CONSTRAINED BEARINGS-ONLY TARGET MOTION ANALYSIS VIA MONTE CARLO MARKOV CHAINS

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Abstract. The aim of this paper is to develop methods for estimating the range of a moving target from bearings-only observations and for weakly observable scenarios, by including constraints about kinematic parameters. It is assumed that the target motion is rectilinear and uniform which leads us to restrict to batch algorithms. Poor observability is generally resulting from a (very) limited amplitude of the observer maneuvers. In these situations, classical methods perform very poorly (especially for range estimation) and including constraints is uneasy and not reliable. We consider here methods for determining a confidence interval for the range based on the Highest Probability Density (HPD) Intervals, by taking into account prior informations about the kinematics parameters. Two types of prior constraints will be considered : first the kinematics parameters are supposed belonging to intervals, without supposing a particular distribution, and second the target trajectory is supposed staying in a known area. The determination of an HPD interval requires a Markov Chain Monte Carlo (MCMC) sampling. The HPD interval method is illustrated by simulation results.

INTRODUCTION

The problem of performing target motion analysis (TMA) using noisy bearings-only (BOT[1]) measurements derived from a single moving observer is addressed. For BOT TMA, these measurements consist of line-of-sight angles. It is perhaps in the passive sonar environment that BOT TMA is most familiar; though it is a challenging problem for other contexts like surface and airborne ASW (Anti Submarine

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[1] BOT means exactly Bearings-Only Tracking

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Warfare) (I.R. sensors) or passive surveillance via Electronic Support Measurement (ESM). TMA is instrumental for many systems like surveillance systems, for evaluating the threats, for performing data association, correlation processing (e.g. track-to-track association), for sensor management, etc. Finally, let us stress that the real difficulty of TMA is due to the passive nature of the observation. Particular importance is attached to range estimation. Also, with target range at hand, course and speed may be estimated by finding position at two distinct times. This clearly motivates this article whose major objective is range estimation for "difficult" scenarios.

The BOT TMA problem is not new and even traces back to the work of C.F. Gauss. Since that time various approaches have been developed. Basically, these methods are able to include simple kinematic constraints. However, they are not designed for estimating Highest Probability Density (HPD) Intervals [3] or dealing with general constraints or multi-modality which are precisely the objectives of this work. This paper is organized as follows. Problem formulation is given in the Section 2; followed by a presentation of the Monte Carlo Markov Chain (MCMC) for constrained estimation in Section 3 (Metropolis and Hit-and-Run algorithms). In this section, accent is put on the application of these general algorithms to the BOT TMA context. Section 4 deals with the HPD estimation.

PROBLEM FORMULATION

In the general two-dimensional problem, the angles of arrival are viewed from an observer confined to a plane which includes the target. Here, we consider the problem of estimating the parameters of the target trajectory under the assumption of rectilinear and uniform (target) motion from a history of noisy passive bearing measurements viewed by a moving observer. The scenario is depicted in fig 1 below :

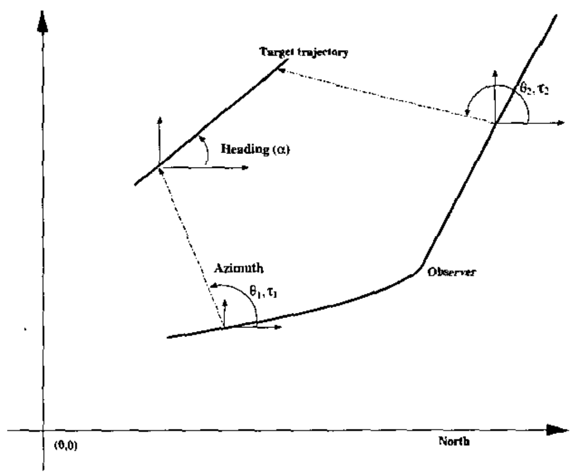

Figure 1: The BOT TMA scenario

The target, whose position is initially defined by its initial range r and initial azimuth θ, moves with a constant velocity v holding heading α relating to North, defining the state vector $\mathbf{x} = (\mathbf{r}, \theta, \mathbf{v}, \alpha)$.

Classically, the *batch* BOT TMA problem is solved by a Maximum Likelihood Estimator (MLE) which consists in finding the state vector which maximizes the likelihood given the observed bearings, i.e. :

$$(\hat{\mathbf{x}}) = \underset{\mathbf{x}}{\text{argmax}}\ L_{(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_N)}(\mathbf{x}) \ = \underset{\mathbf{x}}{\text{argmax}}\ f_{(\mathbf{x})}(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_N) \qquad (1)$$

where $(\hat{\theta}_1, \cdots, \hat{\theta}_N)$ are N (noisy) bearing observations at time periods $\tau_1, \cdots, \tau_N$. The bearing measurements $\hat{\theta}_i$ are the exact bearings corrupted by a sequence of independent and identically (normally) distributed (i.i.d.) noises. Thus, the mean[2] of $\hat{\theta}_i$ is θ_i, while its variance is σ^2.

The likelihood of a state vector given the observed bearings is just the density function of the given observed bearings seen as a function of the state vector e.g. the density of the measured observations under the hypothesis that (r, θ, v, α) is the true state vector.

Since there does not exist an explicit method for solving eq. (1), iterative solutions are generally used for solving this non-linear regression problem. Best known methods are Gradient [2] and Modified Instrumental Variable [2],[1] (MIV) ones. So, a vast literature has been devoted to this subject. The aim of this article is to replace the search for the *maximum* of the likelihood functional by *the search for an interval*. The main idea developed here consists in incorporating prior informations e.g. the parameters we want to estimate belong to given intervals. The (r, θ, v, α) parametrization is particularly well adapted for including operational constraints and will be in use **throughout** this paper. Prior information has to be used in a Bayesian context. Given observations, the posterior distribution of the state vector is related to prior information and likelihood function via the Bayes's relation. For determining an interval including the "greatest" values of r, it is necessary to access to the marginal posterior distribution $\pi(r \mid (\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_N))$ of the posterior density state vector $\pi((r, \theta, v, \alpha) \mid (\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_N))$. The whole difficulty lies in the fact that the marginal posterior density $\pi(r \mid (\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_N))$ is defined by the integral of the posterior density $\pi((r, \theta, v, \alpha) \mid (\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_N))$ with regard to θ, v and α and **not under closed-form.**

We propose a method of providing an interval for r based on *Highest Probability Density* (HPD) interval. For a given probability content, say $1-\alpha$, the HPD method allows to obtain a confidence interval for the range r with a probability content of $1-\alpha$ by using a sample from the marginal posterior density $\pi\left(r \mid (\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_N)\right)$. This sample is obtained by generating a *Markov Chain Monte Carlo* (MCMC) sample from the marginal posterior distribution, sample that is itself obtained by generating a MCMC sample from the posterior distribution $\pi\left((r, \theta, v, \alpha) \mid (\hat{\theta}_1, \cdots, \hat{\theta}_N)\right)$.

[2]Note that the $\left\{\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_N\right\}$ is not identically distributed since its mean is time-varying.

THE MCMC METROPOLIS-HASTINGS AND MCMC HIT-AND-RUN ALGORITHMS

The aim of this section will be to provide a particular setting for the application of the Markov Chain Monte Carlo (MCMC) methods [5] for constrained estimation [8]. In this article, two types of constraints will be considered. First, we consider that each of the state variables (r, θ, v, α) belongs to a given interval, without assuming a particular distribution in these intervals. These constraints will be taken into account in *the Metropolis-Hastings Algorithm*. Second, we consider that all the target trajectory belongs to a known area (implying complex constraints on (r, θ, v, α)). This second type of constraints will be used throughout *the Hit-and-Run* Monte Carlo sampling. For both type of constraints, MCMC method will be the common workhorse.

The MCMC Metropolis-Hastings algorithm

Due to operational considerations, it is natural to assume minimal and maximal bounds about range r (namely $r_{\min}$ and $r_{\max}$). In order to reduce the search width around θ, we shall suppose that the first bearing measured $\hat{\theta}_1$ is not too far from the right bearing θ_1. So, the state variable θ will be supposed to be in the interval $\left[\hat{\theta}_1 - m\,\sigma, \hat{\theta}_1 + m\,\sigma\right]$ where σ is the standard deviation of the measurements. Typically, we choose $m = 5$. Indeed, the first measured bearing $\hat{\theta}_1$ allows to give a good idea of the true bearing θ_1 since the interval $\left[\hat{\theta}_1 - 5\sigma, \hat{\theta}_1 + 5\sigma\right]$ contains the true value θ_1 with a probability of 99.9999%. In the absence of any strong prior about target parameters, uniform priors are the more convenient:

$$\left\{ \begin{array}{lllllll} \pi(r) & \sim & U[r_{\min}, r_{\max}] & , & \pi(\theta) & \sim & U\left[\hat{\theta}_1 - m\,\sigma, \hat{\theta}_1 + m\sigma\right] , \\ \pi(v) & \sim & U[v_{\min}, v_{\max}] & , & \pi(\alpha) & \sim & U[-\pi, +\pi] . \end{array} \right. \tag{2}$$

Then, we consider the following distribution of the state vector:

$$\begin{aligned} \pi(r, \theta, v, \alpha) &= \pi(r) \times \pi(\theta) \times \pi(v) \times \pi(\alpha) , \qquad (3)\\ &= C\, \mathbb{1}_{[r_{\min}, r_{\max}]}(r) \times \mathbb{1}_{[\theta_{\min}, \theta_{\max}]}(\theta) \times \mathbb{1}_{[v_{\min}, v_{\max}]}(v) , \\ \text{where } C &= \frac{1}{4m\,\pi\,\sigma\ (r_{\max} - r_{\min})(v_{\max} - v_{\min})} , \end{aligned}$$

while $\mathbb{1}$ is the indicator function. The following step is to define the transition kernel to move from a state $(r, \theta, v, \alpha)^{(t)}$ to a state $(r', \theta', v', \alpha')$, so as to ease the simulation process. To that aim, we consider the following transition kernel:

$$q\left((r', \theta', v', \alpha') \mid (r, \theta, v, \alpha)^{(t)}\right) = q\left(r' \mid r^{(t)}\right) \times \cdots \times q\left(\alpha' \mid \alpha^{(t)}\right) . \tag{4}$$

In order to check the symmetry condition (i.e. $q(y|x) = q(x|y)$), conditional distributions are chosen uniform. A factor κ (e.g. 20) is introduced, so as to reduce the interval length, thus avoiding that new values $(r', \theta', v', \alpha')$ be drawn out of the posterior distribution support and be rejected. Which lead us to consider the following densities:

$$\begin{array}{lcl} q\left(r' \mid r^{(t)}\right) & \sim & U\left[r^{(t)} - \left(\frac{r_{\max}-r_{\min}}{\kappa}\right), r^{(t)} + \left(\frac{r_{\max}-r_{\min}}{\kappa}\right)\right] \\ q\left(\theta' \mid \theta^{(t)}\right) & \sim & U\left[\theta^{(t)} - \frac{10\,\sigma}{\kappa}, \theta^{(t)} + \frac{10\,\sigma}{\kappa}\right] \\ q\left(v' \mid v^{(t)}\right) & \sim & U\left[v^{(t)} - \left(\frac{v_{\max}-v_{\min}}{\kappa}\right), v^{(t)} + \left(\frac{v_{\max}-v_{\min}}{\kappa}\right)\right] \\ q\left(\alpha' \mid \alpha^{(t)}\right) & \sim & U\left[\alpha^{(t)} - \frac{2\pi}{\kappa}, \alpha^{(t)} + \frac{2\pi}{\kappa}\right] . \end{array} \tag{5}$$

Let D be the set of observed bearings ; i.e. $D \stackrel{\text{def}}{=} (\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_N)$. Now the next step is the computation of $\rho\left((r,\theta,v,\alpha)^{(t)}, (r',\theta',v',\alpha')\right)$ which needs the computation of the ratio of the posterior densities:

$$\begin{aligned} \frac{\pi\left((r',\theta',v',\alpha') \mid D\right)}{\pi\left((r,\theta,v,\alpha)^{(t)} \mid D\right)} &= \frac{L_{(D)}\left(r',\theta',v',\alpha'\right) \times \pi\left(r',\theta',v',\alpha'\right)}{L_{(D)}\left(r,\theta,v,\alpha\right)^{(t)} \times \pi\left(r,\theta,v,\alpha\right)^{(t)}} , \\ &= \frac{L_{(D)}\left(r',\theta',v',\alpha'\right)}{L_{(D)}\left(r,\theta,v,\alpha\right)^{(t)}} \times \mathbb{1}_{[r_{\min},r_{\max}]}\left(r'\right) \times \mathbb{1}_{[\theta_{\min},\theta_{\max}]}\left(\theta'\right) \times \mathbb{1}_{[v_{\min},v_{\max}]}\left(v'\right) , \end{aligned} \tag{6}$$

itself requiring the calculation of the measurement likelihood. Assuming that bearing measurements at various time-periods are independent, we have:

$$L_{(\hat{\theta}_1,\ldots,\hat{\theta}_N)}\left(r,\theta,v,\alpha\right) = \prod_{i=1}^{N} L_{\hat{\theta}_i}\left(r,\theta,v,\alpha\right) . \tag{7}$$

Let $\theta_i \mid (r,\theta,v,\alpha)$ be the exact bearing at the time period i, conditionally to the state vector (r,θ,v,α); elementary geometrical considerations yield[3]:

$$\theta_i \mid (r,\theta,v,\alpha) = 2\arctan\left(\frac{\Delta y_i}{\Delta x_i + \sqrt{\Delta x_i^2 + \Delta y_i^2}}\right) , \tag{8}$$

where Δx_i and ΔY_i are the relative positions of the target and the observer. Assuming that the observations are normally distributed; i.e. $\hat{\theta}_i = \theta_i + \epsilon_i$ with $\epsilon_i \sim N\left(0,\sigma^2\right)$, we have:

$$\begin{aligned} \frac{\pi\left((r',\theta',v',\alpha') \mid D\right)}{\pi\left((r,\theta,v,\alpha)^{(t)} \mid D\right)} &= \prod_{i=1}^{N}\left\{\frac{\exp\left[-\frac{1}{2\sigma^2}\left(\hat{\theta}_i - \theta_i \mid (r',\theta',v',\alpha')\right)^2\right]}{\exp\left[-\frac{1}{2\sigma^2}\left(\hat{\theta}_i - \theta_i \mid (r,\theta,v,\alpha)^{(t)}\right)^2\right]}\right\} \\ &\quad \times \mathbb{1}_{[r_{\min},\, r_{\max}]}\left(r'\right) \times \mathbb{1}_{[\theta_{\min},\, \theta_{\max}]}\left(\theta'\right) \times \mathbb{1}_{[v_{\min},\, v_{\max}]}\left(v'\right) . \end{aligned} \tag{9}$$

[3] Note that the usual expression is $\theta_i \mid (r,\theta,v,\alpha) = \arctan\left(\frac{\Delta Y_i}{\Delta X_i}\right)$, eq. 8 is used to avoid ambiguities of the arctan function.

At present, we have seen the way to generate a MCMC sample from the posterior distribution $\pi\left((r,\theta,v,\alpha) \mid (\hat{\theta}_1,\hat{\theta}_2,\cdots,\hat{\theta}_N)\right)$ by taking into account constraints given by intervals, where priors are uniformly distributed. In the following paragraph, we shall consider again MCMC sampling but, this time, with much more general constraints.

The MCMC Hit-and-Run algorithm

Actually, operational considerations usually leads us to consider that the target trajectory lies in a given domain. The Hit-and-Run sampler is a MCMC method for generating random samples from an arbitrary continuous density f over its support by drawing from a time-reversible Markov chain. The Markov chain transitions are defined by choosing a random direction and then moving to a new point whose likelihood depends on f in that direction. The convergence is based on convergence in distribution of realizations to their asymptotic distribution f. The Hit-and-Run sampler, which generates a continuous-state Markov chain sample path $\{\mathbf{x}_i, i \geq 0\}$ from S, the support of f, is given below:

Algorithm: Hit-and-Run sampler

1. Choose a starting point $\mathbf{x}_0 \in S$, and set $i=0$
2. Generate a direction $\mathbf{d}_i$, from a distribution on the surface of the unit sphere.
3. Find the set $S_i(\mathbf{d}_i,\mathbf{x}_i) \triangleq \{\lambda \in \mathbb{R} \mid \mathbf{x}_i + \lambda\mathbf{d}_i \in S\}$
4. Generate a signed distance : λ_i from density $g_i(\lambda \mid \mathbf{d}_i,\mathbf{x}_i)$, where $\lambda_i \in S_i$
5. Set $\mathbf{y} = \mathbf{x}_i + \lambda_i\mathbf{d}_i$

$$\mathbf{x}_{i+1} = \begin{cases} \mathbf{y} & \text{with probability } a_i(\mathbf{y} \mid \mathbf{x}_i) \\ \mathbf{x}_i & \text{otherwise.} \end{cases}$$

6. Set $i+1$, and go to Step 2.

It exists various choices for the distribution of $\mathbf{d}_i$, the densities g_i, and the probabilities a_i [6]. It is not necessary that the density function f or its support be bounded (proved by Chen and Schmeiser). In practice, there are three jump strategies (i.e. choices of $g_i(\lambda \mid \mathbf{d}_i,\mathbf{x}_i)$ and $a_i(y \mid \mathbf{x}_i)$) allowing the resulting Markov chain to have a probability transition kernel and to be time reversible with respect to f. As we consider in this article a target moving in a bounded area, the following strategy will be used:

- $g_i(\lambda \mid \mathbf{d}_i,\mathbf{x}_i) = \dfrac{1}{m(S_i(\mathbf{d}_i,\mathbf{x}_i))}$ for $\lambda \in S_i(\mathbf{d}_i,\mathbf{x}_i)$ where m is Lebesgue measure.

- $a_i = \begin{cases} \dfrac{f(\mathbf{y})}{f(\mathbf{x}_i)+f(\mathbf{y})} & \text{(Barker's method)} \\ \min\left(1, \dfrac{f(\mathbf{y})}{f(\mathbf{x}_i)}\right) & \text{(Metropolis's method)} \end{cases}$

Directions $\mathbf{d}_i \in \mathbb{R}^n$ on the surface of the unit sphere are obtained by drawing $D_i^j \sim U[-1,1]$ for $j = 1 \dots n$ (where U represents the uniform law) and by normalizing to 1 so that $\mathbf{d}_i^j = D_i^j / \sqrt{\sum_{k=1}^n (D_i^k)^2}$.

Rectilinear target trajectory.

Our problem is to draw samples $(r, \theta, v, \alpha)^i$ from the posterior density $\pi((r, \theta, v, \alpha) \mid (\hat{\theta}_1, \dots, \hat{\theta}_N))$ by considering the prior information expressed by the constraint that the target's trajectory stands in a supposed known area and a constraint on the maximum target velocity. Under the assumption of a rectilinear and uniform motion, the target trajectory (denoted $\mathcal{T}$) can be defined by its two extremal points (P_{init} and P_{end}), i.e. :

$$P_{init} = \begin{pmatrix} r_{x,init} \\ r_{y,init} \end{pmatrix} \in \mathbb{R}^2 , \quad P_{end} = \begin{pmatrix} r_{x,end} \\ r_{y,end} \end{pmatrix} \in \mathbb{R}^2 , \quad \mathcal{T} = \begin{pmatrix} P_{init} \\ P_{end} \end{pmatrix} \tag{10}$$

Let us denote C the area where stand all the target trajectories without constraint on the maximal speed and let be : $S \in \mathbb{R}^2 \times \mathbb{R}^2$ the subset of C representing all the target trajectories satisfying both the constraint on the maximum target speed and the constraint seen before on the target trajectory. S is then defined by the relation:

$$\mathcal{T} = \begin{pmatrix} P_{init} \\ P_{end} \end{pmatrix} \in S \quad \Longleftrightarrow \quad \begin{cases} P_{init} \in C \\ P_{end} \in C \\ \|P_{end} - P_{init}\| \leq L , \end{cases} \tag{11}$$

where L is the maximal length of a trajectory defined from the total measurement duration Δt and the maximal supposed target speed v_{max}; so that $L = v_{max} \times \Delta t$.

In this setup, the Hit-and-Run algorithm is applied to TMA by associating a point x in S to a feasible trajectory $\mathcal{T}$ in the set S of the whole feasible trajectories, then taking the following form :

1. Choose an initial arbitrary trajectory satisfying the constraints. Define $\mathcal{T}_0$ as:

$$\mathcal{T}_0 = \begin{pmatrix} P_{init} \\ P_{end} \end{pmatrix}^{(0)} \in S \text{ and set } i = 0$$

2. Generate 4 uniform variables in $[-1, 1]$: $\{D_1, D_2, D_3, D_4\}$ that represent the evolutions of directions of the points P_{init} and P_{end} and define the *normalized* $\mathbf{d}_i$ vector ($\|\mathbf{d}_i\| = 1$) by:

$$\mathbf{d}_i^T = (d_1, d_2, d_3, d_4)^{(i)} . \tag{12}$$

3. Determine the λ set : $S_i(\mathbf{d}_i, \mathcal{T}_i) \stackrel{def}{=} \{\lambda \in \mathbb{R} \mid \mathcal{T}_i + \lambda \cdot \mathbf{d}_i \in S\}$. Then the constraints on λ stand as follows:

$$\mathcal{T}_i + \lambda \cdot \mathbf{d}_i = \begin{pmatrix} r_{x,init}^{(i)} + \lambda \cdot d_1^{(i)} \\ r_{y,init}^{(i)} + \lambda \cdot d_2^{(i)} \\ r_{x,end}^{(i)} + \lambda \cdot d_3^{(i)} \\ r_{y,end}^{(i)} + \lambda \cdot d_4^{(i)} \end{pmatrix} \in S$$

4. Draw a signed distance λ_i from the density :

$$g_i(\lambda \mid \mathbf{d}_i, \mathcal{T}_i) = \frac{1}{m(S_i(\mathbf{d}_i, \mathcal{T}_i))} \mathbb{1}_{S_i(\mathbf{d}_i, \mathcal{T}_i)}(\lambda)$$

where m is the (Lebesgue) measure of the segment $S_i(\mathbf{d}_i, \mathcal{T}_i)$.

5. Define the Φ vector by $\Phi = \mathcal{T}_i + \lambda_i \cdot \mathbf{d}_i$ and define $\mathcal{T}_{i+1}$ by:

$$\mathcal{T}_{i+1} = \begin{cases} \Phi \text{ with probability } \min\left(1, \dfrac{f(\Phi)}{f(\mathcal{T}_i)}\right) , \\ \mathcal{T}_i \text{ otherwise ,} \end{cases} \tag{13}$$

where $f(\Phi)$ and $f(\mathcal{T}_i)$ are the posterior distributions of the trajectories Φ and $\mathcal{T}_i$ respectively. In fact, the ratio of the posterior densities $\frac{f(\Phi)}{f(\mathcal{T}_i)}$ reduces to the ratio the likelihoods since the prior informations are already naturally taken into account in the generation of feasible trajectories by the algorithm. So :

$$\frac{f(\Phi)}{f(\mathcal{T}_i)} = \frac{L_{(\hat{\theta}_1,\dots,\hat{\theta}_N)}(\Phi)}{L_{(\hat{\theta}_1,\dots,\hat{\theta}_N)}(\mathcal{T}_i)} \tag{14}$$

with $L_{(\hat{\theta}_1,\dots,\hat{\theta}_N)}(\Phi)$ and $L_{(\hat{\theta}_1,\dots,\hat{\theta}_N)}(\mathcal{T}_i)$ likelihoods of trajectories Φ and $\mathcal{T}_i$.

HIGHEST PROBABILITY DENSITY (HPD) INTERVALS

Computing HPD intervals is the main point in this article for providing interval of confidence; both under the rectilinear uniform hypothesis and the leg-by-leg one.

We present here the calculation of the HPD intervals method and their connection with MCMC methods. Consider a Bayesian posterior density having the form:

$$\pi(\lambda, \varphi \mid D) = \frac{L_D(\lambda, \varphi) \times \pi(\lambda, \varphi)}{c(D)} ,$$

where D denotes *data*, λ is one-dimensional, φ may be a multidimensional vector of parameters other than λ. The functional $L_D(\lambda, \varphi)$ is a likelihood function given D, $\pi(\lambda, \varphi)$ is a prior and $c(D)$ is a normalization constant. The major aim of Bayesian posterior inference is precisely to investigate posterior marginal densities. Based on the main properties of the HPD interval, Chen and Shao [4] (1999) propose the following procedure for calculating an HPD interval for λ :

Chen-Shao HPD Estimation Algorithm

1. Obtain an MCMC sample $\{\lambda_i\}_{i=1,2,\dots,n}$ from $\pi(\lambda \mid D)$
2. Sort $\{\lambda_i\}_{i=1,2,\dots,n}$ to obtain the ordered values:
$$\lambda_{(1)} \leq \lambda_{(2)} \leq \dots \leq \lambda_{(n)}.$$
3. Compute the $100(1-\alpha)\%$ credible intervals:
$$R_j(n) = (\lambda_{(j)}, \lambda_{(j+[(1-\alpha)n])})$$
for $j = 1, 2, \dots, n - [(1-\alpha)n]$ (entire part)
4. The $100(1 - \alpha)\%$ HPD interval is the one, denoted by $R_{j^*}(n)$, with the smallest interval.

RESULTS

The general scenario is depicted on fig. 2, below (left):

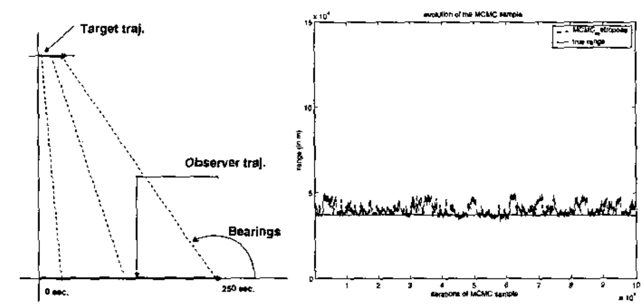

Figure 2: Left: The scenario parameters. Target parameters: $r = 37040$ m., $v_{\text{tgt}} \approx .10.3$ m/sec., $\theta = 90.$ deg., $\alpha = 0.$ deg. Right: MCMC samples vs time.

First, we consider interval constraints for the kinematics parameters (see section 3). The true target state vector is defined by: $r = 37\,040$ m., $v \approx 10.3$ m/sec. (20 knots), $\theta = 90.$ deg., $\alpha = 0.$ deg, (at time 0). The observer speed is 92.6 m/sec, motion is rectilinear and uniform. Fifty (50) bearings are measured every 5 sec. The measurement error is modelled by a centered Gaussian noise, with a 1. deg. standard deviation. Prior constraints stand as follows: $r \in [500, 200\,000]$m, $\theta \in [\hat{\theta}_1 - 5\sigma, \hat{\theta}_1 + 5\sigma]$, $v \in [7.716, 12.86]$ and $\alpha \in [-\pi, \pi]$.
The Metropolis algorithm generates a sample vector of dimension 100 000. We present on fig. 2 (right) the evolution of the MCMC sample for range generated from the marginal posterior density, taking into account the above constraints on

state vector. The true range ($37\,040m$) is drawn in solid line. It appears that range estimations seem to be (only) slightly greater than the true value. However, taking into account that the range is (almost) unobservable, this clearly gives us an idea of the potential of this method.

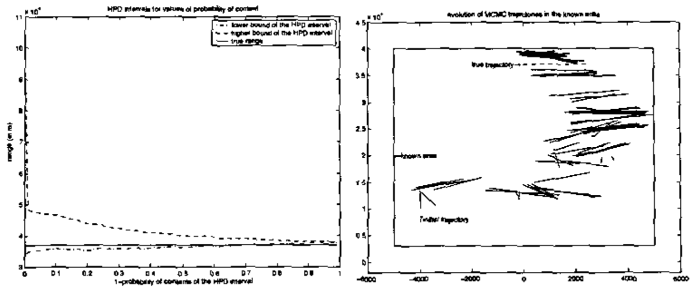

Figure 3: Left: HPD interval width vs probability of content. Right: 500 iterations of the MCMC Hit-and-Run

Fig. 3 (left) illustrates the HPD intervals for probabilities of content. Actually, it is the complementary of this probability which is plotted on the x-axis (i.e. $1 - P(\text{prob. of content})$). The second figure (right) concerns the implementation of the Hit-and-Run algorithm using constraints about target trajectories (trajectory domain and target speed). The scenario is the same than previously.

CONCLUSION

In this paper, we have shown that target motion analysis under demanding operational requirements becomes feasible if the target prior is properly taken into account. The algoritm we developed is based on MCMC methods. It is both reliable and feasible. A definite advantage is also to obtain HPD for parameter estimation, as a by-product.

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