t+1)$ elementary blocks. We study each of these elementary blocks (denoted $C_{0:t}(k,l)$):

$$C_{0:t}(k,l) = \int (\hat{Y}_k - Y_k)\nabla^*_{Y_l} p(Z_{1:t}, Y_{0:t}) d(Z_{1:t}, Y_{0:t}) \ \ , k \in \{1, \cdots, n_y\} \ , \ l \in \{1, \cdots, n_y\}. \tag{19}$$

Before integrating by parts, let us introduce the following notation:

Notation 2 *For a function $F : \mathbb{R}^d \to \mathbb{R}^n$, U, $\mathcal{U}^-$ and $\mathcal{U}^+$ three $\mathbb{R}^d$-vectors such that $U = \left[U_1, \ldots, U_d\right]^*$, $\mathcal{U}^- = \left[\mathcal{U}_1^-, \ldots, \mathcal{U}_d^-\right]^*$ and $\mathcal{U}^+ = \left[\mathcal{U}_1^+, \ldots, \mathcal{U}_d^+\right]^*$, then we can define:*

$$[F(U)]_{\mathcal{U}^-}^{\mathcal{U}^+} = \lim_{U \to \mathcal{U}^+} F(U) - \lim_{U \to \mathcal{U}^-} F(U) \tag{20}$$

where $\lim_{U \to \mathcal{U}^+} F(U)$ and $\lim_{U \to \mathcal{U}^-} F(U)$ are defined using notation 1.

Integrating by parts and using the previous notation, a matrix element of $C_{0:t}$ given by eq.(19) can be rewritten:

$$C_{0:t}(k,l) = Id_{n_y}\delta_{k=l} + \int \left[(\hat{Y}_k - Y_k) p(Z_{1:t}, Y_{0:t})\right]_{\mathcal{Y}_l^-}^{\mathcal{Y}_l^+} d(Z_{1:t}, Y_{0:t}^{-\{l\}}) \ , \tag{21}$$

where $Y_{0:t}^{-\{l\}}$ is a whole target trajectory except the term Y_l. Now, if limit and integral operators can be reversed, we have:

$$C_{0:t}(k,l) = Id_{n_y}\delta_{k=l} + \int \left[\int (\hat{Y}_k - Y_k) p(Z_{1:t}, Y_{0:t}) dZ_{1:t}\right]_{\mathcal{Y}_l^-}^{\mathcal{Y}_l^+} dY_{0:t}^{-\{l\}} \ . \tag{22}$$

Using bias notation previously introduced, we finally obtain:

$$C_{0:t}(k,l) \quad = \quad Id_{n_y}\delta_{k=l} + \int \left[B(Y_{0:t}) p(Y_{0:t})\right]_{\mathcal{Y}_l^-}^{\mathcal{Y}_l^+} dY_{0:t}^{-\{l\}} \ . \tag{23}$$

Thus, under assumption 1, $C_{0:t}$ is the identity matrix.□

Then we can apply proposition 1 to the BOT problem if "*asymptotic unbiasedness assumption*" is satisfied. More precisely, this assumption assures that the term $C_{0:t}$ is the identity matrix. Let us now study the validity of this hypothesis in the BOT context.

3.2 Validity of the "asymptotic unbiasedness assumption" in BOT context

We show in this section that the *"asymptotic unbiasedness assumption"* is not always true in the BOT context. More precisely, this classical result can not be used when measurements obtained by the observer are close to $-\frac{\pi}{2}$ or $\frac{\pi}{2}$. First let us remind that proposition 1 is true under a technical assumption named "*asymptotic unbiasedness assumption*". According to the previous section, $C_{0:t}$ given by eq.(17) is not the identity matrix if this assumption is not verified. Van Trees showed in [16] that there is a more general result which is valid without this technical assumption:

Proposition 2 (PCRB) *for a filtering problem given by eq.(9)*

$$ECM_{0:t} \succcurlyeq C_{0:t}J_{0:t}^{-1}C_{0:t}^{*} \text{ with } C_{0:t} = \mathbb{E}\left\{(\hat{Y}_{0:t} - Y_{0:t})\nabla_{Y_{0:t}}^{*} \ln p(Z_{1:t}, Y_{0:t})\right\}. \tag{24}$$

Proof of proposition 2 The result is a direct application of lemma 1 with $A_{0:t}$, $B_{0:t}$ and $C_{0:t}$ given by eq.(17).□
Proposition 2 is a classical result which is not often used in practice because this bound depends on the estimator i.e. $\hat{Y}_{0:t}$ via $C_{0:t}$. This last term must be estimated using Monte-Carlo methods. Here, an original result is proposed which specifies proposition 2 in the bearings-only context. In particular, we propose a simple formula for $C_{0:t}$.

Proposition 3 (PCRB) *For a filtering problem given by eq.(9),*

$$ECM_{0:t} \succcurlyeq C_{0:t}^{*}J_{0:t}^{-1}C_{0:t}$$

where $C_{0:t}$ *is a* $n_y(t+1) \times n_y(t+1)$ *block diagonal matrix where diagonal terms are expressed as follows:*

$$C_{0:t}(l,l) = \begin{bmatrix} 1 - \pi p(\beta_l)\Big|_{\frac{\pi}{2}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \forall l \in \{0, \ldots, t\}. \tag{25}$$

where $p(\beta_l)$ *is the PDF of* β_l.

Proposition 3 is adapted from proposition 2 to fit to the BOT context. More precisely, proposition 3 gives a more simple formula for $C_{0:t}$. This result is quite intuitive. When bearing measurements are close to a bound (i.e. $-\frac{\pi}{2}$ or $\frac{\pi}{2}$) there is an overlapping phenomenon due to the arctan definition as the underlying Probability Density Function (PDF) is not Gaussian but something like that function represented in fig.1. Finally let us notice that $p(\beta_l)$ is not defined in $\frac{\pi}{2}$ because β_l is in $]-\frac{\pi}{2}, \frac{\pi}{2}[$. However, the limit exists.

Proof of proposition 3 The complete proof of proposition 3 is given in Appendix B with three intermediate results skipped in sub-Appendix B1, B2 and B3. The idea of the proof consists of studying $C_{0:t}$ using the formula given by eq.(22) in Prop. 1 proof. To study eq.(22), the PDF of Y_{t+1} given Y_t i.e. $p(Y_{t+1}|Y_t)$ and the PDF of Z_t given Y_t i.e. $p(Z_t|Y_t)$ are derived in Appendix B1 and B2. Then, a technical lemma allows us to end the proof.□

In the filtering context, we are generally not interested in $ECM_{0:t}$ but only in the right lower block $ECM_t = \|\hat{Y}_t - Y_t\|^2$. Thus, it is not the whole matrix $C_{0:t}J_{0:t}^{-1}C_{0:t}^{*}$ which is of interest but just the right lower block. As $C_{0:t}$ is a diagonal matrix according to Pro. 3, we have:

$$ECM_t \succcurlyeq C_t J_t^{-1} C_t^* ,$$

with:

$$C_t = \begin{bmatrix} 1 - \pi p(\beta_t)\Big|_{\frac{\pi}{2}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{26}$$

Matrix J_t^{-1} is the right lower block of $J_{0:t}$-inverse, given by eq.(11). Now from a practical point of view, the problem is to be able to estimate J_t^{-1} and C_t. Concerning the first one, J_t^{-1} is classically obtained by means of Tichavský's recursive formula via Monte-Carlo methods. Looking at eq.(26), we can see that C_t only modifies the PCRB linked to the first component of the target state β_t. The PCRB associated to this component is overestimated because $p(\beta_t)\Big|_{\frac{\pi}{2}}$ is not zero all the time. When bearing measurements are sufficiently far from the bounds $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, C_t is the identity matrix, so that the classical PCRB is given by the FIM inverse.

Assumption 2 *(Side assumption) For a filtering problem given by eq.(9), the* side assumption *is defined as:*

$$p(\beta_l)\Big|_{\frac{\pi}{2}} = 0 \;\; \forall l \in \{0, \dots, T\} , \tag{27}$$

where $p(\beta_l)$ is the PDF of β_l.

Proposition 4 (PCRB) *Under assumption 2,*

$$ECM_t \succcurlyeq J_t^{-1} . \tag{28}$$

Proof of proposition 4 Proposition 4 is easily derived from proposition 3.□

Comparing proposition 1 and 4, we see that we have given a more concrete meaning to the assumption we use for PCRB calculation.

4 Closed-form formulation for Tichavský's formula in the LPC coordinates system

We have derived in the previous section a PCRB adapted to the BOT context, given by eq.(29). Now it is necessary to estimate J_t^{-1}. The classical approach consists of using J_t^{-1} recursive formula proposed by Tichavský's et al. However, some terms involved in this formula must be estimated using Monte Carlo methods. We demonstrate here that all these terms have closed-form expressions if the PCRB is derived using the LPC system, so that J_t^{-1} can be computed exactly via Tichavský's formula. In section A, Tichavský's recursive formula is reminded. We remark in section B that no closed-form expressions for the terms involved in this formula can be obtained using Cartesian or MPC framework. Then we show in section C that closed-form calculation can be derived in the new LPC system.

4.1 Tichavský's formula

Tichavský et al. proposed a recursive formula in [23] for the right lower block of the FIM inverse noted J_t^{-1}.

Proposition 5 (Tichavský's formula) *For a filtering problem given by eq.(9), the right lower block of the FIM inverse noted J_t^{-1} has a recursive formula:*

$$J_{t+1}^{-1} = \left(D_t^{22} + D_t^{33} - D_t^{21} \left(J_t^{-1} + D_t^{11} \right)^{-1} D_t^{12} \right)^{-1} ,$$

where D_t^{11}, D_t^{12}, D_t^{21}, D_t^{22}, D_t^{33} are defined by:

$$\left\{ \begin{array}{lcl} D_t^{11} & = & \mathbb{E}\{\nabla_{Y_t} \ln p(Y_{t+1}|Y_t) \nabla_{Y_t}^* \ln p(Y_{t+1|Y_t}\}\} , \\ D_t^{21} & = & \mathbb{E}\{\nabla_{Y_{t+1}} \ln p(Y_{t+1}|Y_t) \nabla_{Y_t}^* \ln p(Y_{t+1}|Y_t)\} , \\ D_t^{12} & = & \mathbb{E}\{\nabla_{Y_t} \ln p(Y_{t+1}|Y_t) \nabla_{Y_{t+1}}^* \ln p(Y_{t+1}|Y_t)\} , \\ D_t^{22} & = & \mathbb{E}\{\nabla_{Y_{t+1}} \ln p(Y_{t+1}|Y_t) \nabla_{Y_{t+1}}^* \ln p(Y_{t+1}|Y_t)\} , \\ D_t^{33} & = & \mathbb{E}\{\nabla_{Y_{t+1}} \ln p(Z_{t+1}|Y_{t+1}) \nabla_{Y_{t+1}}^* \ln p(Z_{t+1}|Y_{t+1})\} . \end{array} \right. \quad (29)$$

Proposition 5 is proved in [23]. However, **for the BOT context**, even if PDF $p(Y_{l+1}|Y_l)$ and $p(Z_t|Y_t)$ are known and simple, D_t^{11}, D_t^{12}, D_t^{21}, D_t^{22} and D_t^{33} do not have closed-form expressions altogether. We shall show now that existence of closed-form expressions is a characteristic of the LPC system, introduced in section II.B.

4.2 Closed form expressions of D_t^{11}, D_t^{12}, D_t^{22}, D_t^{21} and D_t^{33} in different coordinates systems

Ristic et al. in [15] have derived the PCRB in the Cartesian coordinates system. Matrices D_t^{11}, D_t^{12}, D_t^{22} and D_t^{21} have closed-form expressions using this system. However D_t^{33} has no closed-form, so that the authors assumed that the process noise makes a very small effect on the PCRB (i.e. $W_t = 0$) for approximating D_t^{33}. Otherwise, the classical PCRB has not been derived in MP coordinates system yet. It seems that no closed-form for D_t^{11}, D_t^{12}, D_t^{22} and D_t^{21} can be expected, though a closed-form of D_t^{33} exists under assumption 2. These results are summed up in tab.1.

	Cartesian	**modified polar**	**logarithmic polar**
D_t^{11}	Yes	No	Yes
D_t^{12}	Yes	No	Yes
D_t^{21}	Yes	No	Yes
D_t^{22}	Yes	No	Yes
D_t^{33}	No	Yes	Yes

Table 1: Closed-forms in different coordinates systems

Now the question is whether we can find a coordinates system allowing closed-forms for all terms. First, it seems that the coordinates system must include β_t so that under assumption 2, D_t^{33} has a closed-form as in the MPC system.

Second, in the Cartesian framework, it seems that the existence of closed-forms for D_t^{11}, D_t^{12}, D_t^{22} and D_t^{21} in eq.(28) are inherited from the linear property of $\nabla_{X_t} \ln p(X_{t+1}|X_t)$ and $\nabla_{X_{t+1}} \ln p(X_{t+1}|X_t)$. First, considering LPC definition given by eq.(6), we can see that β_t is one of the component of the state. Second, we can show that gradients $\nabla_{Y_t} \ln p(X_{t+1}|X_t)$ and $\nabla_{Y_{t+1}} \ln p(X_{t+1}|X_t)$ are quadratic forms in X_t, X_{t+1}. Indeed, we have:

$$\begin{cases} \nabla_{Y_t}^* \ln p(X_{t+1}|X_t) = (X_{t+1} - AX_t - HU_t)^* Q^{-1} A \nabla_{Y_t}\{X_t\} \ , \\ \nabla_{Y_{t+1}}^* \ln p(X_{t+1}|X_t) = (X_{t+1} - AX_t - HU_t)^* Q^{-1} \nabla_{Y_{t+1}}\{X_{t+1}\} \ , \end{cases} \tag{30}$$

where $\nabla_{Y_t}\{X_t\}$ and $\nabla_{Y_t+1}\{X_{t+1}\}$ are LPC-to-Cartesian mapping function derivatives at time t and $t+1$ (LPC-to-Cartesian mapping function is given by eq.(7)). These two terms can be expressed using the Cartesian framework:

$$\begin{cases} \nabla_{Y_t}\{X_t\} = \begin{bmatrix} r_y(t) & -r_x(t) & 0 & 0 \\ r_x(t) & r_y(t) & 0 & 0 \\ v_y(t) & -v_x(t) & r_y(t) & -r_x(t) \\ v_x(t) & v_y(t) & r_x(t) & r_y(t) \end{bmatrix} , \\ \nabla_{Y_t+1}\{X_{t+1}\} = \begin{bmatrix} r_y(t+1) & -r_x(t+1) & 0 & 0 \\ r_x(t+1) & r_y(t+1) & 0 & 0 \\ v_y(t+1) & -v_x(t+1) & r_y(t+1) & -r_x(t+1) \\ v_x(t+1) & v_y(t+1) & r_x(t+1) & r_y(t+1) \end{bmatrix} . \end{cases} \tag{31}$$

so that $\nabla_{Y_t}\{X_t\}$ and $\nabla_{Y_t+1}\{X_{t+1}\}$ given by eq.(31) are linear operators in X_t, X_{t+1}.

4.3 An algorithm for calculating a closed-form PCRB, in the LPC system

Based on previous sections, Section C1, C2, C3 and C4 give closed-forms for D_t^{11}, D_t^{12}, D_t^{22} and D_t^{33} in the LPC framework. Moreover, we show that these closed-forms can be written in a recursive manner. The algorithm that calculates the closed-form PCRB is summed up in fig.2. We can see that calculation of D_t^{11}, D_t^{12} and D_t^{22} is splitted in two steps. In step 1, the auxiliary matrices Γ_t^{11}, Γ_t^{12} and Γ_t^{22}, defined by eqs.(36,40,44), are computed via a linear system. Then, D_t^{11}, D_t^{12} and D_t^{22} are extracted from Γ_t^{11}, Γ_t^{12}, Γ_t^{22} in step 2. This algorithm will be compared in the simulations section with the classical PCRB summed up in fig.3.

4.3.1 D_t^{11} closed-form

We show in Appendix D that D_t^{11} can be expressed as an expectation of a simple function in the Cartesian coordinates system:

$$D_t^{11} = \frac{1}{\sigma_{max}^2} \mathbb{E}\left\{ F_{X_t}^* A^* Q^{-1} A F_{X_t} \right\} \text{ with } F_{X_t} = \nabla_{Y_t}\{X_t\} \ . \tag{32}$$

The problem is now to compute this expectation. We show now that no “direct” recursive formula can be derived for D_t^{11} but the latter can be obtained as the by product of a general linear system in Prop.6.1. First let us investigate the

- Initialization of J_0^{-1} using the initial error covariance matrix given by eq.(56).
- Initialization of Γ_0^{11}, Γ_0^{12} and Γ_0^{22} using eqs.(38,42,46).
- J_1^{-1} is calculated using only step 2 and 3 with $t = 0$.
- For $t = 1$ to T

  1. Calculation of auxiliary matrices Γ_t^{11}, Γ_t^{12} and Γ_t^{22}

     (a) Calculate Λ_{t-1}^{11}, Λ_{t-1}^{12} and Λ_{t-1}^{22} using eqs.(37,41,45) if observer maneuvers (else these terms are equal to zero).

     (b) $$\begin{cases} \Gamma_t^{11} = \Omega^{11} + \Psi\ \Gamma_{t-1}^{11}\ (+\boldsymbol{\Lambda_{t-1}^{11}})\ , \\ \Gamma_t^{12} = \Omega^{12} + \Psi\ \Gamma_{t-1}^{12}\ (+\boldsymbol{\Lambda_{t-1}^{12}})\ , \\ \Gamma_t^{22} = \Omega^{22} + \Psi\ \Gamma_{t-1}^{22}\ (+\boldsymbol{\Lambda_{t-1}^{22}})\ . \end{cases}$$

     Remark : Ω^{11}, Ω^{12} *and* Ω^{22} *are given by eqs.(37,41,45).* Ψ *is given by eq.(37).*

  2. Calculation of D_t^{11}, D_t^{12} and D_t^{22}

     (a) If observer maneuvers, computeΥ_t^{12} and Υ_t^{22} using eq.(39) and eq.(43) (else these terms are equal to zero).

     (b) $$\begin{cases} D_t^{11} = \begin{bmatrix} Id_{n_y} & 0_{n_y \times 3n_y} \end{bmatrix} \Gamma_t^{11}\ , \\ D_t^{12} = -\begin{bmatrix} Id_{n_y} & 0_{n_y \times 3n_y} \end{bmatrix} \Gamma_t^{12}\ (-\boldsymbol{\Upsilon_t^{12}})\ , \\ D_t^{22} = \begin{bmatrix} Id_{n_y} & 0_{n_y \times 3n_y} \end{bmatrix} \Gamma_t^{22} + \mathcal{C}\ (+\boldsymbol{\Upsilon_t^{22}})\ . \end{cases} .$$

     Remark : $\mathcal{C}$ *is given by eq.(43) and* D_t^{21} *is given by the relation* $D_t^{21} = \left(D_t^{12}\right)^*$.

     (c) D_t^{33} is given by eq.(47).

  3. Calculate J_{t+1}^{-1} using Tichavský's formula:

$$J_{t+1}^{-1} = \left(D_t^{22} + D_t^{33} - D_t^{21}\left(J_t^{-1} + D_t^{11}\right)^{-1} D_t^{12}\right)^{-1}\ .$$

Figure 2: Closed-form calculation of the PCRB.

- Initialisation of J_0^{-1} using the initial error covariance matrix given by eq.(56).

- For $t = 0$ to T

  1. Approximation of D_t^{11}, D_t^{12} and D_t^{22} by Monte Carlo.
  2. D_t^{21} is given by the relation $D_t^{21} = \left(D_{t+1}^{12}\right)^*$ and D_t^{33} is given by eq.(47).
  3. Compute J_{t+1}^{-1} using Tichavský's formula:

$$J_{t+1}^{-1} = \left(D_t^{22} + D_t^{33} - D_t^{21}\left(J_t^{-1} + D_t^{11}\right)^{-1} D_t^{12}\right)^{-1} .$$

Figure 3: Classical computation of the PCRB.

non maneuvering case. In this case, using the statistical properties of X_{t+1} given X_t and the linear property of f, eq.(32) can be rewritten:

$$D_t^{11} = \underbrace{\frac{1}{\sigma_{max}^2}\mathbb{E}\left\{F_{X_t-AX_{t-1}}^* A^* Q^{-1} A F_{X_t-AX_{t-1}}\right\}}_{constant} + \frac{1}{\sigma_{max}^2}\mathbb{E}\left\{F_{AX_{t-1}}^* A^* Q^{-1} A F_{AX_{t-1}}\right\} \tag{33}$$

The first term can be calculated remarking that $X_t - AX_{t-1} \sim \mathcal{N}(0, Q)$ and F is a linear operator. We derived in appendix D from the linear property of F that:

$$\left\{\begin{array}{lcl} F_{AX_t} & = & F_{X_t} + \delta_t G_{X_t} \ , \\ G_{AX_t} & = & G_{X_t} \ , \end{array}\right. \quad \text{where} \quad \left\{\begin{array}{l} F_{X_t} = \nabla_{Y_t}\{X_t\} \ , \\ G_{X_t} = Id_2 \otimes \begin{pmatrix} v_y(t) & -v_x(t) \\ v_x(t) & v_y(t) \end{pmatrix} \end{array}\right. . \tag{34}$$

Incorporating eq.(34) in eq.(33), we obtain:

$$\begin{aligned} D_t^{11} &= constant + \underbrace{\frac{1}{\sigma_{max}^2}\mathbb{E}\left\{F_{X_{t-1}}^* A^* Q^{-1} A F_{X_{t-1}}\right\}}_{=D_{t-1}^{11}} + \frac{\delta_t^2}{\sigma_{max}^2}\mathbb{E}\left\{G_{X_{t-1}}^* A^* Q^{-1} A G_{X_{t-1}}\right\} \\ &+ \frac{\delta_t}{\sigma_{max}^2}\mathbb{E}\left\{F_{X_{t-1}}^* A^* Q^{-1} A G_{X_{t-1}}\right\} + \frac{\delta_t}{\sigma_{max}^2}\mathbb{E}\left\{G_{X_{t-1}}^* A^* Q^{-1} A F_{X_{t-1}}\right\} . \end{aligned} \tag{35}$$

Looking at eq.(35), It seems that no "direct" recursive formula can be derived for D_t^{11}. However, we can propose an original recursive formula for D_t^{11} via a joint matrix Γ_t^{11} formed with the four terms involved in eq.(35) which is valid in the general case including the maneuvering case:

$$\left\{\begin{array}{l} D_t^{11} = \left[Id_{n_y} \quad 0_{n_y \times 3n_y}\right] \Gamma_t^{11} \ , \\ \Gamma_t^{11} = \frac{1}{\sigma_{max}^2}\begin{pmatrix} \mathbb{E}\left\{F_{X_t}^* A^* Q^{-1} A F_{X_t}\right\} \\ \mathbb{E}\left\{F_{X_t}^* A^* Q^{-1} A G_{X_t}\right\} \\ \mathbb{E}\left\{G_{X_t}^* A^* Q^{-1} A F_{X_t}\right\} \\ \mathbb{E}\left\{G_{X_t}^* A^* Q^{-1} A G_{X_t}\right\} \end{pmatrix} \text{ where } F_{X_t} \text{ and } G_{X_t} \text{ are defined by eq.(34).} \end{array}\right. \tag{36}$$

We can see that D_t^{11} is just one block of Γ_t^{11}. Now the following proposition assumes that we have a recursive formula for Γ_t^{11}, so that D_t^{11} is obtained as a by product.

proposition 6.1(Γ_t^{11} formula) *For a filtering problem given by eq.(9), we have the following recursive formula for* Γ_t^{11}*:*

$$\boxed{\Gamma_t^{11} \;=\; \Omega^{11} + \; \Psi\,\Gamma_{t-1}^{11} \qquad (+ \boldsymbol{\Lambda_{t-1}^{11}})}$$

where

$$\begin{cases} \Psi = \begin{pmatrix} 1 & \delta_t & \delta_t & \delta_t^2 \\ 0 & 1 & 0 & \delta_t \\ 0 & 0 & 1 & \delta_t \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes Id_4 \;, \\ \Omega^{11} = \begin{pmatrix} 2\alpha_3 A^* Q^{-1} A + 2\alpha_1 B A^* Q^{-1} A B^* + 2\alpha_2 B A^* Q^{-1} A + 2\alpha_2 A^* Q^{-1} A B^* \\ 2\alpha_1 B A^* Q^{-1} A + 2\alpha_2 A^* Q^{-1} A \\ 2\alpha_1 A^* Q^{-1} A B^* + 2\alpha_2 A^* Q^{-1} A \\ 2\alpha_1 A^* Q^{-1} A \end{pmatrix} \end{cases}$$

and

$$\boldsymbol{\Lambda_{t-1}^{11}} = \begin{cases} 0_{4n_y \times n_y} & \textit{if}\; U_{t-1} = 0 \;, \\ \frac{1}{\sigma_{max}^2} \begin{pmatrix} F_{\mathbb{E}X_t}^* A^* Q^{-1} A F_{\mathbb{E}X_t} - F_{A\mathbb{E}X_{t-1}}^* A^* Q^{-1} A F_{A\mathbb{E}X_{t-1}} \\ F_{\mathbb{E}X_t}^* A^* Q^{-1} A G_{\mathbb{E}X_t} - F_{A\mathbb{E}X_{t-1}}^* A^* Q^{-1} A G_{A\mathbb{E}X_{t-1}} \\ G_{\mathbb{E}X_t}^* A^* Q^{-1} A F_{\mathbb{E}X_t} - G_{A\mathbb{E}X_{t-1}}^* A^* Q^{-1} A F_{A\mathbb{E}X_{t-1}} \\ G_{\mathbb{E}X_t}^* A^* Q^{-1} A G_{\mathbb{E}X_t} - G_{A\mathbb{E}X_{t-1}}^* A^* Q^{-1} A G_{A\mathbb{E}X_{t-1}} \end{pmatrix} & \textit{if}\; U_{t-1} \neq 0 \;. \end{cases} \qquad (37)$$

We refer to eq.(2), for a definition of the various terms $\{A, B, Q, \alpha_1, \alpha_2, \alpha_3\}$ *involved in this closed form. For definitions of* F *and* G *see eq.(34).*

Let us now make some remarks about the previous proposition. We can see that the recursive formula for Γ_t^{11} given by eq.(37) is just a simple linear equation, where all the terms have closed-form expressions. Moreover, if the maneuvering term U_{t-1} is zero, then $\mathbb{E}X_t = A\mathbb{E}X_{t-1}$. As a consequence, $\boldsymbol{\Lambda_{t-1}^{11}}$ is zero if the maneuvering term U_{t-1} is zero. If this condition does not hold, $\boldsymbol{\Lambda_{t-1}^{11}}$ can be computed exactly using $\mathbb{E}(X_0)$ and the recursion $\mathbb{E}(X_t) = A\mathbb{E}(X_{t-1}) + HU_{t-1}$.

Finally, we must pay attention to the initialization of Γ_t^{11}. We show in Appendix F that Γ_0^{11} can be expressed as a function of the first moments of target state in LPC system, more precisely we have:

$$\Gamma_0^{11} = \frac{\mathbb{E}r_0^2}{\sigma_{max}^2} \begin{pmatrix} \Sigma^{11} \otimes RR^{11} + \Sigma_{\nwarrow}^{11} \otimes VV^{11} + \Sigma_{\uparrow}^{11} \otimes VR^{11} + \Sigma_{\leftarrow}^{11} \otimes RV^{11} \\ \Sigma_{\uparrow}^{11} \otimes VV^{11} + \Sigma^{11} \otimes RV^{11} \\ \Sigma_{\leftarrow}^{11} \otimes VV^{11} + \Sigma^{11} \otimes VR^{11} \\ \Sigma^{11} \otimes VV^{11} \end{pmatrix}$$

$\Sigma^{11} = \begin{bmatrix} 1 & 0 \\ \delta_t & 1 \end{bmatrix} \Sigma^{-1} \begin{bmatrix} 1 & \delta_t \\ 0 & 1 \end{bmatrix}$	$\Sigma^{12} = \begin{bmatrix} 1 & 0 \\ \delta_t & 1 \end{bmatrix} \Sigma^{-1}$	$\Sigma^{22} = \Sigma^{-1} = \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_1 \end{bmatrix}^{-1}$
$\Sigma^{11}_{\uparrow} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Sigma^{11}$	$\Sigma^{12}_{\uparrow} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Sigma^{12}$	$\Sigma^{22}_{\uparrow} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Sigma^{22}$
$\Sigma^{11}_{\leftarrow} = \Sigma^{11} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	$\Sigma^{12}_{\leftarrow} = \Sigma^{12} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	$\Sigma^{22}_{\leftarrow} = \Sigma^{22} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$
$\Sigma^{11}_{\nwarrow} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Sigma^{11} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	$\Sigma^{12}_{\nwarrow} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Sigma^{11} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	$\Sigma^{22}_{\nwarrow} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Sigma^{11} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

Table 2: Initialization of the Γ_t^{ij} recursion ($t = 0$).

where Σ^{11}, $\Sigma^{11}_{\nwarrow}$, $\Sigma^{11}_{\uparrow}$ and $\Sigma^{11}_{\leftarrow}$ are given in tab.2 and

$$\begin{cases} RR^{11} = Id_2 \quad , VV^{11} = \left(\mathbb{E}\dot{\beta}_0^2 + \mathbb{E}\frac{\dot{r}_0^2}{r_0^2}\right) Id_2 \quad , \\ RV^{11} = \begin{pmatrix} \mathbb{E}\frac{\dot{r}_0}{r_0} & -\mathbb{E}\dot{\beta}_0 \\ \mathbb{E}\dot{\beta}_0 & \mathbb{E}\frac{\dot{r}_0}{r_0} \end{pmatrix} \quad , VR^{11} = \begin{pmatrix} \mathbb{E}\frac{\dot{r}_0}{r_0} & \mathbb{E}\dot{\beta}_0 \\ -\mathbb{E}\dot{\beta}_0 & \mathbb{E}\frac{\dot{r}_0}{r_0} \end{pmatrix} \quad . \end{cases} \tag{38}$$

From a practical point of view, RR^{11}, VV^{11}, RV^{11}, VR^{11} can be derived from the initial PDF in LPC system i.e. $p(Y_0)$.

4.3.2 D_t^{12} closed-form

Using the same approach as in the previous section, we show in Appendix D that:

$$D_t^{12} \quad = \quad \underbrace{-\frac{1}{\sigma_{max}^2}\mathbb{E}\left\{F^*_{X_t} A^* Q^{-1} F_{AX_t}\right\}}_{(\star)} \quad \left(\; - \boldsymbol{\Upsilon_t^{12}} \; \right) ,$$

with

$$\boldsymbol{\Upsilon_t^{12}} = \begin{cases} 0_{n_y \times n_y} & \text{if } U_t = 0 \; , \\ \frac{1}{\sigma_{max}^2}\left(F^*_{\mathbb{E}X_t} A^* Q^{-1} F_{EX_{t+1}} - F^*_{\mathbb{E}X_t} A^* Q^{-1} F_{AEX_t}\right) & \text{if } U_t \neq 0 \end{cases} \tag{39}$$

where operator F is defined by eq.(34). Comparing eq.(39) with eq.(32), we can notice that we have now **two** terms to compute. The term $\boldsymbol{\Upsilon_t^{12}}$ can be easily calculated. We can remark that the latter is zero if U_t is zero. If this condition is not verified, $\mathbb{E}(X_t)$ is computed for any value of t using $\mathbb{E}(X_0)$ and the relation $\mathbb{E}(X_t) = A\mathbb{E}(X_{t-1}) + HU_{t-1}$. Otherwise, $(\star)$ can be computed recursively using the same approach as for D_t^{11}. D_t^{12} is deduced from Γ_t^{12}, via:

$$\begin{cases} D_t^{12} = \begin{bmatrix} Id_{n_y} & 0_{n_y \times 3n_y} \end{bmatrix} \Gamma_t^{12} \quad \left(\; + \boldsymbol{\Upsilon_t^{12}} \; \right) , \\ \Gamma_t^{12} = \frac{1}{\sigma_{max}^2} \begin{pmatrix} \mathbb{E}\left\{F^*_{X_t} A^* Q^{-1} F_{AX_t}\right\} \\ \mathbb{E}\left\{F^*_{X_t} A^* Q^{-1} G_{AX_t}\right\} \\ \mathbb{E}\left\{G^*_{X_t} A^* Q^{-1} F_{AX_t}\right\} \\ \mathbb{E}\left\{G^*_{X_t} A^* Q^{-1} G_{AX_t}\right\} \end{pmatrix} \end{cases} \tag{40}$$

where operators F and G are given by eq.(34). Again, we have a recursive formula for Γ_t^{12}, yielding D_t^{12} as a by product.

proposition 6.2(Γ_t^{12} formula) *For a filtering problem given by eq.(9), we have the following recursive formula for* Γ_t^{12} *:*

$$\boxed{\Gamma_t^{12} \;=\; \Omega^{12} \;+\; \Psi\,\Gamma_{t-1}^{12} \qquad (\,+\,\mathbf{\Lambda_{t-1}^{12}}\,)}$$

where:

$$\Omega^{12} = \begin{pmatrix} 2(\alpha_3+\delta_t\alpha_2)A^*Q^{-1} + 2\alpha_1 BA^*Q^{-1}B^* + 2(\alpha_2+\delta_t\alpha_1)BA^*Q^{-1} + 2\alpha_2 A^*Q^{-1}B^* \\ 2\alpha_1 BA^*Q^{-1} + 2\alpha_2 A^*Q^{-1} \\ 2\alpha_1 A^*Q^{-1}B^* + 2(\alpha_2+\delta_t\alpha_1)A^*Q^{-1} \\ 2\alpha_1 A^*Q^{-1} \end{pmatrix}$$

and:

$$\mathbf{\Lambda_{t-1}^{12}} = \begin{cases} 0_{4n_y\times n_y} & \textit{if}\, U_{t-1}=0\ , \\ \frac{1}{\sigma_{max}^2}\begin{pmatrix} F^*_{\mathbb{E}X_t}A^*Q^{-1}F_{A\mathbb{E}X_t} - F^*_{A\mathbb{E}X_{t-1}}A^*Q^{-1}F_{A^2\mathbb{E}X_{t-1}} \\ F^*_{\mathbb{E}X_t}A^*Q^{-1}G_{A\mathbb{E}X_t} - F^*_{A\mathbb{E}X_{t-1}}A^*Q^{-1}G_{A^2\mathbb{E}X_{t-1}} \\ G^*_{\mathbb{E}X_t}A^*Q^{-1}F_{A\mathbb{E}X_t} - G^*_{A\mathbb{E}X_{t-1}}A^*Q^{-1}F_{A^2\mathbb{E}X_{t-1}} \\ G^*_{\mathbb{E}X_t}A^*Q^{-1}G_{A\mathbb{E}X_t} - G^*_{A\mathbb{E}X_{t-1}}A^*Q^{-1}G_{A^2\mathbb{E}X_{t-1}} \end{pmatrix} & \textit{if}\, U_{t-1}\neq 0\ . \end{cases} \tag{41}$$

Ψ *is given by eq.(37). We refer to eq.(2), for a definition of the various terms* $\{A,B,Q,\alpha_1,\alpha_2,\alpha_3\}$ *involved in this closed form. For definitions of* F *and* G *see eq.(34).*

Again, the recursion giving Γ_t^{12} is linear and have a closed-form. Similarly to Γ_t^{11} recursion, $\mathbf{\Lambda_{t-1}^{12}}$ is zero if no maneuver occurs ($\mathbb{E}X_t = A\mathbb{E}X_{t-1}$). Else, $\mathbf{\Lambda_{t-1}^{12}}$ is updated from $\mathbb{E}(X_0)$. Considering the initialization of the Γ_t^{12} recursion, we show in Appendix F that Γ_0^{12} can be expressed as a function of the first moments of target state in LPC, as:

$$\Gamma_0^{12} = \frac{\mathbb{E}r_0^2}{\sigma_{max}^2}\begin{pmatrix} \Sigma^{12}\otimes RR^{12} + \Sigma^{12}_{\nwarrow}\otimes VV^{12} + \Sigma^{12}_{\leftarrow}\otimes VR^{12} + \Sigma^{12}_{\uparrow}\otimes RV^{12} \\ \Sigma^{12}_{\uparrow}\otimes VV^{12} + \Sigma^{12}\otimes RV^{12} \\ \Sigma^{12}_{\leftarrow}\otimes VV^{12} + \Sigma^{12}\otimes VR^{12} \\ \Sigma^{12}\otimes VV^{12} \end{pmatrix}$$

where Σ^{12}, $\Sigma^{12}_{\nwarrow}$, $\Sigma^{12}_{\uparrow}$ and $\Sigma^{12}_{\leftarrow}$ are given in tab.2 and

$$\begin{cases} RR^{12} = Id_2 + \delta_t\begin{pmatrix} \mathbb{E}\frac{\dot r_0}{r_0} & -\mathbb{E}\dot\beta_0 \\ \mathbb{E}\dot\beta_0 & \mathbb{E}\frac{\dot r_0}{r_0}\end{pmatrix}\ ,\ VV^{12} = \left(\mathbb{E}\dot\beta_0^2 + \mathbb{E}\frac{\dot r_0^2}{r_0^2}\right)Id_2 \\ RV^{12} = \begin{pmatrix} \mathbb{E}\frac{\dot r_0}{r_0} & -\mathbb{E}\dot\beta_0 \\ \mathbb{E}\dot\beta_0 & \mathbb{E}\frac{\dot r_0}{r_0}\end{pmatrix}\ ,\ VR^{12} = \begin{pmatrix} \mathbb{E}\frac{\dot r_0}{r_0} & \mathbb{E}\dot\beta_0 \\ -\mathbb{E}\dot\beta_0 & \mathbb{E}\frac{\dot r_0}{r_0}\end{pmatrix} + \delta_t\left(\mathbb{E}\dot\beta_0^2 + \mathbb{E}\frac{\dot r_0^2}{r_0^2}\right)Id_{2\times 2}\ . \end{cases} \tag{42}$$

4.3.3 D_t^{22} closed-form

Using the same approach as in the previous section, we show in Appendix D that:

$$D_t^{22} = \underbrace{\frac{1}{\sigma_{max}^2}\mathbb{E}\left\{F_{AX_t}^* Q^{-1} F_{AX_t}\right\}}_{(\star)} + \mathcal{C} \;\; (\; + \;\; \boldsymbol{\Upsilon_t^{22}} \;\;) \; ,$$

where:

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_1 & 0 & 0 & 0 \\ 0 & 16 + \mathcal{C}_1 & 0 & \mathcal{C}_3 \\ 0 & 0 & \mathcal{C}_2 & 0 \\ 0 & \mathcal{C}_3 & 0 & \mathcal{C}_2 \end{pmatrix}$$

with:

$$\begin{cases} \mathcal{C}_1 = \frac{576\alpha_3^2}{\delta_t^6} + \frac{672\alpha_2^2}{\delta_t^4} + \frac{64\alpha_1^2}{\delta_t^2} - \frac{1152\alpha_3\alpha_2}{\delta_t^5} + \frac{288\alpha_3\alpha_1}{\delta_t^4} - \frac{384\alpha_2\alpha_1}{\delta_t^3} \; , \\ \mathcal{C}_2 = \frac{144\alpha_3^2}{\delta_t^4} + \frac{32\alpha_2^2}{\delta_t^2} - \frac{192\alpha_3\alpha_2}{\delta_t^3} + \frac{32\alpha_3\alpha_1}{\delta_t^2} \; , \\ \mathcal{C}_3 = -\frac{288\alpha_3^2}{\delta_t^5} - \frac{192\alpha_2^2}{\delta_t^3} + \frac{480\alpha_3\alpha_2}{\delta_t^4} - \frac{96\alpha_3\alpha_1}{\delta_t^3} + \frac{64\alpha_2\alpha_1}{\delta_t^2} \end{cases}$$

and

$$\boldsymbol{\Upsilon_t^{22}} = \begin{cases} 0_{n_y \times n_y} & \text{if } U_t = 0 \; , \\ \frac{1}{\sigma_{max}^2}\left(F_{\mathbb{E}X_{t+1}}^* Q^{-1} F_{\mathbb{E}X_{t+1}} - F_{A\mathbb{E}X_t}^* Q^{-1} F_{A\mathbb{E}X_t}\right) & \text{if } U_t \neq 0 \; . \end{cases} \tag{43}$$

where the operator F is defined by eq.(34). As we can see above, $\mathcal{C}$ is just a constant term and $\boldsymbol{\Upsilon_t^{22}}$ is a maneuvering term which can be calculated using the same approach as for $\boldsymbol{\Upsilon_t^{12}}$ in section B.2. Otherwise, $(\star)$ in eq.(43) can be calculated recursively. The matrix D_t^{22} is deduced from Γ_t^{22} via:

$$\begin{cases} D_{t+1}^{22} = \begin{bmatrix} Id_{n_y \times n_y} & 0_{n_y \times 3n_y} \end{bmatrix} \Gamma_{t+1}^{22} + \mathcal{C} \; (\; + \boldsymbol{\Upsilon_t^{22}} \;) \; , \\ \Gamma_t^{22} = \frac{1}{\sigma_{max}^2} \begin{pmatrix} \mathbb{E}\left\{F_{AX_t}^* Q^{-1} F_{AX_t}\right\} \\ \mathbb{E}\left\{F_{AX_t}^* Q^{-1} G_{AX_t}\right\} \\ \mathbb{E}\left\{G_{AX_t}^* Q^{-1} F_{AX_t}\right\} \\ \mathbb{E}\left\{G_{AX_t}^* Q^{-1} G_{AX_t}\right\} \end{pmatrix} \end{cases} \tag{44}$$

where operators F and G are given by eq.(34). Again, the following proposition yields a closed-form recursive formula for Γ_t^{22}, and for D_t^{22} as a by product.

proposition 6.3(Γ_t^{22} formula) *For a filtering problem given by eq.(9), a closed-form recursive formula for* Γ_t^{22} *is given by:*

$$\boxed{\Gamma_t^{22} \; = \; \Omega^{22} \; + \Psi \;\; \Gamma_{t-1}^{22} \; (\; + \boldsymbol{\Lambda_{t-1}^{22}} \;)}$$

where:

$$\Omega^{22} = \begin{pmatrix} 2(\alpha_3 + 2\delta_t\alpha_2 + \delta_t^2\alpha_1)Q^{-1} + 2\alpha_1 B Q^{-1} B^* + 2(\alpha_2 + \delta_t\alpha_1)\left(BQ^{-1} + Q^{-1}B^*\right) \\ 2\alpha_1 B Q^{-1} + 2(\alpha_2 + \delta_t\alpha_1)Q^{-1} \\ 2\alpha_1 Q^{-1} B^* + 2(\alpha_2 + \delta_t\alpha_1)Q^{-1} \\ 2\alpha_1 Q^{-1} \end{pmatrix}$$

and:

$$\mathbf{\Lambda_{t-1}^{22}} = \begin{cases} 0_{n_y \times n_y} & \textit{if}\, U_{t-1} = 0 \ , \\ \frac{1}{\sigma_{max}^2} \begin{pmatrix} F^*_{A\mathbb{E}X_t} Q^{-1} F_{A\mathbb{E}X_t} - F^*_{A^2\mathbb{E}X_{t-1}} Q^{-1} F_{A^2\mathbb{E}X_{t-1}} \\ F^*_{A\mathbb{E}X_t} Q^{-1} G_{A\mathbb{E}X_t} - F^*_{A^2\mathbb{E}X_{t-1}} Q^{-1} G_{A^2\mathbb{E}X_{t-1}} \\ G^*_{A\mathbb{E}X_t} Q^{-1} F_{A\mathbb{E}X_t} - G^*_{A^2\mathbb{E}X_{t-1}} Q^{-1} F_{A^2\mathbb{E}X_{t-1}} \\ G^*_{A\mathbb{E}X_t} Q^{-1} G_{A\mathbb{E}X_t} - G^*_{A^2\mathbb{E}X_{t-1}} Q^{-1} G_{A^2\mathbb{E}X_{t-1}} \end{pmatrix} & \textit{if}\, U_{t-1} \neq 0 \ . \end{cases} \tag{45}$$

Always using the same approach, we show in Appendix F that Γ_0^{22} can be expressed as a function of the first moments of target state in LPC system, i.e. :

$$\Gamma_0^{22} = \frac{\mathbb{E}r_0^2}{\sigma_{max}^2} \begin{pmatrix} \Sigma^{22} \otimes RR^{22} + \Sigma^{22}_{\nwarrow} \otimes VV^{22} + \Sigma^{22}_{\uparrow} \otimes VR^{22} + \Sigma^{22}_{\leftarrow} \otimes RV^{22} \\ \Sigma^{22}_{\uparrow} \otimes VV^{22} + \Sigma^{22} \otimes RV^{22} \\ \Sigma^{22}_{\leftarrow} \otimes VV^{22} + \Sigma^{22} \otimes VR^{22} \\ \Sigma^{22} \otimes VV^{22} \end{pmatrix}$$

where Σ^{22}, $\Sigma^{22}_{\nwarrow}$, $\Sigma^{22}_{\uparrow}$ and $\Sigma^{22}_{\leftarrow}$ are given in tab.2 and

$$\begin{cases} RR^{22} = (1 + 2\delta_t \mathbb{E}\frac{\dot{r}_0}{r_0} + \delta_t^2 \mathbb{E}\dot{\beta}_0^2 + \delta_t^2 \mathbb{E}\frac{\dot{r}_0^2}{r_0^2}) Id_{2\times 2} \ , \ VV^{22} = \left(\mathbb{E}\dot{\beta}_0^2 + \mathbb{E}\frac{\dot{r}_0^2}{r_0^2}\right) Id_{2\times 2} \ , \\ RV^{22} = \begin{pmatrix} \mathbb{E}\frac{\dot{r}_0}{r_0} & -\mathbb{E}\dot{\beta}_0 \\ \mathbb{E}\dot{\beta}_0 & \mathbb{E}\frac{\dot{r}_0}{r_0} \end{pmatrix} + \delta_t \left(\mathbb{E}\dot{\beta}_0^2 + \mathbb{E}\frac{\dot{r}_0^2}{r_0^2}\right) Id_{2\times 2} \ , VR^{22} = \begin{pmatrix} \mathbb{E}\frac{\dot{r}_0}{r_0} & \mathbb{E}\dot{\beta}_0 \\ -\mathbb{E}\dot{\beta}_0 & \mathbb{E}\frac{\dot{r}_0}{r_0} \end{pmatrix} + \delta_t \left(\mathbb{E}\dot{\beta}_0^2 + \mathbb{E}\frac{\dot{r}_0^2}{r_0^2}\right) Id_{2\times 2} \ . \end{cases} \tag{46}$$

4.3.4 D_t^{33} closed-form

We show in Appendix D that D_t^{33} is simply:

$$D_t^{33} = \begin{pmatrix} \frac{1}{\sigma_\beta^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ . \tag{47}$$

5 About active measurements scheduling for state estimation

We assume now that additionally to (passive) bearing measurements, there is an other sub-system which can produce a noise corrupted range measurement at time t noted d_t:

$$d_t = r_t + \eta_t \quad \text{where } \eta_t \sim \mathcal{N}\left(0, \sigma_r^2\right) \ . \tag{48}$$

where σ_r is the range standard deviation. However, active measurements have a cost so that the total active measurements budget is fixed. The aim of measurement scheduling is to optimize the time-distribution of active measurements to obtain an accurate target state estimate.

The general problem of optimizing the time-distribution of measurements has a long history. Avitzour et al. in [29] have proposed an algorithm to optimize the time-distribution of measurements when estimating a scalar random variable by solving a nonquadratic minimization problem. This result has been extended by Shakeri et al in [30] to discrete-time stochastic processes. However, this previous approach is devoted to linear systems when the BOT is highly nonlinear. Then, Le Cadre has proposed to use the CRB to solve the problem in [31] for nonlinear systems where the state equation is deterministic. We show in this section that a closed-form PCRB derived can be used for active measurement scheduling.

In the previous section, a closed-form PCRB has been derived for bearings-only measurements. What happens if range measurements are included ? We show in this section that the PCRB has still a closed-form. First, looking at eq.(29), we can see that only D_t^{33} depends on the measurement equation. Then, only the latter has to be modified. If the sensor produces a range measurement at time t, then:

$$\mathcal{D}_t^{33} = \mathbb{E}\{\nabla_{Y_{t+1}} \ln p(Z_{t+1}, d_{t+1}|Y_{t+1}) \nabla^*_{Y_{t+1}} \ln p(Z_{t+1}, d_{t+1}|Y_{t+1})\} \ . \tag{49}$$

Using the independence property between bearings and range measurements, eq.(49) can be rewritten:

$$\mathcal{D}_t^{33} = \underbrace{\mathbb{E}\{\nabla_{Y_{t+1}} \ln p(Z_{t+1}|Y_{t+1}) \nabla^*_{Y_{t+1}} \ln p(Z_{t+1}|Y_{t+1})\}}_{=D_t^{33}} + \mathbb{E}\{\nabla_{Y_{t+1}} \ln p(d_{t+1}|Y_{t+1}) \nabla^*_{Y_{t+1}} \ln p(d_{t+1}|Y_{t+1})\} \ . \tag{50}$$

Using D_t^{33} given by eq.(47) and range measurement equation given by eq.(48), we obtain

$$\mathcal{D}_t^{33} = \begin{bmatrix} \frac{1}{\sigma_\beta^2} & 0 & 0 & 0 \\ 0 & \frac{\mathbb{E}r_{t+1}^2}{\sigma_r^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \ . \tag{51}$$

Consequently, the problem is to compute $\mathbb{E}r_{t+1}^2$. We show now that there is no "direct" recursive formula to calculate $\mathbb{E}r_{t+1}^2$ but the latter can be obtained as a by product of a linear system. First let us address the non maneuvering case. Using the state equation given by eq.(3) and the statistical properties of W_t, elementary calculations yield:

$$\begin{aligned} \mathbb{E}r_{t+1}^2 &= \mathbb{E}\{r_x^2(t+1) + r_y^2(t+1)\} \\ &= 2\sigma_{max}^2 \alpha_3 + \underbrace{\mathbb{E}\{r_x^2(t) + r_y^2(t)\}}_{=\mathbb{E}r_t^2} + 2\delta_t \mathbb{E}\{v_x(t) r_x(t) + v_y(t) r_y(t)\} + \delta_t^2 \mathbb{E}\left\{v_x^2(t) + v_y^2(t)\right\} \ . \end{aligned} \tag{52}$$

Then looking at eq.(52), It seems that no "direct" recursive formula can be derived for $\mathbb{E}r_{t+1}^2$. However, we can propose an original recursive formula for the latter via a joint matrix Γ_t^{33} formed with the three terms involved in eq.(52) which is valid in the general case including the maneuvering case:

$$\left\{ \begin{aligned} &\mathbb{E}r_{t+1}^2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \Gamma_t^{33} \ , \\ &\Gamma_t^{33} = \begin{bmatrix} \mathbb{E}\{r_x^2(t+1) + r_y^2(t+1)\} \\ \mathbb{E}\{v_x(t+1) r_x(t+1) + v_y(t+1) r_y(t+1)\} \\ \mathbb{E}\left\{v_x^2(t+1) + v_y^2(t+1)\right\} \end{bmatrix} \ . \end{aligned} \right. \tag{53}$$

We can see that $\mathbb{E}r_{t+1}^2$ is the first component of Γ_t^{33}. We have a simple recursive formula for Γ_t^{33} given by:

Proposition 7(Γ_t^{33} formula)

$$\boxed{\Gamma_t^{33} = \Omega^{33} \ + \ \Phi \ \Gamma_{t-1}^{33} \qquad (+ \boldsymbol{\Lambda_{t-1}^{33}} \ )}$$

where

$$\begin{cases} \Omega^{33} = 2\sigma_{max}^2 \begin{bmatrix} \alpha_3 \\ \alpha_2 \\ \alpha_1 \end{bmatrix} , \\ \Phi = \begin{bmatrix} 1 & 2\delta_t & \delta_t^2 \\ 0 & 1 & \delta_t \\ 0 & 0 & 1 \end{bmatrix} , \end{cases}$$

and

$$\boldsymbol{\Lambda_{t-1}^{33}} = \begin{bmatrix} 2\delta_t \begin{bmatrix} \mathbb{E}r_x(t) \\ \mathbb{E}r_y(t) \end{bmatrix}^* U_t + 2\delta_t^2 \begin{bmatrix} \mathbb{E}v_x(t) \\ \mathbb{E}v_y(t) \end{bmatrix}^* U_t + \delta_t^2 U_t^* U_t \\ \begin{bmatrix} \mathbb{E}r_x(t) \\ \mathbb{E}r_y(t) \end{bmatrix}^* U_t + 2\delta_t \begin{bmatrix} \mathbb{E}v_x(t) \\ \mathbb{E}v_y(t) \end{bmatrix}^* U_t + \delta_t U_t^* U_t \\ 2 \begin{bmatrix} \mathbb{E}r_x(t) \\ \mathbb{E}r_y(t) \end{bmatrix}^* U_t + U_t^* U_t \end{bmatrix} \tag{54}$$

We refer to eq.(2), for a definition of the various terms $\{\alpha_1, \alpha_2, \alpha_3\}$ *involved in this closed form.*

Proof of proposition 7 We incorporate the diffusion equation given by eq.(3) in Γ_t^{33} given by eq.(53). Finally, we obtain eq.(54) using the statistical properties of W_t.$\square$

Λ_{t-1}^{33} is zero if no maneuver occurs. Concerning the initialization, Γ_0^{33} can be expressed using the first moments:

$$\Gamma_0^{33} = \Phi \mathbb{E} r_0^2 \begin{pmatrix} 1 \\ \mathbb{E}\frac{\dot{r}_0}{r_0} \\ \mathbb{E}\dot{\beta}_0^2 + \mathbb{E}\frac{\dot{r}_0^2}{r_0^2} \end{pmatrix} + \boldsymbol{\Lambda_{-1}^{33}} \ . \tag{55}$$

Eq.(55) is obtained by applying the Cartesian-to-LPC mapping function given by eq.(8) to eq.(53) with $t = 0$ and prop.7. The algorithm is summed up in fig.4 and will be illustrated by simulation results in the following section.

6 Simulations

We have shown in the section IV that under assumption 2, the PCRB has a closed-form. We have presented the algorithm in fig.2. The aim of this section is double. First, we show that these original formulas are valid and allow to compute accurately the PCRB without high computation load. Second, this bound can be used for optimal scheduling of active measurements in a sensor management context.

To check formulas, the closed-form PCRB is compared with the classical one using two scenarios. In the first one, the observer goes straight line while in the second one, the observer maneuvers. For the sake of completeness, all the constants involved in the two scenarios are presented in tab.3. For these two scenarios, the standard deviation of the process noise in the state equation σ_{max} is fixed to 0.05 ms^{-1} so that target trajectory strongly departs from a straight line. The classical PCRB algorithm is reminded in fig.3 (the sample size to approximate D_t^{11}, D_t^{12}, D_t^{22} and D_t^{21} by Monte-Carlo methods is 1000). For all the algorithms, the initial FIM inverse is computed using the initial error covariance matrix. The latter is computed using Monte-Carlo methods. More precisely, N initial target states in LPC, noted $\{Y_0^{(i)}\}_{i\in\{1,\dots,N\}}$, are sampled by using the initial range, bearing and speed standard deviations which are respectively set to $\sigma_{r_0} = 2$ km, $\sigma_{\beta_0} = 0.05$ rad (about 3 deg.) and $\sigma_s = 1$ ms^{-1}. Then, we obtain J_0^{-1} using the following approximation:

$$J_0^{-1} \approx \mathbb{E}\left\{(Y_0 - \mathbb{E}\left\{Y_0\right\})^*(Y_0 - \mathbb{E}\left\{Y_0\right\})\right\} \approx \frac{1}{N}\sum_{i=1}^{N}(Y_0^{(i)} - Y_0)^*(Y_0^{(i)} - Y_0) \tag{56}$$

The first scenario is presented in fig. 5. An example of trajectory is presented in fig. 5(a1), while the set of bearing measurements is presented in Fig. 5(b1). Fig. 6 presents the comparison of PCRB obtained by the algorithms given fig.2 and fig.3 for the four components of the target state. The closed-formed PCRB and the classical one produce the same results which verify formulas. Moreover, the computation load difference between the two methods is important. The approximated PCRB takes **about 600 seconds** when closed-form PCRB takes **about 3 seconds**. Now looking at $\ln r_t$'s bound given fig. 6.b , it is a bit surprising to see that the two PCRBs decrease while r_t is weakly observable. The fact is that $\ln r_t$ is not a meaningful component such that the bound given fig. 6.b for $ECM_{\ln r_t}$ (i.e. the error covariance matrix related to $\ln r_t$) is not intuitive. A bound for ECM_{r_t} (i.e. the error covariance matrix related to r_t) would be more meaningful. Using a Taylor series, we can demonstrate that:

$$ECM_{r_t} \approx e^{2\mathbb{E}(\ln r_t)} ECM_{\ln r_t} \tag{57}$$

so that

$$ECM_{r_t} \geq e^{2\mathbb{E}(\ln r_t)} FIM_{\ln r_t}\ . \tag{58}$$

Consequently, we can use the PCRB related to $\ln r_t$ to derive a bound for the error covariance matrix related to r_t. The problem is that $\mathbb{E}(\ln r_t)$ is generally weakly observable. We have computed in fig. 9 the bound given by eq.(58) using the true r_t. We can see that the bound increases over time which matches theoretical observability results.

In the second scenario, the closed-form PCRB is checked when maneuvering terms appear. We consider that the observer follows a leg-by-leg trajectory. Its velocity vector is constant on each leg:

$$1500 \leq t \leq 4500 \begin{pmatrix} v_x^{obs}(t) \\ v_y^{obs}(t) \end{pmatrix} = \begin{pmatrix} 4 \text{ ms}^{-1} \\ 12 \text{ ms}^{-1} \end{pmatrix},\ \ 4500 \leq t \leq \text{end} \begin{pmatrix} v_x^{obs}(t) \\ v_y^{obs}(t) \end{pmatrix} = \begin{pmatrix} 8 \text{ ms}^{-1} \\ -7 \text{ ms}^{-1} \end{pmatrix}. \tag{59}$$

An example of trajectory for the second scenario is presented in fig. 5(a2), while the set of bearing measurements is presented in fig. 5(b2). Fig. 7 presents a comparison of PCRB obtained by the algorithms given in fig.2 and fig.3. We obtain the same results. Then the closed-form PCRB is valid in the maneuvering case. As for the previous scenario, we compute the bound given by eq.(58) which is given by fig. 10. As expected, the PCRB dramatically decreases when the observer maneuvers at time periods 1500 and 4500.

Consequently, we can now compute the PCRB accurately and quickly, making it suitable for sensor management applications. We have proposed in section V an algorithm given by fig.4 which calculates the closed-form PCRB for active measurement scheduling application. Fig. 8 presents a comparison based on the first scenario of the closed-form PCRB with active measurements produced every 80 seconds with the closed-form when no active measurements are produced. In simulations, The range standard deviation is set to $\sigma_r = 100$ m. As we can see in fig. 8.b. $\ln r_t$ bound falls when the sensor produces a range measurement. Fig. 11 presented the related bounds for r_t given by eq.(58).

7 Conclusion

Along this paper, an innovative analysis of the PCRB in the bearings-only context has been presented. In particular, strong results were shown with regards to the PCRB calculation; namely we derived an original closed-form PCRB. This powerful result, asserted by various simulations, cascades down from an original frame that consists in a new coordinates system: the Logarithmic Polar Coordinates system. Computing the PCRB then becomes an accurate and time-varying technique of particular interest for real-time sensor management issues.

Appendix A: About the bias

Bias definition as given by eq.(12) may appear surprising at first. A more natural definition could be $\mathbb{E}\{\hat{Y}_{0:t} - Y_{0:t}\}$ where $\hat{Y}_{0:t}$ is an estimator of $Y_{0:t}$, function of $Z_{1:t}$. This is this point of view we are now going to enlighten through a decomposition of the mean square error related to the estimation of $Y_{0:t}$. When estimating a deterministic parameter, the mean square error can be classically decomposed in estimation variance and bias. However, in the stochastic case, using eq.(10), we only have the following relation:

$$ECM_{0:t} = \left\| Y_{0:t} - \mathbb{E}\left\{ \hat{Y}_{0:t} \middle| Y_{0:t} \right\} \right\|^2 + \left\| \mathbb{E}\left\{ \hat{Y}_{0:t} \middle| Y_{0:t} \right\} - \hat{Y}_{0:t} \right\|^2 . \tag{60}$$

The mean square error is then equal to the covariance estimation error if and only if

$$\left\| Y_{0:t} - \mathbb{E}\left\{ \hat{Y}_{0:t} \middle| Y_{0:t} \right\} \right\|^2 = 0 . \tag{61}$$

Assumption (61) is equivalent to:

$$\mathbb{E}\left\{ Y_{0:t} - \hat{Y}_{0:t} \middle| Y_{0:t} \right\} = 0, \text{ for almost } Y_{0:t} . \tag{62}$$

which is the retained definition of an unbiased estimator.

Appendix B: Proof of proposition 3

Proposition 3 is adapted from proposition 2 to BOT context. More precisely, proposition 3 gives a more simple formula for $C_{0:t}$. The idea of proof is to study this term. Looking at eq.(22) in proposition 1 proof, each $n_y \times n_y$-matrix term of $C_{0:t}$ can be rewritten:

$$C_{0:t}(k,l) = Id_{n_y \times n_y} \delta_{k=l} + \int \Theta(k,l) d(Z_{1:t}, Y_{0:t}^{-\{l\}}) ,$$

where

$$\Theta(k,l) = \Big[(\hat{Y}_k - Y_k)p(Z_{1:t}, Y_{0:t})\Big]_{\mathcal{Y}_l^-}^{\mathcal{Y}_l^+} . \tag{63}$$

Remark that $\mathcal{Y}_l^-$ and $\mathcal{Y}_l^+$ are n_y-vectors, so that $\Theta(k,l)$ is a $n_y \times n_y$-matrix (notation $[\quad]_{\mathcal{Y}_l^-}^{\mathcal{Y}_l^+}$ defined in eq.(20). First, let us rewrite $\Theta(k,l)$ using the statistical property of stochastic system (9). The idea is to use the following relation:

$$p(Z_{1:t}, Y_{0:t}) \quad = \quad \prod_{j=1}^{t} \left\{ p(Z_j|Y_j)p(Y_j|Y_{j-1}) \right\} p(Y_0) \ , \tag{64}$$

which is true under two assumptions. First, the measurement at time t depends only on the target state at time t. Second, $\{Y_t\}_{t\in\mathbb{N}}$ is a Markovian process. These two assumptions are easily deduced from the formulation of the BOT problem given by eq.(9). Then using eq.(64), eq.(63) is equivalent to :

$$\Theta(k,l) = \left[(\hat{Y}_k - Y_k) \prod_{j=1}^{t} \left\{ p(Z_j|Y_j)(Y_j|Y_{j-1}) \right\} p(Y_0) \right]_{\mathcal{Y}_l^-}^{\mathcal{Y}_l^+} . \tag{65}$$

Now, one can see that some terms in eq.(65) do not depend on Y_l so that they can be factorized. Then we obtain:

$$\Theta(k,l) = \begin{cases} \theta(k,l)p(Z_{l+1:t}, Y_{l+2:t}|Y_{l+1}), \text{ if } l = 0 \ , \\ \theta(k,l)p(Z_{l+1:t}, Y_{l+2:t}|Y_{l+1})p(Y_{l-1}), \text{ if } l = 1 \ , \\ \theta(k,l)p(Z_{l+1:t}, Y_{l+2:t}|Y_{l+1})p(Z_{1:l-1}, Y_{0:l-1}), \text{ if } 1 < l < t \ , \\ \theta(k,l)p(Z_{1:l-1}, Y_{0:l-1}), \text{ if } l = t \end{cases}$$

where:

$$\theta(k,l) = \begin{cases} \Big[(\hat{Y}_k - Y_k)p(Y_{l+1}|Y_l)p(Y_l)\Big]_{\mathcal{Y}_l^-}^{\mathcal{Y}_l^+}, \text{ if } l = 0 \ , \\ \Big[(\hat{Y}_k - Y_k)p(Z_l|Y_l)p(Y_{l+1}|Y_l)p(Y_l|Y_{l-1})\Big]_{\mathcal{Y}_l^-}^{\mathcal{Y}_l^+}, \text{ if } 0 < l < t \ , \\ \Big[(\hat{Y}_k - Y_k)p(Z_l|Y_l)p(Y_l|Y_{l-1})\Big]_{\mathcal{Y}_l^-}^{\mathcal{Y}_l^+}, \text{ if } l = t. \end{cases} \tag{66}$$

We are thus reduced to calculate $\theta(k,l)$. Thus, the following limits must be studied:

$$\lim_{Y_l \to \mathcal{Y}_l^+} p(Y_l|Y_{l-1}) \ , \lim_{Y_l \to \mathcal{Y}_l^-} p(Y_l|Y_{l-1}) \quad , \quad \lim_{Y_l \to \mathcal{Y}_l^+} p(Y_{l+1}|Y_l) \ , \lim_{Y_l \to \mathcal{Y}_l^-} p(Y_{l+1}|Y_l) \ , \tag{67}$$
$$\lim_{Y_l \to \mathcal{Y}_l^+} p(Z_l|Y_l) \quad , \quad \lim_{Y_l \to \mathcal{Y}_l^-} p(Z_l|Y_l).$$

To study the first four limits, $p(Y_{l+1}|Y_l)$ derived in Appendix B1 is needed:

$$p(Y_{t+1}|Y_t) = r_{t+1}^4 p(X_{t+1}|X_t)\alpha(Y_t) \ ,$$

where:

$$\begin{cases} p(X_{t+1}|X_t) = \frac{1}{4\pi^2\sqrt{det(Q)}} e^{-\frac{1}{2}\|X_{t+1} - AX_t - HU_t\|_Q^2} \ , \\ \alpha(Y_t) = \mathbb{P}(r_y(l) > 0|Y_l)\mathbb{1}_{\{r_y(l)>0\}} + \mathbb{P}(r_y(l) < 0|Y_l)\mathbb{1}_{\{r_y(l)<0\}} \ . \end{cases} \tag{68}$$

We can notice that in eq.(68), $p(X_{t+1}|X_t)$ is just the PDF of the diffusion process given by eq.(3). The PDF of Y_{t+1} given Y_t is less simple than in Cartesian coordinates system because we do not have a direct bijection between the two coordinates systems.

Now let us remark that Y_l takes its values in $]-\frac{\pi}{2},\frac{\pi}{2}[\times\mathbb{R}^3$ so that $\mathcal{Y}_l^- = \left[-\frac{\pi}{2},-\infty,-\infty,-\infty\right]$ and $\mathcal{Y}_l^+ = \left[\frac{\pi}{2},+\infty,+\infty,+\infty\right]$. According to eq.(57), we must study $\lim_{Y_l\to\mathcal{Y}_l^-} p(X_{t+1}|X_t)$ and $\lim_{Y_l\to\mathcal{Y}_l^+} p(X_{t+1}|X_t)$ to derive the first four limits of eq.(67). Using f_{lp}^c definition given by eq.(7), we can obtain $\lim_{Y_l\to\mathcal{Y}_l^-} X_t$ and $\lim_{Y_l\to\mathcal{Y}_l^+} X_t$ via $\lim_{Y_l\to\mathcal{Y}_l^-} f_{lp}^c(Y_l)$ and $\lim_{Y_l\to\mathcal{Y}_l^+} f_{lp}^c(Y_l)$ and finally derive:

$$\begin{cases} \lim_{Y_l\to\mathcal{Y}_l^-} p(X_t|X_{t-1}) = \begin{bmatrix} p(X_t|X_{t-1})\Big|_{\beta_l=-\frac{\pi}{2}} & 0 & 0 & 0\end{bmatrix}, \\ \lim_{Y_l\to\mathcal{Y}_l^+} p(X_t|X_{t-1}) = \begin{bmatrix} p(X_t|X_{t-1})\Big|_{\beta_l=\frac{\pi}{2}} & 0 & 0 & 0\end{bmatrix}, \\ \lim_{Y_l\to\mathcal{Y}_l^-} p(X_{t+1}|X_t) = \begin{bmatrix} p(X_{t+1}|X_t)\Big|_{\beta_l=-\frac{\pi}{2}} & 0 & 0 & 0\end{bmatrix}, \\ \lim_{Y_l\to\mathcal{Y}_l^+} p(X_{t+1}|X_t) = \begin{bmatrix} p(X_{t+1}|X_t)\Big|_{\beta_l=\frac{\pi}{2}} & 0 & 0 & 0\end{bmatrix}. \end{cases} \tag{69}$$

Now using eq.(69) and notice that $\mathbb{P}(r_y(l)>0|Y_l)$ and $\mathbb{P}(r_y(l)<0|Y_l)$ are bounded functions, we obtain:

$$\begin{cases} \lim_{Y_l\to\mathcal{Y}_l^+} p(Y_l|Y_{l-1}) = \begin{bmatrix} p(Y_l|Y_{l-1})\Big|_{\beta_l=\frac{\pi}{2}} & 0 & 0 & 0\end{bmatrix}, \\ \lim_{Y_l\to\mathcal{Y}_l^-} p(Y_l|Y_{l-1}) = \begin{bmatrix} p(Y_l|Y_{l-1})\Big|_{\beta_l=-\frac{\pi}{2}} & 0 & 0 & 0\end{bmatrix}, \\ \lim_{Y_l\to\mathcal{Y}_l^+} p(Y_{l+1}|Y_l) = \begin{bmatrix} p(Y_{l+1}|Y_l)\Big|_{\beta_l=\frac{\pi}{2}} & 0 & 0 & 0\end{bmatrix}, \\ \lim_{Y_l\to\mathcal{Y}_l^-} p(Y_{l+1}|Y_l) = \begin{bmatrix} p(Y_{l+1}|Y_l)\Big|_{\beta_l=-\frac{\pi}{2}} & 0 & 0 & 0\end{bmatrix}. \end{cases} \tag{70}$$

We have studied the four first limits of eq.(67). Now, let us turn toward the two last ones. The PDF of Z_l given Y_l is derived in Appendix B2:

$$p(Z_l|Y_l) = \frac{1}{\sqrt{2\pi}\sigma_\beta}\left(e^{-\frac{(Z_l-\beta_l)^2}{2\sigma_v^2}} + e^{-\frac{(Z_l-\beta_l-\pi)^2}{2\sigma_v^2}} + e^{-\frac{(Z_l-\beta_l+\pi)^2}{2\sigma_v^2}}\right)\mathbb{1}_{-\frac{\pi}{2}<Z_t<\frac{\pi}{2}}\,. \tag{71}$$

We deduce from eq.(71) that:

$$\begin{cases} \lim_{Y_l\to\mathcal{Y}_l^+} p(Z_l|Y_l) = \begin{bmatrix} p(Z_l|\beta_l)\Big|_{\beta_l=\frac{\pi}{2}} & p(Z_l|\beta_l) & p(Z_l|\beta_l) & p(Z_l|\beta_l)\end{bmatrix}, \\ \lim_{Y_l\to\mathcal{Y}_l^-} p(Z_l|Y_l) = \begin{bmatrix} p(Z_l|\beta_l)\Big|_{\beta_l=-\frac{\pi}{2}} & p(Z_l|\beta_l) & p(Z_l|\beta_l) & p(Z_l|\beta_l)\end{bmatrix}. \end{cases} \tag{72}$$

Using limits given by eq.(70) and eq.(72), $\theta(k,l)$ given by eq.(67) can be rewritten:

$$\theta(k,l) = \begin{cases} \begin{bmatrix}\left[(\hat{Y}_k - Y_k)p(Y_{l+1}|Y_l)p(Y_l)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} & 0_{n_y\times(n_y-1)}\end{bmatrix} \text{ if } l=0, \\ \begin{bmatrix}\left[(\hat{Y}_k - Y_k)p(Z_l|Y_l)p(Y_{l+1}|Y_l)p(Y_l|Y_{l-1})\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} & 0_{n_y\times(n_y-1)}\end{bmatrix} \text{ if } 1<l<t, \\ \begin{bmatrix}\left[(\hat{Y}_k - Y_k)p(Z_l|Y_l)p(Y_l|Y_{l-1})\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} & 0_{n_y\times(n_y-1)}\end{bmatrix} \text{ if } l=t. \end{cases} \tag{73}$$

Consequently, lots of terms in $\theta(k,l)$ are equal to zero without any technical assumption. The problem is now to study more precisely the first column of $\theta(k,l)$. The following result assure a more simple formulation for this column.

Lemma 2 *For a filtering problem given by eq.(9)*

$$\begin{cases} \lim_{\beta_l \to -\frac{\pi}{2}} p(Z_l|Y_l) \approx \lim_{\beta_l \to \frac{\pi}{2}} p(Z_l|Y_l) \ , \\ \lim_{\beta_l \to -\frac{\pi}{2}} p(Y_l|Y_{l-1}) = \lim_{\beta_l \to \frac{\pi}{2}} p(Y_l|Y_{l-1}) \ , \\ \lim_{\beta_l \to -\frac{\pi}{2}} p(Y_{l+1}|Y_l) = \lim_{\beta_l \to \frac{\pi}{2}} p(Y_{l+1}|Y_l). \end{cases} \tag{74}$$

Lemma 2 is proved in Appendix B3. Using previous lemma, $\theta(k,l)$ formula given by eq.(73) becomes:

$$\theta(k,l) = \delta_{\{k=l\}} \begin{bmatrix} -\pi\zeta(l) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where:

$$\zeta(l) = \begin{cases} p(Y_{l+1}|Y_l)p(Y_l)\Big|_{\beta_l = \frac{\pi}{2}} & \text{if } l = 0 \ , \\ p(Z_l|Y_l)p(Y_{l+1}|Y_l)p(Y_l|Y_{l-1})\Big|_{\beta_l = \frac{\pi}{2}} & \text{if } 0 < l < t \ , \\ p(Z_{1:t}, Y_{0:t})\Big|_{\beta_l = \frac{\pi}{2}} & \text{if } l = t \ . \end{cases} \tag{75}$$

Incorporating $\theta(k,l)$ new formula given by eq.(75) in $\Theta(k,l)$ formulation given by eq.(66), yields:

$$\Theta(k,l) = \delta_{\{k=l\}} \begin{bmatrix} -\pi p(Z_{1:t}, Y_{0:t})\Big|_{\beta_l = \frac{\pi}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{76}$$

Putting the new expression of $\Theta(k,l)$ given by eq.(76) in $C_{0:t}$ formula given by eq.(63), we deduce that $C_{0:t}$ is a diagonal matrix with diagonal element:

$$C_{0:t}(l,l) = \begin{bmatrix} 1 - \pi p(\beta_l)\Big|_{\frac{\pi}{2}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{77}$$

□

Appendix B1: A closed-form for for $p(Y_{l+1}|Y_l)$.

The aim of this section is to derive the PDF of Y_{l+1} given Y_l. The classical approach consists of proving that there exists a function $g_{Y_l}(.)$ such that:

$$\mathbb{P}(Y_{l+1} \in A|Y_l) = \int_A g_{Y_l}(y_{l+1}) d\lambda(y_{l+1}), \ \ \forall A \in \mathcal{B}(] - \frac{\pi}{2}, \frac{\pi}{2}[\times \mathbb{R}^3) \tag{78}$$

where $\mathcal{B}(]-\frac{\pi}{2},\frac{\pi}{2}[\times\mathbb{R}^3)$ is the σ-algebra of Borel subsets of $]-\frac{\pi}{2},\frac{\pi}{2}[\times\mathbb{R}^3$ and $\lambda(.)$ is Lebesgue measure. If this property is true then $g_{Y_l}(.)$ is the distribution density function of Y_{l+1} given Y_l. To obtain this result we will use the distribution density function of X_{l+1} given X_l. However, computation is not easy because there is no direct bijection between Cartesian and LPC system. We only have eq.(7) and eq.(8). Then we have:

$$\begin{aligned}
\mathbb{P}(Y_{l+1}\in A|Y_l) &= \mathbb{P}(f_c^{lp}(X_{l+1})\in A|Y_l) \\
&= \mathbb{P}(f_c^{lp}(X_{l+1})\in A|Y_l,\{r_y(l)>0\})\mathbb{P}(\{r_y(l)>0\}|Y_l) \qquad (79)\\
&\quad +\mathbb{P}(f_c^{lp}(X_{l+1})\in A|Y_l,\{r_y(l)<0\})\mathbb{P}(\{r_y(l)<0\}|Y_l) \qquad (80)
\end{aligned}$$

Then, using the PDF of X_{l+1} given X_l and the Change of Variable Theorem, we obtain the PDF of Y_{l+1} given Y_l:

$$p(Y_{l+1}|Y_l)=r_{l+1}^4\,p(X_{l+1}|X_l)\alpha(Y_l)$$

with:

$$\begin{cases} p(X_{l+1}|X_l)=\frac{1}{4\pi^2\sqrt{det(Q)}}e^{-\frac{1}{2}\|X_{l+1}-AX_l-HU_l\|_Q^2}\ , \\ \alpha(Y_l)=\mathbb{1}_{\{r_y(l)>0\}}\mathbb{P}(\{r_y(l)>0\}|Y_l)+\mathbb{1}_{\{r_y(l)<0\}}\mathbb{P}(\{r_y(l)<0\}|Y_l)\ . \end{cases} \qquad (81)$$

One can remark that the Jacobian term is r_{l+1}^4 where r_{l+1} is the relative range at time $t+1$. Moreover $p(X_{l+1}|X_l)$ is the PDF of the diffusion process given by eq.(3). This term can be rewritten as function of Y_l and Y_{l+1} using Cartesian-to-LPC state mapping function given by eq.(7).

Appendix B2: A closed-form for $p(Z_l|Y_l)$.

The aim of this section is to derive the PDF of Z_l given Y_l. To obtain this result we will use eq.(9). Let us remind that:

$$Z_l=\begin{cases} \beta_l+V_l+\pi & \text{if} \quad \beta_t+V_l<-\frac{\pi}{2}\ , \\ \beta_l+V_l & \text{if} \quad -\frac{\pi}{2}<\beta_t+V_l<\frac{\pi}{2}\ , \\ \beta_l+V_l-\pi & \text{if} \quad \frac{\pi}{2}<\beta_t+V_l\ . \end{cases} \qquad (82)$$

Then the PDF of Z_l given Y_l is:

$$p(Z_l|Y_l)=\frac{1}{\sqrt{2\pi}\sigma_\beta}\left(e^{-\frac{(Z_l-\beta_l)^2}{2\sigma_\beta^2}}+e^{-\frac{(Z_l-\beta_l-\pi)^2}{2\sigma_\beta^2}}+e^{-\frac{(Z_l-\beta_l+\pi)^2}{2\sigma_\beta^2}}\right)\mathbb{1}_{-\frac{\pi}{2}<Z_t<\frac{\pi}{2}}\ . \qquad (83)$$

We can see examples of PDF of Z_l given Y_l in fig.1.

Appendix B3: lemma 2 proof

First relation of lemma 2

Using $p(Z_l|Y_l)$ definition given by eq.(83), we can remark the second relation of lemma 2 is satisfied if

$$\begin{cases} e^{-\frac{(Z_l-\frac{3\pi}{2})^2}{2\sigma_\beta^2}}\approx 0\ , \\ e^{-\frac{(Z_l+\frac{3\pi}{2})^2}{2\sigma_\beta^2}}\approx 0\ , \end{cases}\quad \forall Z_l\in]-\frac{\pi}{2},\frac{\pi}{2}[\ . \qquad (84)$$

We can see easily that this assumption is equivalent to $e^{-\frac{2\pi^2}{\sigma_\beta^2}} \approx 0$, so that the first relation of lemma 2 is true if σ_β is not too large.

Second relation of lemma 2

Looking at eq.(81), we can see that we have just to prove that:

$$\lim_{\beta_l \to -\frac{\pi}{2}} p(X_l|X_{l-1}) = \lim_{\beta_l \to \frac{\pi}{2}} p(X_l|X_{l-1}) \ . \tag{85}$$

Then we need to express X_l as a function which depends on Y_l. Using eq.(7), we obtain:

$$\begin{cases} \lim_{\beta_l \to -\frac{\pi}{2}} p(X_l|X_{l-1}) & = & \lim_{\beta_l \to -\frac{\pi}{2}} p(f^c_{lp}(Y_l)|X_{l-1}) \mathbb{1}_{r_y(l)>0} + \lim_{\beta_l \to -\frac{\pi}{2}} p(-f^c_{lp}(Y_l)|X_{l-1}) \mathbb{1}_{r_y(l)<0} \ , \\ \lim_{\beta_l \to \frac{\pi}{2}} p(X_l|X_{l-1}) & = & \lim_{\beta_l \to \frac{\pi}{2}} p(f^c_{lp}(Y_l)|X_{l-1}) \mathbb{1}_{r_y(l)>0} + \lim_{\beta_l \to \frac{\pi}{2}} p(-f^c_{lp}(Y_l)|X_{l-1}) \mathbb{1}_{r_y(l)<0} \ . \end{cases} \tag{86}$$

Now if we note

$$X_l^{\frac{\pi}{2}} = \begin{bmatrix} r_l & 0 & r_l \frac{\dot{r}_l}{r_l} & -r_l \dot{\beta}_l \end{bmatrix}^* \ , \tag{87}$$

we finally obtain

$$\begin{cases} \lim_{\beta_l \to -\frac{\pi}{2}} p(X_l|X_{l-1}) = p(X_l^{\frac{\pi}{2}}|X_{l-1}) + p(-X_l^{\frac{\pi}{2}}|X_{l-1}) \ , \\ \lim_{\beta_l \to \frac{\pi}{2}} p(X_l|X_{l-1}) = p(-X_l^{\frac{\pi}{2}}|X_{l-1}) + p(X_l^{\frac{\pi}{2}}|X_{l-1}) \ , \end{cases} \tag{88}$$

so that the second relation of lemma 2 is true.

Third relation of lemma 2

Looking at eq.(81), we can see that we have to prove that:

$$\lim_{\beta_l \to -\frac{\pi}{2}} p(X_{l+1}|X_l)\alpha(Y_l) = \lim_{\beta_l \to \frac{\pi}{2}} p(X_{l+1}|X_l)\alpha(Y_l) \ . \tag{89}$$

The proof is a little bit more difficult because we need to study $\alpha(Y_l)$ limit. First let us remark that $\alpha(Y_t)$ definition given by eq.(81) can rewritten as:

$$\alpha(Y_t) = \mathbb{P}(r_y(l) > 0||r_y(l)|)\mathbb{1}_{\{r_y(l)>0\}} + \mathbb{P}(r_y(l) < 0||r_y(l)|)\mathbb{1}_{\{r_y(l)<0\}} \ . \tag{90}$$

Now to study $\alpha(Y_l)$ limit, we need the following lemma.

Lemma 3 *For X a scalar random variate*

$$\begin{cases} \mathbb{P}\left(X > 0 \middle| |X| = x\right) = \frac{p_X(x)}{p_X(x)+p_X(-x)} \ , \\ \mathbb{P}\left(X < 0 \middle| |X| = x\right) = \frac{p_X(-x)}{p_X(x)+p_X(-x)} \end{cases} \tag{91}$$

where p_X is the PDF of X.

Lemma 3 proof First let us remark that for a positive ϵ, we can write:

$$\mathbb{P}\left(X>0\Big||X|\in[x-\epsilon,x+\epsilon]\right) \;=\; \frac{\int_{x-\epsilon}^{x+\epsilon}p_X(x)dx}{\int_{x-\epsilon}^{x+\epsilon}p_X(x)dx+\int_{-x-\epsilon}^{-x+\epsilon}p_X(x)dx} \tag{92}$$

so that

$$M_\epsilon^- \leq \mathbb{P}\left(X>0\Big||X|\in[x-\epsilon,x+\epsilon]\right)\leq M_\epsilon^+$$

with

$$\begin{cases} M_\epsilon^- = \frac{\inf_{[x-\epsilon,x+\epsilon]}p_X(x)}{\sup_{[x-\epsilon,x+\epsilon]}p_X(x)+\sup_{[-x-\epsilon,-x+\epsilon]}p_X(x)} \ , \\ M_\epsilon^+ = \frac{\sup_{[x-\epsilon,x+\epsilon]}p_X(x)}{\inf_{[x-\epsilon,x+\epsilon]}p_X(x)+\inf_{[-x-\epsilon,-x+\epsilon]}p_X(x)} \ . \end{cases} \tag{93}$$

Then let ϵ converge to zero so that the first relation of the lemma is proved. The second relation is straight forward. □

Applying lemma 3 with $X=r_y(l)$ and finally remarking that $\lim_{\beta_l\to-\frac{\pi}{2}}r_y(l)=\lim_{\beta_l\to\frac{\pi}{2}}r_y(l)=0$, we obtain:

$$\lim_{\beta_l\to-\frac{\pi}{2}}\alpha(Y_t)=\lim_{\beta_l\to\frac{\pi}{2}}\alpha(Y_t)=\frac{1}{2} \tag{94}$$

so that

$$\begin{cases} \lim_{\beta_l\to-\frac{\pi}{2}}p(X_{l+1}|X_l)\alpha(Y_l) & = & \frac{1}{2}p(X_{l+1}|-X_l^{\frac{\pi}{2}})+\frac{1}{2}p(X_{l+1}|X_l^{\frac{\pi}{2}}) \ , \\ \lim_{\beta_l\to\frac{\pi}{2}}p(X_l|X_{l-1})\alpha(Y_l) & = & \frac{1}{2}p(X_{l+1}|X_l^{\frac{\pi}{2}})+\frac{1}{2}p(X_{l+1}|-X_l^{\frac{\pi}{2}}) \end{cases} \tag{95}$$

with $X_l^{\frac{\pi}{2}}$ defined by eq.(87). The third relation of lemma is proven.

Appendix C: Properties of operators F and G

Operators F and G are defined by eq.(34). Before investigating the properties of such operators, let us remark that these operators can be rewritten using direct tensor product. First, let us study F_{X_t} which represents the derivative of the LPC to Cartesian mapping w.r.t. state in LPC. Using eq.(7), we have:

$$F_{X_t}=\nabla_{Y_t}\{X_t\}=\begin{cases}\nabla_{Y_t}f_{lp}^c(Y_t)\text{ if }r_y(t)>0\ , \\ -\nabla_{Y_t}f_{lp}^c(Y_t)\text{ if }r_y(t)<0\ .\end{cases} \tag{96}$$

Using now f_{lp}^c definition given by eq.(7) , we have:

$$\nabla_{Y_t}f_{lp}^c(Y_t)=r_t\begin{bmatrix}\cos\beta_t & -\sin\beta_t & 0 & 0 \\ \sin\beta_t & \cos\beta_t & 0 & 0 \\ \frac{\dot r_t}{r_t}\cos\beta_t-\dot\beta_t\sin\beta_t & -\frac{\dot r_t}{r_t}\sin\beta_t-\dot\beta_t\cos\beta_t & \cos\beta_t & -\sin\beta_t \\ \frac{\dot r_t}{r_t}\sin\beta_t+\dot\beta_t\cos\beta_t & \frac{\dot r_t}{r_t}\cos\beta_t-\dot\beta_t\sin\beta_t & \sin\beta_t & \cos\beta_t\end{bmatrix}. \tag{97}$$

We can notice the block structure of $\nabla_{Y_t}f_{lp}^c(Y_t)$. Then using eq.(96) and eq.(97), F_{X_t} can be rewritten using Kronecker products, so that eq.(34) can be rewritten as:

$$\begin{cases} F_{X_t}=Id_{2\times2}\otimes R_{X_t}+\begin{bmatrix}0&0\\1&0\end{bmatrix}\otimes V_{X_t}\ , \\ G_{X_t}=Id_{2\times2}\otimes V_{X_t}\end{cases}$$

where:

$$R_{X_t} = \begin{bmatrix} r_y(t) & -r_x(t) \\ r_x(t) & r_y(t) \end{bmatrix} \text{ and } V_{X_t} = \begin{bmatrix} v_y(t) & -v_x(t) \\ v_x(t) & v_y(t) \end{bmatrix} . \tag{98}$$

Now let us detail the basic properties of $F_{\cdot}$ and $G_{\cdot}$ operators.

Property 1 *$G_{\cdot}$ and $F_{\cdot}$ are linear operators i.e. let X_t and $\tilde{X}_t$ to state vector, then $F_{X_t+\tilde{X}_t} = F_{X_t} + F_{\tilde{X}_t}$ and $G_{X_t+\tilde{X}_t} = G_{X_t} + G_{\tilde{X}_t}$.*

Property 2 *Reminding that $A = \begin{bmatrix} 1 & \delta_t \\ 0 & 1 \end{bmatrix} \otimes Id_{2\times 2}$, terms $G_{A^k X_t}$ and $F_{A^k X_t}$ stand as follows:*

$$\begin{cases} F_{A^k X_t} &= F_{X_t} + k\delta_t G_{X_t} \ , \\ G_{A^k X_t} &= G_{X_t} \ . \end{cases} \tag{99}$$

Proofs are omitted.

Appendix D: Closed-forms for D_t^{11}, D_t^{12} and D_t^{22} and D_t^{33}

We show in this section that eq.(29) can be rewritten as:

$$\begin{cases} D_t^{11} &= \mathbb{E}\left\{F_{X_t}^* A^* Q^{-1} A F_{X_t}\right\} \ , \\ D_t^{12} &= -\mathbb{E}\left\{F_{X_t}^* A^* Q^{-1} F_{AX_t}\right\} - \Upsilon_t^{12} \ , \\ D_t^{22} &= \mathbb{E}\left\{F_{AX_t}^* Q^{-1} F_{AX_t}\right\} + \mathcal{C} + \Upsilon_t^{22} \ , \\ D_t^{33} &= \begin{bmatrix} \frac{1}{\sigma_\beta^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \end{cases} \tag{100}$$

with

$$\begin{cases} \Upsilon_t^{12} &= F_{\mathbb{E}X_t}^* A^* Q^{-1} F_{EX_{t+1}} - F_{\mathbb{E}X_t}^* A^* Q^{-1} F_{AEX_t} \ , \\ \Upsilon_t^{22} &= F_{\mathbb{E}X_{t+1}}^* Q^{-1} F_{\mathbb{E}X_{t+1}} - F_{A\mathbb{E}X_t}^* Q^{-1} F_{A\mathbb{E}X_t} \ , \\ \mathcal{C} &= \begin{pmatrix} \mathcal{C}_1 & 0 & 0 & 0 \\ 0 & 16+\mathcal{C}_1 & 0 & \mathcal{C}_3 \\ 0 & 0 & \mathcal{C}_2 & 0 \\ 0 & \mathcal{C}_3 & 0 & \mathcal{C}_2 \end{pmatrix} \end{cases}$$

and

$$\begin{cases} \mathcal{C}_1 &= \frac{576\alpha_3^2}{\delta_t^6} + \frac{672\alpha_2^2}{\delta_t^4} + \frac{64\alpha_1^2}{\delta_t^2} - \frac{1152\alpha_3\alpha_2}{\delta_t^5} + \frac{288\alpha_3\alpha_1}{\delta_t^4} - \frac{384\alpha_2\alpha_1}{\delta_t^3} \ , \\ \mathcal{C}_2 &= \frac{144\alpha_3^2}{\delta_t^4} + \frac{32\alpha_2^2}{\delta_t^2} - \frac{192\alpha_3\alpha_2}{\delta_t^3} + \frac{32\alpha_3\alpha_1}{\delta_t^2} \ , \\ \mathcal{C}_3 &= -\frac{288\alpha_3^2}{\delta_t^5} - \frac{192\alpha_2^2}{\delta_t^3} + \frac{480\alpha_3\alpha_2}{\delta_t^4} - \frac{96\alpha_3\alpha_1}{\delta_t^3} + \frac{64\alpha_2\alpha_1}{\delta_t^2} \ . \end{cases} \tag{101}$$

Considering at D_t^{11}, D_t^{12} and D_t^{22} and D_t^{33} formulas given by eq.(29), it is necessary to derive $p(Y_{t+1}|Y_t)$ and $p(Z_t|Y_t)$. According to Appendix B1 and B2:

$$\begin{cases} p(Y_{t+1}|Y_t) = r_{t+1}^4 p(X_{t+1}|X_t)\alpha(Y_t) \ , \\ p(Z_t|Y_t) = \frac{1}{\sqrt{2\pi}\sigma_\beta}\left(e^{-\frac{(Z_l-\beta_l)^2}{2\sigma_\beta^2}} + e^{-\frac{(Z_l-\beta_l-\pi)^2}{2\sigma_\beta^2}} + e^{-\frac{(Z_l-\beta_l+\pi)^2}{2\sigma_\beta^2}}\right)\mathbb{1}_{-\frac{\pi}{2}<Z_t<\frac{\pi}{2}} . \end{cases} \tag{102}$$

More precisely, according to eq.(29), we need $\nabla_{Y_t}\ln p(Y_{t+1}|Y_t)$, $\nabla_{Y_{t+1}}\ln p(Y_{t+1}|Y_t)$ and $\nabla_{Y_t}\ln p(Z_t|Y_t)$. Using $p(Y_{t+1}|Y_t)$ as given by eq.(102) and remarking that $\nabla_{Y_t}\alpha(Y_t) = 0$, we obtain:

$$\begin{cases} \nabla_{Y_t} p(Y_{t+1}|Y_t) & = & -r_{t+1}^4 F_{X_t}^* A^* Q^{-1}(X_{t+1} - AX_t - HU_t)p(X_{t+1}|X_t)\alpha(Y_t) \ , \\ \nabla_{Y_{t+1}} p(Y_{t+1}|Y_t) & = & r_{t+1}^4\left(F_{X_{t+1}}^* Q^{-1}(X_{t+1} - AX_t - HU_t) + \begin{bmatrix}0 & 4 & 0 & 0\end{bmatrix}^*\right)p(X_{t+1}|X_t)\alpha(Y_t) \end{cases} \tag{103}$$

where F_{X_t} is defined by eq.(34). Then, using eq.(102) and eq.(103), we obtain:

$$\begin{cases} \nabla_{Y_t}\ln p(Y_{t+1}|Y_t) & = & F_{X_t}^* A^* Q^{-1}(X_{t+1} - AX_t - HU_t) \ , \\ \nabla_{Y_{t+1}}\ln p(Y_{t+1}|Y_t) & = & -F_{X_{t+1}}^* Q^{-1}(X_{t+1} - AX_t - HU_t) - \begin{bmatrix}0 & 4 & 0 & 0\end{bmatrix}^* \ , \\ \nabla_{Y_t}\ln p(Z_t|Y_t) & = & \begin{bmatrix}\frac{Z_t-\beta_t-\pi\mathbb{1}_{\frac{\pi}{2}<Z_t-\beta_t<\frac{3\pi}{2}}+\pi\mathbb{1}_{-\frac{3\pi}{2}<Z_t-\beta_t<-\frac{\pi}{2}}}{\sigma_\beta^2} & 0 & 0 & 0\end{bmatrix}^* . \end{cases} \tag{104}$$

Incorporating $\nabla_{Y_t}\ln p(Z_t|Y_t)$ given by (104) in eq.(29) and using the statistical properties of Z_t given Y_t, we obtain D_t^{33} formula given eq.(101) . Otherwise, incorporating $\nabla_{Y_t}\ln p(Y_{t+1}|Y_t)$, $\nabla_{Y_{t+1}}\ln p(Y_{t+1}|Y_t)$ given by (104) in eq.(29), we obtain:

$$\begin{cases} D_t^{11} & = & \mathbb{E}\left\{F_{X_t}^* A^* Q^{-1}(X_{t+1} - AX_t - HU_t)(X_{t+1} - AX_t - HU_t)^* Q^{-1} A F_{X_t}\right\} \ , \\ D_t^{12} & = & -\mathbb{E}\left\{F_{X_t}^* A^* Q^{-1}(X_{t+1} - AX_t - HU_t)(X_{t+1} - AX_t - HU_t)^* Q^{-1} F_{X_{t+1}}\right\} \ , \\ D_t^{22} & = & \mathbb{E}\left\{F_{X_{t+1}}^* Q^{-1}(X_{t+1} - AX_t - HU_t)(X_{t+1} - AX_t - HU_t)^* Q^{-1} F_{X_{t+1}}\right\} \\ & + & \mathbb{E}\left\{F_{X_{t+1}}^* Q^{-1}(X_{t+1} - AX_t - HU_t)\right\}\begin{bmatrix}0 & 4 & 0 & 0\end{bmatrix} \\ & + & \begin{bmatrix}0 & 4 & 0 & 0\end{bmatrix}^* \mathbb{E}\left\{(X_{t+1} - AX_t - HU_t)^* Q^{-1} F_{X_{t+1}}\right\} \ . \end{cases} \tag{105}$$

Now, we are dealing with the calculation of each elementary term of eq.(105) separately.

D_t^{11} formula

Let us rewrite D_t^{11} as given by eq.(105), we have:

$$\begin{aligned} D_t^{11} & = \mathbb{E}\left\{F_{X_t}^* A^* Q^{-1}(X_{t+1} - AX_t - HU_t)(X_{t+1} - AX_t - HU_t)^* Q^{-1} A F_{X_t}\right\}, \\ & = \mathbb{E}\{F_{X_t}^* A^* Q^{-1}\underbrace{\mathbb{E}\left\{(X_{t+1} - AX_t - HU_t)(X_{t+1} - AX_t - HU_t)^*|X_t\right\}}_{=Q} Q^{-1} A F_{X_t}\} \ . \end{aligned} \tag{106}$$

Then using the statistical property of X_{t+1} given X_t i.e. $\mathcal{N}(AX_t + HU_t, Q)$ given by eq.(3), we obtain D_t^{11} formula as given by eq.(101).

D_t^{12} formula

Our aim is now to render explicit D_t^{12} given by eq.(105). Let us first use the linear property of $F_{.}$:

$$\begin{aligned} D_t^{12} &= -\overbrace{\mathbb{E}\left\{F^*_{X_t} A^* Q^{-1}(X_{t+1} - AX_t - HU_t)(X_{t+1} - AX_t - HU_t)^* Q^{-1} F_{X_{t+1}-AX_t-HU_t}\right\}}^{=0} \\ &- \mathbb{E}\left\{F^*_{X_t} A^* Q^{-1}(X_{t+1} - AX_t - HU_t)(X_{t+1} - AX_t - HU_t)^* Q^{-1} F_{AX_t+HU_t}\right\} . \end{aligned} \tag{107}$$

Using the statistical property of X_{t+1} i.e X_{t+1} given X_t is a $\mathcal{N}\left(AX_t + HU_t, Q\right)$, we obtain:

$$D_t^{12} = -\mathbb{E}\left\{F^*_{X_t} A^* Q^{-1} F_{AX_t}\right\} - F^*_{\mathbb{E}X_t} A^* Q^{-1} F_{HU_t}. \tag{108}$$

Now remarking that $HU_t = \mathbb{E}X_{t+1} - X_t$ and the linearity of operator F, we obtain D_t^{12} expression given by eq.(101).

D_t^{22} formula

Starting from D_t^{22} given by eq.(105) and using again the linearity of $F_{.}$:

$$\begin{aligned} D_t^{22} &= \overbrace{\mathbb{E}\left\{F^*_{AX_t+HU_t} Q^{-1}(X_{t+1} - AX_t - HU_t)(X_{t+1} - AX_t - HU_t)^* Q^{-1} F_{X_{t+1}-AX_t-HU_t}\right\}}^{=0} , \\ &+ \mathbb{E}\left\{F^*_{AX_t+HU_t} Q^{-1}(X_{t+1} - AX_t - HU_t)(X_{t+1} - AX_t - HU_t)^* Q^{-1} F_{AX_t+HU_t}\right\} + \mathcal{C} \end{aligned} \tag{109}$$

with:

$$\begin{aligned} \mathcal{C} &= \mathbb{E}\{F^*_{X_{t+1}-AX_t-HU_t} Q^{-1}(X_{t+1} - AX_t - HU_t)(X_{t+1} - AX_t - HU_t)^* Q^{-1} F_{X_{t+1}-AX_t-HU_t}\} , \\ &+ \mathbb{E}\{F^*_{X_{t+1}-AX_t-HU_t} Q^{-1}(X_{t+1} - AX_t - HU_t)\} \begin{pmatrix} 0 & 4 & 0 & 0 \end{pmatrix} , \\ &+ \mathbb{E}\{\begin{pmatrix} 0 & 4 & 0 & 0 \end{pmatrix}^* \mathbb{E}(X_{t+1} - AX_t - HU_t)^* Q^{-1} F_{X_{t+1}-AX_t-HU_t}\} , \\ &+ \begin{pmatrix} 0 & 4 & 0 & 0 \end{pmatrix}^* \begin{pmatrix} 0 & 4 & 0 & 0 \end{pmatrix} . \end{aligned}$$

Let us notice that we can show using F definition given by eq.(34) and the statistical property of X_{t+1} (i.e. X_{t+1} given X_t is $\mathcal{N}\left(AX_t + HU_t, Q\right)$ distributed) that the $\mathcal{C}$ definition given by eq.(110) is equivalent to the $\mathcal{C}$ definition given by eq.(101). Now, using again the statistical property of X_{t+1}, we obtain:

$$D_t^{22} = \mathbb{E}\left\{F^*_{AX_t+HU_t} Q^{-1}(X_{t+1} - AX_t - HU_t)(X_{t+1} - AX_t - HU_t)^* Q^{-1} F_{AX_t+HU_t}\right\} + \mathcal{C} . \tag{110}$$

To end the proof, the linearity of the operator F and the equality $HU_t = \mathbb{E}X_{t+1} - X_t$ allow us to infer eq.(101) from eq.(110).

Appendix E1: Proof of proposition 6.1

The proof of proposition 6.1 is based on the properties of F_{X_t} and G_{X_t} investigated in Appendix C. Developing Γ_t^{11} given by eq.(36) and using the linearity of operator F, we obtain

$$\Gamma_t^{11} = \Omega^{11} + \begin{pmatrix} \mathbb{E}\left\{F^*_{(AX_{t-1}+HU_{t-1})} A^* Q^{-1} A F_{(AX_{t-1}+HU_{t-1})}\right\} \\ \mathbb{E}\left\{F^*_{(AX_{t-1}+HU_{t-1})} A^* Q^{-1} A G_{(AX_{t-1}+HU_{t-1})}\right\} \\ \mathbb{E}\left\{G^*_{(AX_{t-1}+HU_{t-1})} A^* Q^{-1} A F_{(AX_{t-1}+HU_{t-1})}\right\} \\ \mathbb{E}\left\{G^*_{(AX_{t-1}+HU_{t-1})} A^* Q^{-1} A G_{(AX_{t-1}+HU_{t-1})}\right\} \end{pmatrix}$$

where

$$\Omega^{11} = \begin{pmatrix} \mathbb{E}\left\{F^*_{(X_t-AX_{t-1}-HU_{t-1})} A^* Q^{-1} A F_{(X_t-AX_{t-1}-HU_{t-1})}\right\} \\ \mathbb{E}\left\{F^*_{(X_t-AX_{t-1}-HU_{t-1})} A^* Q^{-1} A G_{(X_t-AX_{t-1}-HU_{t-1})}\right\} \\ \mathbb{E}\left\{G^*_{(X_t-AX_{t-1}-HU_{t-1})} A^* Q^{-1} A F_{(X_t-AX_{t-1}-HU_{t-1})}\right\} \\ \mathbb{E}\left\{G^*_{(X_t-AX_{t-1}-HU_{t-1})} A^* Q^{-1} A G_{(X_t-AX_{t-1}-HU_{t-1})}\right\} \end{pmatrix} . \tag{111}$$

Now remarking that $HU_{t-1} = \mathbb{E}X_t - A\mathbb{E}X_{t-1}$ and using linear property of operator F, we obtain:

$$\Gamma_t^{11} = \Omega^{11} + \begin{pmatrix} \mathbb{E}\left\{F^*_{AX_{t-1}} A^* Q^{-1} A F_{AX_{t-1}}\right\} \\ \mathbb{E}\left\{F^*_{AX_{t-1}} A^* Q^{-1} A G_{AX_{t-1}}\right\} \\ \mathbb{E}\left\{G^*_{AX_{t-1}} A^* Q^{-1} A F_{AX_{t-1}}\right\} \\ \mathbb{E}\left\{G^*_{AX_{t-1}} A^* Q^{-1} A G_{AX_{t-1}}\right\} \end{pmatrix} + \boldsymbol{\Lambda^{11}_{t-1}} \tag{112}$$

where $\boldsymbol{\Lambda^{11}_{t-1}}$ is defined by eq.(37). According to Appendix C, $F_{AX_{t-1}} = F_{X_{t-1}} + \delta_t G_{X_{t-1}}$ and $G_{AX_{t-1}} = G_{X_{t-1}}$, so that:

$$\Gamma_t^{11} = \Omega^{11} + \Psi\ \Gamma_{t-1}^{11} + \boldsymbol{\Lambda^{11}_{t-1}} \tag{113}$$

where Ψ is defined by eq.(37). It remains to show that Ω^{11} has a more simple formula using the following lemma:

Lemma 4 *For X and Y two state vectors, let us define*

$$\Theta = \begin{pmatrix} \mathbb{E}\left(F_X^*(\Sigma \otimes Id_{2\times 2}) F_Y\right) \\ \mathbb{E}\left(F_X^*(\Sigma \otimes Id_{2\times 2}) G_Y\right) \\ \mathbb{E}\left(G_X^*(\Sigma \otimes Id_{2\times 2}) F_Y\right) \\ \mathbb{E}\left(G_X^*(\Sigma \otimes Id_{2\times 2}) G_Y\right) \end{pmatrix} \tag{114}$$

where operators F and G are defined by eq.(34). Then

$$\Theta = \begin{pmatrix} \Sigma \otimes \mathbb{E}\left\{R_X^* R_Y\right\} + \Sigma_{\nwarrow} \otimes \mathbb{E}\left\{V_X^* V_Y\right\} + \Sigma_{\downarrow} \otimes \mathbb{E}\left\{V_X^* R_Y\right\} + \Sigma_{\uparrow} \otimes \mathbb{E}\left\{R_X^* V_Y\right\} \\ \Sigma_{\uparrow} \otimes \mathbb{E}\left\{V_X^* V_Y\right\} + \Sigma \otimes \mathbb{E}\left\{R_X^* V_Y\right\} \\ \Sigma_{\leftarrow} \otimes \mathbb{E}\left\{V_X^* V_Y\right\} + \Sigma \otimes \mathbb{E}\left\{V_X^* R_Y\right\} \\ \Sigma \otimes \mathbb{E}\left\{V_X^* V_Y\right\} \end{pmatrix}$$

where:

$$\Sigma_{\uparrow} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Sigma \,, \Sigma_{\leftarrow} = \Sigma \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \,, \Sigma_{\nwarrow} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Sigma \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} . \tag{115}$$

Proof of lemma 4 We just have to rewrite eq.(114) using F and G formulas given by eq.(34). We prove lemma 4 using direct tensor product properties. □

To end the proof, Lemma 4 is applied with:

$$\begin{cases} X = X_t - AX_{t-1} - HU_{t-1} \,, \\ Y = X_t - AX_{t-1} - HU_{t-1} \,, \\ \Sigma \otimes Id_{2\times 2} = A^* Q^{-1} A \,. \end{cases} \tag{116}$$

Then, using the statistical property of X_t i.e X_t given X_{t-1} is $\mathcal{N}\left(AX_{t-1} + HU_{t-1}, Q\right)$-distributed, we obtain:

$$\begin{cases} \mathbb{E}\left\{R_X^* R_Y\right\} = 2\alpha_3 Id_{2\times 2} \;\;, \mathbb{E}\left\{R_X^* V_Y\right\} = 2\alpha_2 Id_{2\times 2} \,, \\ \mathbb{E}\left\{V_X^* R_Y\right\} = 2\alpha_2 Id_{2\times 2} \;\;, \mathbb{E}\left\{V_X^* V_Y\right\} = 2\alpha_1 Id_{2\times 2} \end{cases} \tag{117}$$

so that Ω^{11} is given by eq.(37).

Appendix E2: Proof of proposition 6.2

Using the same approach as in proposition 6.1 proof, we have:

$$\Gamma_t^{12} = \Omega^{12} + \Psi \; \Gamma^{12}(t-1) + \boldsymbol{\Lambda_{t-1}^{12}}$$

where Ψ and $\boldsymbol{\Lambda_{t-1}^{12}}$ given by eq.(37) and eq.(45) and

$$\Omega^{12} = \begin{pmatrix} \mathbb{E}\left\{F^*_{(X_t - AX_{t-1} - HU_{t-1})} A^* Q^{-1} F_{A(X_t - AX_{t-1} - HU_{t-1})}\right\} \\ \mathbb{E}\left\{F^*_{(X_t - AX_{t-1} - HU_{t-1})} A^* Q^{-1} G_{A(X_t - AX_{t-1} - HU_{t-1})}\right\} \\ \mathbb{E}\left\{G^*_{(X_t - AX_{t-1} - HU_{t-1})} A^* Q^{-1} F_{A(X_t - AX_{t-1} - HU_{t-1})}\right\} \\ \mathbb{E}\left\{G^*_{(X_t - AX_{t-1} - HU_{t-1})} A^* Q^{-1} G_{A(X_t - AX_{t-1} - HU_{t-1})}\right\} \end{pmatrix} . \tag{118}$$

Lemma 4 is again the key for simplifying Ω^{12}, and is used with:

$$\begin{cases} X = X_t - AX_{t-1} - HU_{t-1} \,, \\ Y = A(X_t - AX_{t-1} - HU_{t-1}) \,, \\ \Sigma \otimes Id_{2\times 2} = A^* Q^{-1} \,. \end{cases} \tag{119}$$

Now, using the statistical property of X_t i.e X_t given X_{t-1} is $\mathcal{N}\left(AX_{t-1} + HU_{t-1}, Q\right)$-distributed, we obtain for Ω^{12} the simple formula given by eq.(41). □

Appendix E3: Proof of proposition 6.3

The proof again mimics that of proposition 6.1. Thus, we first obtain:

$$\Gamma_t^{22} = \Omega^{22} + \Psi\ \Gamma_{t-1}^{22} + \boldsymbol{\Lambda_t^{22}}\ ,$$

where Ψ and $\boldsymbol{\Lambda_{t-1}^{22}}$ given by eq.(37) and eq.(45), and:

$$\Omega^{22} = \begin{pmatrix} \mathbb{E}\left\{F^*_{A(X_t-AX_{t-1}-HU_{t-1})}Q^{-1}F_{A(X_t-AX_{t-1}-HU_{t-1})}\right\} \\ \mathbb{E}\left\{F^*_{A(X_t-AX_{t-1}-HU_{t-1})}Q^{-1}G_{A(X_t-AX_{t-1}-HU_{t-1})}\right\} \\ \mathbb{E}\left\{G^*_{A(X_t-AX_{t-1}-HU_{t-1})}Q^{-1}F_{A(X_t-AX_{t-1}-HU_{t-1})}\right\} \\ \mathbb{E}\left\{G^*_{A(X_t-AX_{t-1}-HU_{t-1})}Q^{-1}G_{A(X_t-AX_{t-1}-HU_{t-1})}\right\} \end{pmatrix} . \tag{120}$$

We prove now that Ω^{22} has a more simple formula using lemma 4 with:

$$\begin{cases} X = A(X_t - AX_{t-1} - HU_{t-1})\ , \\ Y = A(X_t - AX_{t-1} - HU_{t-1})\ , \\ \Sigma \otimes Id_{2\times 2} = Q^{-1}\ . \end{cases} \tag{121}$$

Then, using the statistical property of X_t i.e X_t given X_{t-1} is $\mathcal{N}\left(AX_{t-1} + HU_{t-1}, Q\right)$-distributed, we obtain for Ω^{22} the formula given by eq.(45). $\Box$

Appendix F: Initialization

Initialization of Γ_0^{11}

We apply lemma 4 with: $\begin{cases} X = X_0\ , \\ Y = X_0\ , \\ \Sigma \otimes Id_{2\times 2} = A^* Q^{-1} A\ . \end{cases}$

Initialization of Γ_0^{12}

We apply lemma 4 with: $\begin{cases} X = X_0\ , \\ Y = AX_0\ , \\ \Sigma \otimes Id_{2\times 2} = A^* Q^{-1}\ . \end{cases}$

Initialization of Γ_0^{22}

We apply lemma 4 with: $\begin{cases} X = AX_0\ , \\ Y = AX_0\ , \\ \Sigma \otimes Id_{2\times 2} = Q^{-1}\ . \end{cases}$

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- Initialization of J_0^{-1} using the initial error covariance matrix given by eq.(56).
- Initialization of Γ_0^{11}, Γ_0^{12}, Γ_0^{22} and Γ_0^{33} using eqs.(38,42,46,54).
- J_1^{-1} is calculated using only step 2 and 3 with $t = 0$.
- For $t = 1$ to T

  1. Calculation of auxiliary matrices Γ_t^{11}, Γ_t^{12}, Γ_t^{22} and Γ_t^{33}

     (a) Calculate Λ_{t-1}^{11}, Λ_{t-1}^{12}, Λ_{t-1}^{22} and Λ_{t-1}^{33} using eqs.(37,41,45) if observer maneuvers (else these terms are null).

     (b) $$\begin{cases} \Gamma_t^{11} = \Omega^{11} + \Psi \ \Gamma_{t-1}^{11} \ (+ \boldsymbol{\Lambda_{t-1}^{11}}) \ , \\ \Gamma_t^{12} = \Omega^{12} + \Psi \ \Gamma_{t-1}^{12} \ (+ \boldsymbol{\Lambda_{t-1}^{12}}) \ , \\ \Gamma_t^{22} = \Omega^{22} + \Psi \ \Gamma_{t-1}^{22} \ (+ \boldsymbol{\Lambda_{t-1}^{22}}) \ , \\ \Gamma_t^{33} = \Omega^{33} + \Phi \ \Gamma_{t-1}^{33} \ (+ \boldsymbol{\Lambda_{t-1}^{33}}) \ . \end{cases}$$

     Remark : Ω^{11}, Ω^{12}, Ω^{22} *and* Ω^{33} *are given by eqs.(37,41,45).* Ψ *and* Φ *are given by eq.(37) and eq.(53).*

  2. Calculation of D_t^{11}, D_t^{12}, D_t^{22}, D_t^{33} and $\mathcal{D}_t^{33}$

     (a) If observer maneuvers, computeΥ_t^{12} and Υ_t^{22} using eq.(39) and eq.(43) (else these terms are null).

     (b) $$\begin{cases} D_t^{11} = \begin{bmatrix} Id_{n_y \times n_y} & 0_{n_y \times 3n_y} \end{bmatrix} \Gamma_t^{11} \ , \\ D_t^{12} = - \begin{bmatrix} Id_{n_y \times n_y} & 0_{n_y \times 3n_y} \end{bmatrix} \Gamma_t^{12} \ (- \boldsymbol{\Upsilon_t^{12}}) \ , \\ D_t^{22} = \begin{bmatrix} Id_{n_y \times n_y} & 0_{n_y \times 3n_y} \end{bmatrix} \Gamma_t^{22} + \mathcal{C} \ (+ \boldsymbol{\Upsilon_t^{22}}) \ . \end{cases}$$

     Remark : $\mathcal{C}$ *is given by eq.(43) and* D_t^{21} *is given by the relation* $D_t^{21} = \left(D_t^{12}\right)^*$.

     (c) Calculation of D_t^{33} using eq.(47) *(passive meas.)*.

     (d) Calculation of $\mathcal{D}_t^{33}$ is given by eq.(51) *(active meas.* $+$ *passive meas.)*

     Remark : $\mathbb{E}r_{t+1}^2$ *is calculated by using eq.(53) and* Γ_t^{33}.

  3. Calculate J_{t+1}^{-1} using Tichavský's formula:

$$J_{t+1}^{-1} = \begin{cases} \left(D_t^{22} + D_t^{33} - D_t^{21} \left(J_t^{-1} + D_t^{11}\right)^{-1} D_t^{12}\right)^{-1} \textit{(passive meas.)} \ , \\ \left(D_t^{22} + \mathcal{D}_t^{33} - D_t^{21} \left(J_t^{-1} + D_t^{11}\right)^{-1} D_t^{12}\right)^{-1} \textit{(active meas. + passive meas.)} \end{cases}$$

Figure 4: Closed-form calculation of the PCRB for active measurements scheduling.

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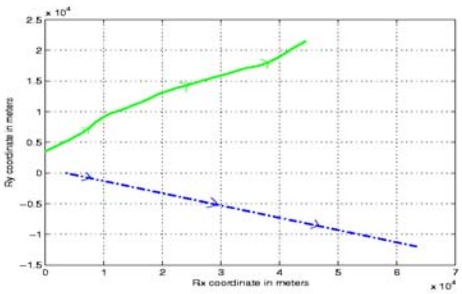
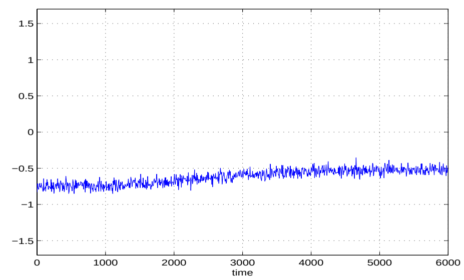
	Scenario 1	**Scenario 2**
$duration$	6000 s	6000 s
$r_x^{obs}(0)$	$3,5$ km	$3,5$km
$r_y^{obs}(0)$	0 km	0 km
$v_x^{obs}(0)$	10 ms^{-1}	10 ms^{-1}
$v_y^{obs}(0)$	-2 ms^{-1}	-2 ms^{-1}
$r_x^{cib}(0)$	0 km	0 km
$r_y^{cib}(0)$	$3,5$ km	$3,5$ km
$v_x^{cib}(0)$	6 ms^{-1}	6 ms^{-1}
$v_y^{cib}(0)$	3 ms^{-1}	3 ms^{-1}
δ_t	6 s	6 s
σ_{max}	0.05 ms^{-1}	0.05 ms^{-1}
σ_β	0.05 rad (about 3 deg.)	0.05 rad (about 3 deg.)
σ_{r_0}	2 km	2 km
σ_{v_0}	1 ms^{-1}	1 ms^{-1}
σ_{β_0}	0.05 rad (about 3 deg.)	0.05 rad (about 3 deg.)

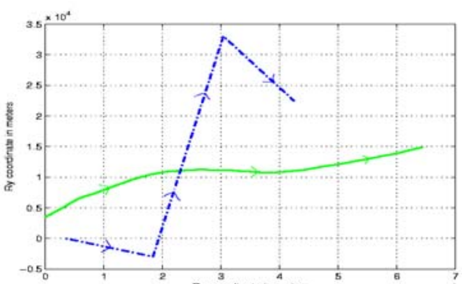
Table 3: Scenarios Constants



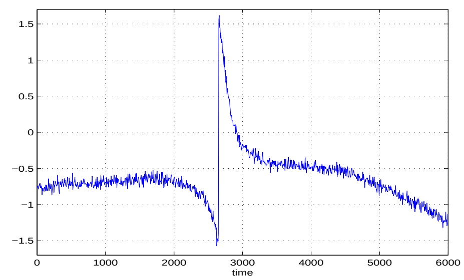
(a1)



(b1)

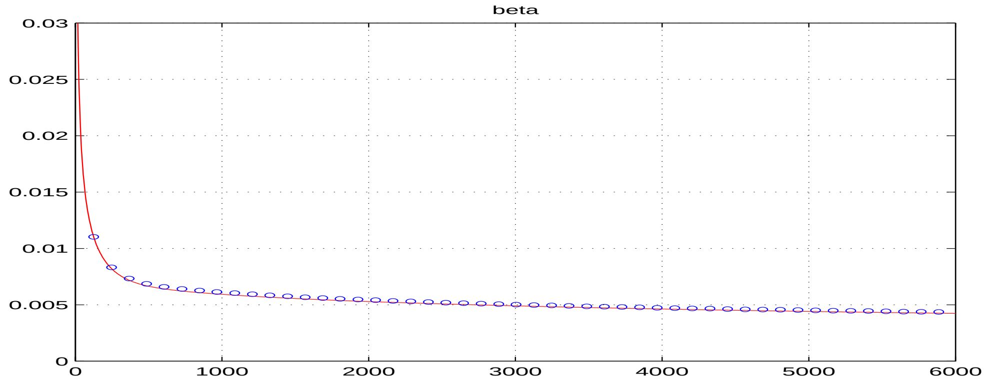


(a2)

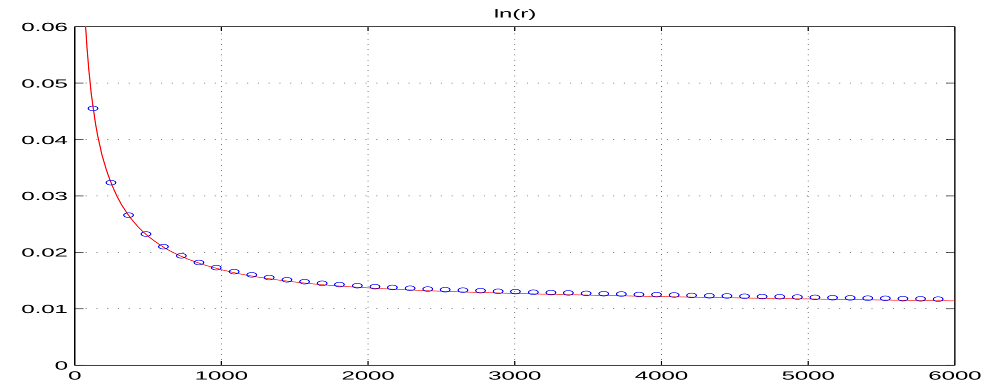


(b2)

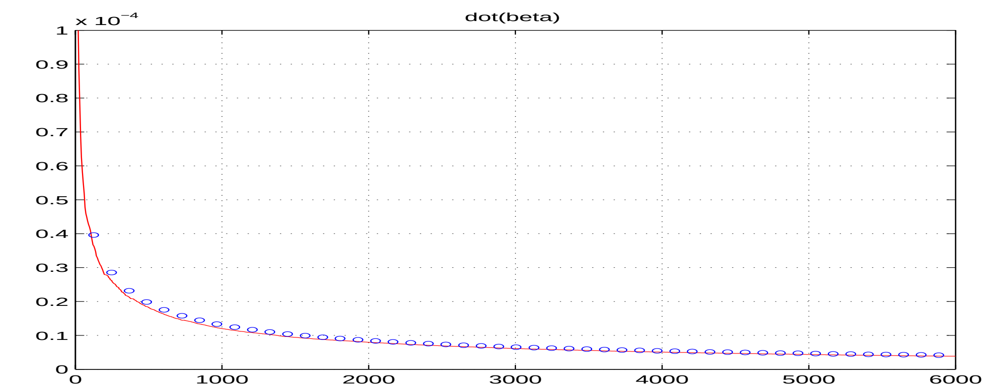
Figure 5: **Scenario 1**:*(a1) example of trajectory of the target (solid line) and the observer (dashed line) (b1) set of bearings measurements* **Scenario 2**: *(a2) example of trajectory of the target (solid line) and the observer (dashed) (b2) set of bearings measurements*



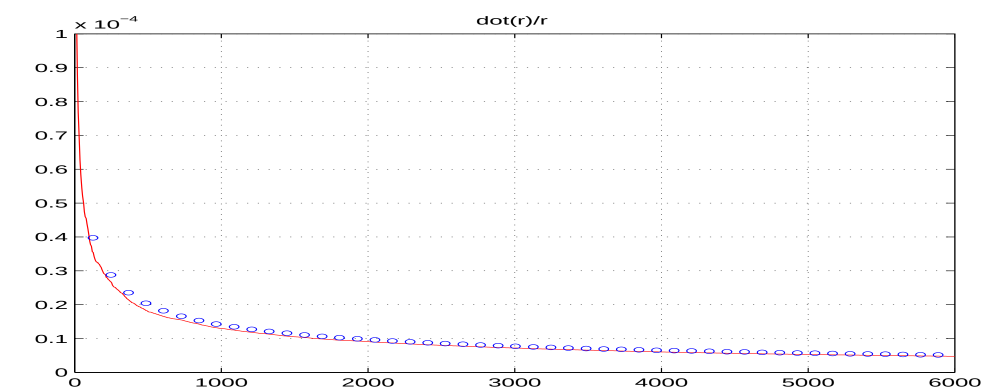
(a)



(b)

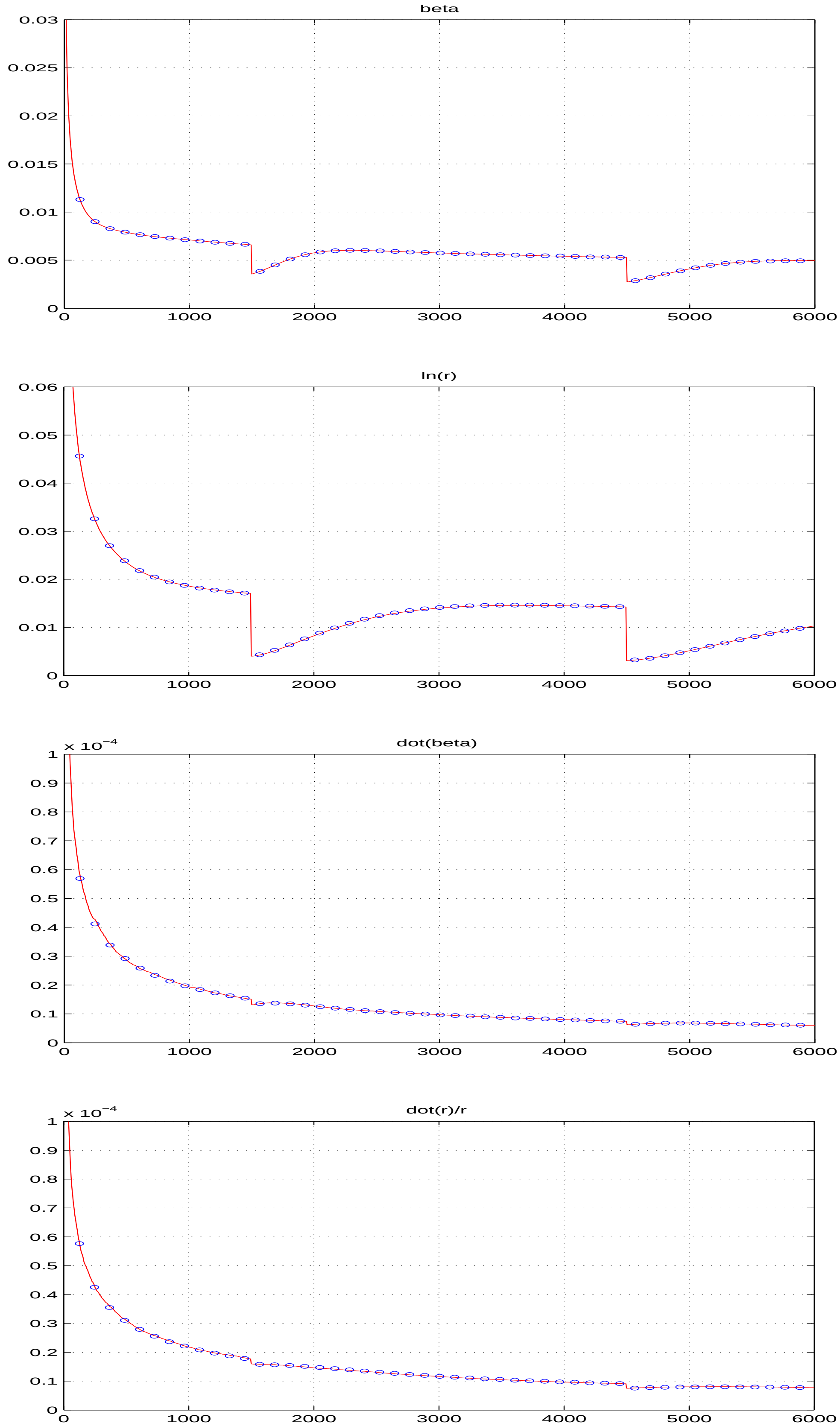


(c)



(d)

Figure 6: *PCRB for (a)* β_t*, (b)* $\ln r_t$*,(c)* $\dot{\beta}_t$*, (d)* $\frac{\dot{r}_t}{r_t}$ *with* **scenario 1***: closed-form PCRB (dashed line) versus approximated PCRB (solid line)*



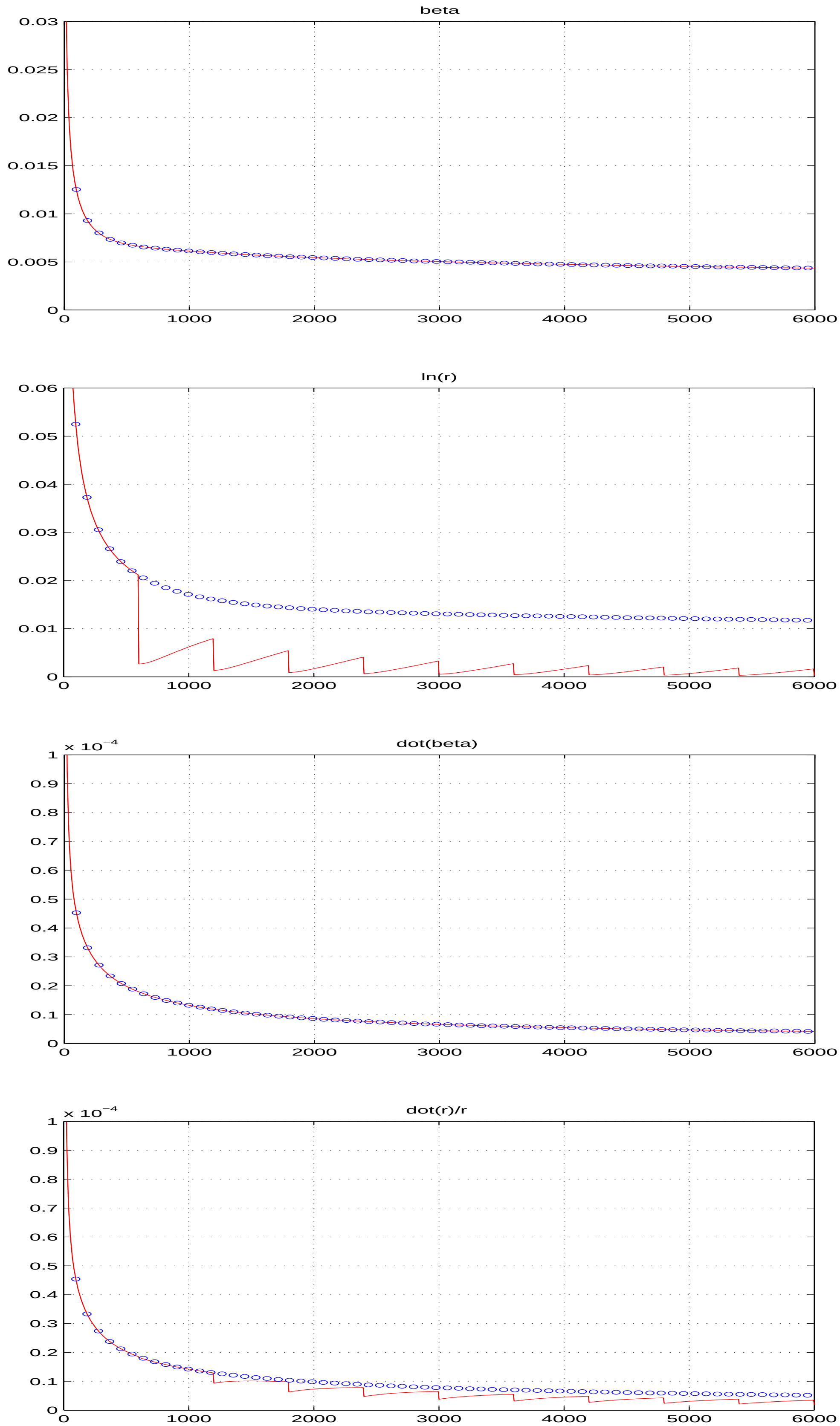
(a)

(b)

(c)

(d)

Figure 7: *PCRB for (a)* β_t*, (b)* $\ln r_t$*,(c)* $\dot{\beta}_t$*, (d)* $\frac{\dot{r}_t}{r_t}$ *with* **scenario 2***: closed-form PCRB (dashed line) versus approximated PCRB (solid line)*



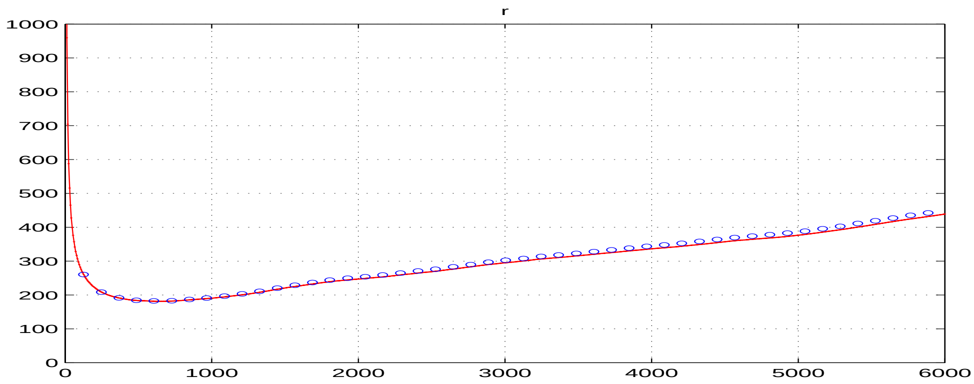
(a)

(b)

(c)

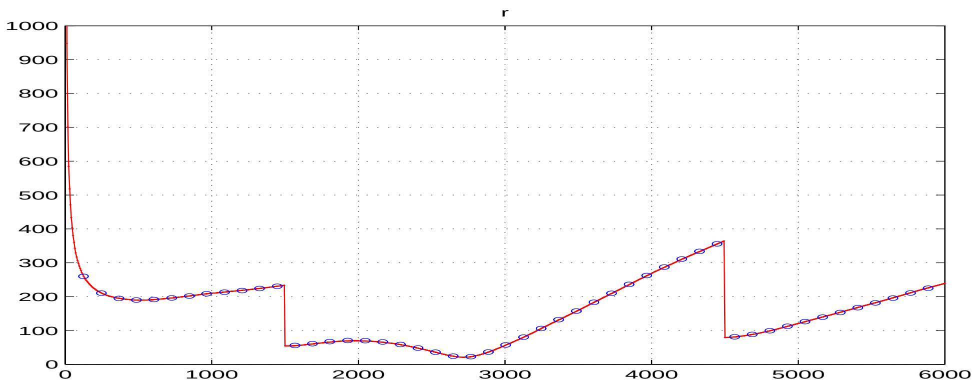
(d)

Figure 8: *Closed-form PCRB with range measurements each 80 seconds (solid line) versus closed-form PCRB (dashed line). (a)* β_t*, (b)* $\ln r_t$*,(c)* $\dot{\beta}_t$*, (d)* $\frac{\dot{r}_t}{r_t}$



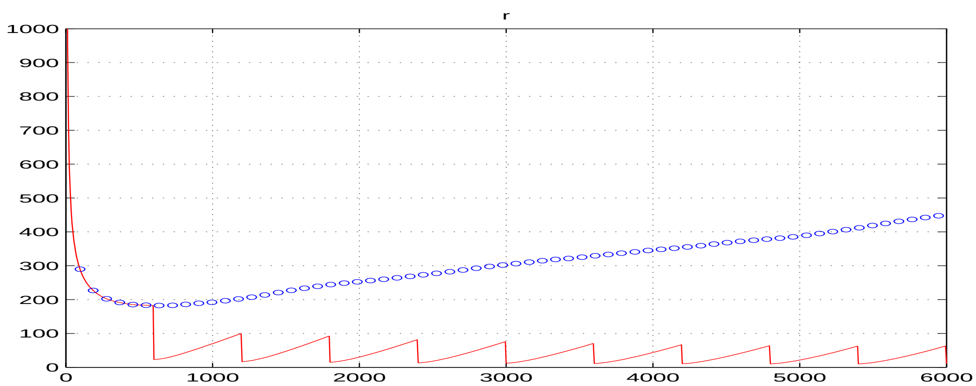
(b)

Figure 9: *PCRB for r_t with* **scenario 1***: closed-form PCRB (dashed line) versus approximated PCRB (solid line)*



(b)

Figure 10: *PCRB for* **scenario 2***: closed-form PCRB for r_t (dashed line) versus approximated PCRB for r_t (solid line)*



(a)

Figure 11: *Closed-form PCRB with range measurements each 80 seconds for r_t (solid line) versus closed-form PCRB for r_t (dashed line)*