Detection of a Markovian Target with Optimization of the Search Efforts under Generalized Linear Constraints

Frédéric Dambreville, Jean-Pierre Le Cadre

IRISA/CNRS, Campus de Beaulieu, 35042 Rennes Cedex, France

Received November 1999; revised June 2001; accepted 1 October 2001

Abstract: This paper deals with search for a target following a Markovian movement or a conditionally deterministic motion. The problem is to allocate the search efforts, when search resources renew with generalized linear constraints. The model obtained is extended to resource mixing management. New optimality equations of *de Guenin*'s style are obtained. Practically, the problem is solved by using an algorithm derived from the FAB method. © 2002 Wiley Periodicals, Inc. Naval Research Logistics 49: 117–142, 2002; DOI 10.1002/nav.10009

Keywords: search theory; detection; optimization; Markovian movement; sensor management

1. INTRODUCTION

Search theory is the discipline which treats the problem of how best to search for an object when the amount of searching efforts is limited and only probabilities of the object's possible position are given. Search theory came into being during World War II with the work of Koopman and his colleagues [8] in the Antisubmarine Warfare Operations Research Group (ASWORG). Since that time, search theory has grown to be a major discipline within the field of operations research. An important literature has been devoted to this subject, interested reader may consult various extensive surveys [2], introductory texts [11], and books [7, 9, 13, 18]. Even quite recently, a meaningful contribution has been done by Hohzaki and Iida [5] in the area of double-layered search optimization. The problem we investigate here share some viewpoints with [5] even if the algorithmic treatment is fundamentally different. More precisely, we consider search optimization in a continuous framework which allows a greater flexibility (especially for a Markovian target) and avoids enumerative optimization.

The situation is characterized by three pieces of data:

i. The probabilities of the searched object (the ‘‘target’’) being in various possible locations.
ii. The local *detection probability* that a particular amount of local search effort should detect the target.
iii. The total amount of searching effort available. The problem is to find the *optimal* distribution of this total effort, i.e. which maximizes the probability of detection.

Correspondence to: F. Dambreville

Solving such problems requires to optimize the processing of optimization problems involving large numbers of variables, e.g., 3600–12,600 for our examples. Decisive improvements have been made for finding search strategies that maximize the probability of detecting a moving target within a fixed amount of time periods. In particular, Brown [3] has proposed an iterative algorithm in which the motion space and the time frame have been discretized, and the search amount available for each period is infinitely divisible between the grid cells of the target motion space. In this approach, the search effort available for each period is bounded above by a constant that does not depend on the allocations made during any other periods.

Even if the general formalism of search theory will be of constant use subsequently, we shall consider now a specific problem. In the framework of ‘‘classical’’ search problems, the amount of search effort available at each period is bounded above by a fixed and known value. For a multiperiod search, the final result (the global probability of detection) is tightly related to the sequence of successive search amounts. Thus, optimizing the sequence of search amounts is quite challenging. However, it is not possible to optimize separately the sequence of search amounts and the search plans (i.e., the distribution of elementary search efforts). Here we shall consider general constraints relative to this sequence. These constraints may take into account specifications relative to the renewal of search resources (see Section 3) as well as general operational requirements, including multitype resource management. Such associated optimization problem is the object of recent developments. For instance, in [5], Hohzaki and Iida also considered generalized constraints (e.g., the total amount of search efforts). This work appears to be a great advance for solving such combined problem. However, their method strongly departs from ours. Indeed, dealing with continuous search variables allows us to solve a very large optimization problem in a Brown–de Guenin framework. Thus, combinatorial difficulties are greatly reduced. Another difference is the type of constraints we consider. For instance, Hohzaki and Iida [5] deal with direct (fixed) bounds on constraints defined at three levels:

- Space and time level: Each local resource for a given cell and a given period is bounded.
- Time level: the (weighted) sum of the local resources for a given period is bounded.
- Global level: the (weighted) sum of all local resources is bounded.

However, a large part of our contribution is centered around the simultaneous optimization of both the sequence of search amounts according to a linear conditioning (temporal optimization) on the first hand and the spatial distribution of search efforts for each time period (spatial optimization) on the other. Moreover, our viewpoint is quite versatile and permits to handle renewable and multi-mode resources [19].

In our method, the general optimality equations are derived by means of a method largely inspired from classical search theory (namely de Guenin’s equations), though they are considerably more complicated (see Section 2.2). An original algorithmic approach has been used for solving the optimization problem. It combines theoretical results of Section 2.2 with a study of the differential changes of the nondetection probability (see Section 4). In order to render the problem feasible, the Markovian hypothesis (relative to the target motion) is instrumental, allowing us to use the Brown’s implementation (see Section 4). Various extensions will then be considered, namely, extension to mixed resources (Section 5) and inequality constraints (Section 6). Finally, our methods and algorithms are illustrated by simulation results (see Section 7).

2. THE SEARCH PROBLEM

2. 1. Search for a Stationary Target

The problem is to detect a target x, lying in a space E, and whose location is characterized by a (known) probability density $\alpha(x)$. To make this detection, a limited amount of search resource ϕ is available. This (total) search effort may be distributed along the whole space E. To describe the distribution of the search effort, we denote by $\varphi(x)$ the search density allocated to $x \in E$. The limitations on search resource inferred by ϕ yields the following condition on the search effort distribution $\varphi(x)$:

$$\int_E \varphi(x)\,dx \leq \phi. \tag{1}$$

When the local search effort $\varphi(x)$ is applied to location x, if the target is at x, then the probability of nondetection of the target is $p_x(\varphi(x))$, a conditional probability. This probability may depend upon x. For x fixed, p_x decreases with the effort used and then $p'_x < 0$. We suppose the detection follows the rule of decreasing return, so that p'_x increases strictly with φ. According to these notations, our problem is to find the search effort distribution φ under the condition (1) in order to minimize $P_{nd}(\varphi)$, the global probability of nondetection:

$$P_{nd}(\varphi) = \int_E \alpha(x)p_x(\varphi(x))\,dx. \tag{2}$$

As the probability of nondetection decreases with the increase of search effort, the optimal solution is obtained by means of an entire use of the efforts. The condition (1) becomes

$$\int_E \varphi(x)\,dx = \phi. \tag{3}$$

From (2), (3) and from the positivity of density φ, the *de Guenin*'s equations (4) are obtained (refer to [4] for a proof). They give optimality conditions on φ, scaled by a scalar term $\eta < 0$, i.e.,

$$\begin{cases} \alpha(x)p'_x(\varphi(x)) = \eta & \text{if } \alpha(x) > \eta/p'_x(0), \\ \varphi(x) = 0 & \text{else.} \end{cases} \tag{4}$$

Using inversion of p'_x in (4), a function φ_η is obtained, defined by

$$\begin{cases} \varphi_\eta(x) = {p'_x}^{-1}\left(\frac{\eta}{\alpha(x)}\right) & \text{if } \alpha(x) > \eta/p'_x(0), \\ \varphi_\eta(x) = 0 & \text{else.} \end{cases}$$

Since p'_x is strictly monotonic increasing, φ_η increases uniformly (i.e., for each $x \in E$) with η and $\int_E \varphi_\eta(x)\,dx$ increases. Then φ_η will satisfy (3) for only one value of η. Once the convenient value η_o of η has been obtained (e.g., by means of a bisectional search), the optimal function φ is φ_{η_o}.

It is remarkable that this method allows to optimize a great number of variables [i.e., each $\varphi(x)$, for $x \in E$]. This massive optimization is made feasible, since the unknown function φ is linked to one single scalar variable, say the dual variable η. This very fast method, tracing back to the seminal work of Koopman and de Guenin, has been extended by Brown and Washburn to deal with multiperiod search for a Markovian moving target. It will be of constant use subsequently.

2. 2. Search for a Moving Target

Our objective is to detect at one (or more) time-period a target moving in a given space E (assuming stationarity for each period). The detection is done within T time-periods and the search ends after the first detection. We define $\vec{x} = (x_1, \dots, x_T)$ the position of the target during the time-periods $1, 2, \dots, T$. We assume that the target motion is probabilistic and Markovian. Because of the Markovian property, the probabilistic density $\alpha(\vec{x}) = \alpha(x_1, \dots, x_T)$ of the target trajectory may be written as a product of elementary densities, i.e.:

$$\alpha(\vec{x}) = \prod_{k=1}^{T-1} \alpha_k(x_k, x_{k+1}). \tag{5}$$

For each time-period k a given amount of search effort ϕ_k is available. It may be distributed along the search space E. The (local) search effort, applied to the point $x_k \in E$ at time k is denoted $\varphi_k(x_k)$. So, at each time period, the following (equality) constraint (6) is commonly considered in the search theory literature:

$$\forall k \in \{1, \dots, T\}, \qquad \int_E \varphi_k(x_k)\, dx_k = \phi_k. \tag{6}$$

Associated with the local effort $\varphi_k(x_k)$, we call $p_{k,x_k}(\varphi_k(x_k))$ the conditional probability not to detect the target within the time period k, when its location is indeed x_k. We still assume that the detection follows the law of diminishing return. Thus for x_k fixed, $p'_{k,x_k} < 0$ and p'_{k,x_k} is strictly increasing. The convexity of each functions p_{k,x_k} is not sufficient however, to ensure the convexity of the whole problem. It is moreover assumed that $\log p_{k,x_k}$ is a convex function, and this hypothesis yields the convexity of the problem (refer to Appendix D for more clarifications and justifications).

The problem is then to find the functions φ_k in order to minimize $\mathbf{P}_{nd}(\varphi)$ the global probability of nondetection, under the constraint (6). Since the elementary detections are independent, $\mathbf{P}_{nd}(\varphi)$ stands as follows:

$$\mathbf{P}_{nd}(\varphi) = \int_{E^T} \alpha(\vec{x}) \prod_{k=1}^{T} p_{k,x_k}(\varphi_k(x_k)) \prod_{k=1}^{T} dx_k. \tag{7}$$

For a particular time-period $\kappa, \mathbf{P}_{nd}(\varphi)$ can also be written:

$$\mathbf{P}_{nd}(\varphi) = \int_E \beta_\kappa^\varphi(x_\kappa) p_{\kappa,x_\kappa}(\varphi_\kappa(x_\kappa))\, dx_\kappa,$$

where

$$\beta_\kappa^\varphi(x_\kappa) = \int_{E^{T-1}} \alpha(\vec{x}) \prod_{1\le k\le T}^{k\ne\kappa} (p_{k,x_k}(\varphi_k(x_k))\, dx_k). \tag{8}$$

This shows that, when the search efforts are fixed for all the time periods, except for a given one denoted κ, the optimization problem may be solved as the following 1-period de Guenin's problem:

$$\begin{array}{ll}\text{Minimize:} & P_{nd}(\varphi_\kappa) = \int_E \beta^\varphi_\kappa(x_\kappa) p_{\kappa,x_\kappa}(\varphi_\kappa(x_\kappa))\, dx_\kappa, \\ \text{subject to:} & \int_E \varphi_\kappa(x_\kappa)\, dx_\kappa = \phi_\kappa \quad \text{and} \quad \varphi_\kappa \geq 0.\end{array} \tag{9}$$

Then, the following de Guenin's conditions are obtained and inverted by the algorithm described in section 2.1:

$$\begin{cases} \beta^\varphi_\kappa(x_\kappa) p'_{\kappa,x_\kappa}(\varphi_\kappa(x_\kappa)) = \eta_\kappa & \text{if } \beta^\varphi_\kappa(x_\kappa) > \eta_\kappa / p'_{\kappa,x_\kappa}(0), \\ \varphi_\kappa(x_\kappa) = 0 & \text{else.} \end{cases} \tag{10}$$

Brown's algorithm follows these general guidelines. The distributions of the search efforts are successively optimized for each time-period (see the following algorithm), the other ones being fixed. An optimal solution for the multiperiod search is obtained as result of these iterative one-period optimizations:

1. Initialization, set $\kappa = 1$.
2. Apply de Guenin algorithm on $\beta^\varphi_\kappa(x_\kappa) p'_{\kappa,x_\kappa}(\varphi_\kappa(x_\kappa)) = \eta_\kappa$ and optimize φ_κ.
3. set $\kappa := \kappa + 1 \bmod T$ (cyclic increment of κ).
4. Return to 2 until convergence.

Convergence requires only a few iterations. A fundamental ingredient of this algorithm is to use basically the Markovian assumption relative to α, so as to drastically reduce the computation requirements for the integral (8) (**F**orward **A**nd **B**ackward algorithm [3, 17]). As we shall see later, this idea will be instrumental for the development of a feasible algorithm for solving the search problem with generalized constraints.

Until now, the constraints we considered were directly related to the values of ϕ_k, the amounts of resources affected to each search period. The aim of this article is to generalize the multiperiod search to more flexible constraints. This includes especially simple cases of resource renewal.

3. GENERALIZED CONSTRAINTS

As seen previously, the Brown's algorithm supposes the time-splitting ϕ of the constraints to be known. For example, Brown's algorithm does not know how to split (optimally) the global amount of search resources between each period of search. A general formulation of resource time sharing will be built below and illustrated by two examples. It will be of constant use subsequently.

3.1. Splitting Nonrenewable Resources

A total amount Φ of search resources, one time only usable, is to be spread over T time periods. As we want to optimize the probability of detection within the T periods, we have to split Φ into T period resources ϕ_k so that $\sum_{k=1}^{T} \phi_k = \Phi$, or, equivalently,

$$A\phi = \psi, \tag{11}$$

where A is the row-matrix $A_{R\infty} = (1 \cdots 1)$ with T elements, ϕ is the T-dimensional vector of search efforts and ψ is the 1-dimensional vector $\psi_{R\infty} = (\Phi)$. In Section 7, we will refer to this matrix as $A_{R\infty}$ and to this vector as $\psi_{R\infty}$, where the subscript $R\infty$ indicates *nonrenewable resources*.

Extension: A related model is possible, for *non-self-renewable resources*, which are *restored by external support*. Imagine, for example, non-self-renewable resources, which are restored (externally) every three periods. Constraints are similar to *nonrenewable* case, but the position is reset every three periods. Constraints are also of the form $\phi_{3k+1} + \phi_{3k+2} + \phi_{3k+3} = \Phi$, last constraint being possibly truncated. The following linear constraint is obtained, for $T = 8$:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \phi \leq \begin{pmatrix} \Phi \\ \Phi \\ \Phi \end{pmatrix}.$$

This kind of constraint should not be confused with constraint for *self-renewable resources*. In particular, the previous matrix should not be confused with matrix A_{R3}, described in next section.

3. 2. Splitting Self-Renewable Resources

Assume now that we have an amount Φ of search resources self-renewable after some time periods (time for replenishment, for moving, deployment constraints of detection devices, etc.). For example, we can assume that resources self-renew after two periods so that the same resource unit cannot be used simultaneously for two consecutive periods. However, a resource unit becomes available again two periods after last use. If we make *summation* of the units, that means for two following periods k and $k+1$ the relation $\phi_k + \phi_{k+1} = \Phi$. These relations are equivalent to the following linear constraints:

$$A\phi = \psi, \tag{12}$$

where A is the band-diagonal matrix $A_{R2} = (a_{R2}(i,j))_{i,j}$ with $T - (2-1)$ rows and T columns defined by

$$\begin{cases} \forall i \in \{1, \ldots, T-1\}, & a_{R2}(i,i) = a_{R2}(i,i+1) = 1, \\ a_{R2}(i,j) = 0 & \text{else.} \end{cases} \tag{13}$$

and ψ is the vector $\psi_{R2} = (\Phi \cdots \Phi)^t$ with $T-1$ components. In Section 7, we will refer to this matrix and its corresponding constraint vector as A_{R2} and ψ_{R2}, where the subscript $R2$ indicates 2-*periods self-renewable resources*.

In an analogous fashion, for a resource that self-renews after three periods there is a similar band-diagonal matrix A_{R3} with $T - (3-1)$ rows and T columns:

$$\begin{cases} \forall i \in \{1, \ldots, T-2\}, & a_{R3}(i,i) = a_{R3}(i,i+1) = a_{R3}(i,i+2) = 1, \\ a_{R3}(i,j) = 0 & \text{else.} \end{cases}$$

In fact, the above modeling appears sufficiently general to handle a variety of constraints related to the use of detection resources (see, e.g., [10, 16]).

3. 3. The General Optimization Problem

More generally, the linear formulation

$$A\phi = \psi \tag{14}$$

of period-sharing constraints seems sufficiently versatile to handle a great variety of resource allocation problems. Matrix A represents the matrix of resource renewal. Object $\psi = (\psi_j)_{1\leq j\leq\Theta}$ is the vector of resource constraints. Of course, a reasonable hypothesis is that $AX = \psi$ admits at least one solution $X \geq 0$. This hypothesis is suitable for constraints not over-determining the variable ϕ. It is generally the case in practice. Anyhow, this theoretical default will be overcome in Section 6, where problems with inequality constraints will be treated. Using the above notations, our optimization problem becomes

$$\begin{array}{rc}
\text{Minimize:} & \mathbf{P}_{nd}(\varphi) = \int_{E^T} \alpha(\vec{x}) \prod_{k=1}^{T} p_{k,x_k}(\varphi_k(x_k)) \prod_{k=1}^{T} dx_k, \\
\text{under constraints:} & \varphi \geq 0, \qquad \phi \geq 0, \\
 & \forall k \in \{1, \dots, T\}, \qquad \int_E \varphi_k(x_k)\, dx_k = \phi_k, \\
 & A\phi = \psi.
\end{array} \tag{15}$$

At this point, it is interesting to place the work of Hohzaki and Iida in our framework. More precisely, the following optimization problem is considered in [5]:

$$\begin{array}{rl}
\text{Maximize:} & f(\varphi_k(x_k)|_{1\leq k\leq T; x_k\in E}), \\
\text{under constraints:} & 0 \leq \varphi_k(x_k) \leq m_k(x_k), \qquad (16) \\
 & \sum_{x_k\in E} c_k(x_k)\varphi_k(x_k) \leq \phi_k, \qquad (17) \\
 & \sum_{k=1}^{T} \sum_{x_k\in E} c_k(x_k)\varphi_k(x_k) \leq \Phi. \qquad (18)
\end{array}$$

Function f represents a general (concave) evaluation function. Target probability α does not explicitly appears (implicitly defined by f), which represents a greater degree of generality. However, our problem formulation (15) is more adapted to massive optimization. We stress that we will deal with optimization of both the $\{\varphi_k(x_k)\}$ and of the sequence of search amounts $\{\phi_k\}$. We thus see that these two problems present strong similarities even if the ways to solve them fundamentally differ. Finally, we will omit the local constraint on search effort (i.e., $\varphi_k(x_k) \leq m_k(x_k)$) from the general presentation. For completeness, it is addressed in Appendix C.

4. NUMERICAL RESOLUTION

4.1. Algorithm

We shall develop now an original numerical method for solving our optimization problem. We refer to the formalism introduced in (15). Then, consider $A^{\sim}$ a matrix such that $\ker A = \mathrm{Im}(A^{\sim})$ and a vector ϕ^0 satisfying to $A\phi^0 = \psi$. These matrices and vectors may be found by means of elementary algorithms (e.g., adaptation of *Gauss method*). As illustration, examples of matrices $A^{\sim}_{R\infty}$ and $A^{\sim}_{R2}$ are given below:

$$A^{\sim}_{R\infty} = \begin{pmatrix} 1 & & 0 \\ -1 & \ddots & \\ & \ddots & 1 \\ 0 & & -1 \end{pmatrix} \quad \text{and} \quad A^{\sim}_{R2} = \begin{pmatrix} 1 \\ -1 \\ \vdots \\ (-1)^T \end{pmatrix}.$$

We denote p the number of columns of $A^{\sim}$. Then, for each vector ϕ fulfilling $A\phi = \psi$, the property $A(\phi - \phi^0) = 0$ holds and there is a vector $\nu \in \mathbb{R}^p$ such that

$$\phi = \phi^0 + A^{\sim}\nu.$$

Likewise $A\phi = \psi$ holds true for all the ϕ vectors of the preceding form. The basis of the algorithm is to optimize variations of the independent variable ν for minimizing $\mathbf{P}_{nd}(\varphi)$. More precisely, define the optimal nondetection probability for a given time-sharing vector ϕ:

$$\mathbf{P}_{nd}(\phi) = \min_{\varphi:\forall k \int \varphi_k = \phi_k} \mathbf{P}_{nd}(\varphi). \tag{19}$$

The problem is to optimize the choice of ν so as to minimize $\mathbf{P}_{nd}(\phi) = \mathbf{P}_{nd}(\phi^0 + A^{\sim}\nu)$. For this purpose, we will study the differential behavior of $\mathbf{P}_{nd}(\phi^0 + A^{\sim}\nu)$ relatively to ν. Moreover, we will have to take into account the resource positivity constraints.

4. 1. 1. *Differential Behavior of* $\mathbf{P}_{nd}(\phi^0 + A^{\sim}\nu)$

Assume φ be an optimal solution for the vector of search effort ϕ, and consider a variation $d\phi$ of this vector. There are two (equivalent) ways to define the variation $d\mathbf{P}_{nd}(\phi)$. The first one is to set $d\mathbf{P}_{nd}(\phi) = \mathbf{P}_{nd}(\varphi + d\varphi) - \mathbf{P}_{nd}(\varphi)$, where $\varphi + d\varphi$ is chosen as the optimal solution for the vector of search effort $\phi + d\phi$. The second way is to set $d\mathbf{P}_{nd}(\phi) = \min_{d\varphi}[\mathbf{P}_{nd}(\varphi + d\varphi) - \mathbf{P}_{nd}(\varphi)]$, where $d\varphi$ accords with the constraints. We elect this second method. From now on, notation $d_\varphi P_{nd}(\varphi) = \mathbf{P}_{nd}(\varphi + d\varphi) - \mathbf{P}_{nd}(\varphi)$ is used. From definition (7), we have

$$d_\varphi \mathbf{P}_{nd}(\varphi) = \int_{E^T} \alpha(\vec{x}) \left(\prod_{j=1}^{T} (p_{j,x_j}((\varphi_j + d\varphi_j)(x_j))\, dx_j) - \prod_{j=1}^{T} (p_{j,x_j}(\varphi_j(x_j))\, dx_j) \right).$$

A first-order expansion relatively to the product gives the more linear form:

$$d_\varphi \mathbf{P}_{nd}(\varphi) = \sum_{k=1}^{T} \int_{E^T} \alpha(\vec{x}) \left(\prod_{j \neq k} (p_{j,x_j}(\varphi_j(x_j))\, dx_j) \right) \times (p_{k,x_k}(\varphi_k(x_k) + d\varphi_k(x_k)) - p_{k,x_k}(\varphi_k(x_k)))\, dx_k,$$

which can be rewritten in the two following ways, according to definitions (8):

$$d_\varphi \mathbf{P}_{nd}(\varphi) = \sum_{k=1}^{T} \int_E \beta_k^\varphi(x_k) p'_{k,x_k}(\varphi_k(x_k))\, d\varphi_k(x_k)\, dx_k,$$

$$d_\varphi \mathbf{P}_{nd}(\varphi) = \sum_{k=1}^{T} \int_E \beta_k^\varphi(x_k) p_{k,x_k}((\varphi_k + d\varphi_k)(x_k))\, dx_k - \sum_{k=1}^{T} \int_E \beta_k^\varphi(x_k) p_{k,x_k}(\varphi_k(x_k))\, dx_k. \tag{20}$$

Only last equation is useful for the moment. In this equation, the second member of the subtraction is constant (φ is known as an optimal solution for ϕ) and only the first sum has to be minimized.

The minimization of this sum reverts to minimizing each of its members. The minimization of $d_\varphi \mathbf{P}_{nd}(\varphi)$ then reduces to optimizing, for each index k, the variation $d\varphi_k$ so as to minimize the following 1-dimensional integrals:

$$\int_E \beta_k^\varphi(x_k) p_{k,x_k}((\varphi_k + d\varphi_k)(x_k))\, dx_k. \tag{21}$$

The above expression is minimized under the constraint $\int_E (\varphi_k + d\varphi_k)(x_k)\, dx_k = \phi_k + d\phi_k$ and positivity of $\varphi_k + d\varphi_k$. This minimization occurs even when φ is maintained constant. Assuming φ constant means that β_k^φ is constant too. Then the optimization of $d\varphi_k$, which is equivalent to optimizing $\varphi_k + d\varphi_k$ in (21), reduces to a 1-period search problem of de Guenin. This problem is clearly solved by means of the de Guenin optimality conditions:

$$\begin{cases} p'_{k,x_k}((\varphi_k + d\varphi_k)(x_k))\beta_k^\varphi(x_k) = c_k & \text{if } \beta_k^\varphi(x_k) > c_k / p'_{k,x_k}(0), \\ (\varphi_k + d\varphi_k)(x_k) = 0 & \text{else.} \end{cases} \tag{22}$$

However, this equation is related to the optimality equations for φ [refer to (10)]:

$$\begin{cases} p'_{k,x_k}(\varphi_k(x_k))\beta_k^\varphi(x_k) = \eta_k & \text{if } \beta_k^\varphi(x_k) > \eta_k / p'_{k,x_k}(0), \\ \varphi_k(x_k) = 0 & \text{else.} \end{cases} \tag{23}$$

The rather rough notation $c_k = \eta_k + d\eta_k$ will be used from now on. Variation $d\eta_k$, which corresponds to a 1-dimensional optimization, must not be confused with the variation of η obtained from the global optimization. We stress that the associated optimality equations must be put together so as to deduce $d\varphi$. Partial derivative of $\mathbf{P}_{nd}(\phi)$ is then deduced. But two cases are distinguished.

A—First Case: $\phi_k > 0$. The sets defined below are instrumental in the forthcoming development:

$$X = \{x_k \in E / \beta_k^\varphi(x_k) > \eta_k / p'_{k,x_k}(0)\},$$

$$Y = \{x_k \in E / \beta_k^\varphi(x_k) > (\eta_k + d\eta_k) / p'_{k,x_k}(0)\},$$

$$F = \{\epsilon \in \mathbb{R}^{-*},\ \text{such that the measure of } \{x_k \in E / \beta_k^\varphi(x_k) p'_{k,x_k}(0) = \epsilon\} \text{ is positive}\}.$$

Because of (22) and (23), X and Y are thus defined so that $\varphi_k(x_k) = 0$ whenever $x_k \notin X$ and $(\varphi_k + d\varphi_k)(x_k) = 0$ whenever $x_k \notin Y$. It is obvious as well (a basic measure property) that the sets $\{x_k \in E / \beta_k^\varphi(x_k) p'_{k,x_k}(0) = \epsilon\}$ have almost always a zero measure. Thus, the set F is discrete and it will be assumed that the problem is sufficiently regular to ensure that F does not contain a point of accumulation. A consequence is that the complement F^c of F is an open set. This property will be useful to handle *borders* $X \backslash Y$ and $Y \backslash X$. Now, optimality equations are differentiated by *subtracting* (23) from (22) and by means of first-order expansion around $\varphi_k(x_k)$ or $\varphi_k(x_k) + d\varphi_k(x_k)$:

$$\begin{cases} \forall x_k \in X \cap Y, & \beta_k^\varphi(x_k) p''_{k,x_k}(\varphi_k(x_k)) d\varphi_k(x_k) = d\eta_k, \\ \forall x_k \in X \backslash Y, & \beta_k^\varphi(x_k) p'_{k,x_k}(0) - \beta_k^\varphi(x_k) p''_{k,x_k}(0) d\varphi_k(x_k) = \eta_k, \\ \forall x_k \in Y \backslash X, & \beta_k^\varphi(x_k) p'_{k,x_k}(0) + \beta_k^\varphi(x_k) p''_{k,x_k}(0) d\varphi_k(x_k) = \eta_k + d\eta_k, \\ \forall x_k \in E \backslash (X \cup Y), & d\varphi_k(x_k) = 0. \end{cases} \tag{24}$$

Some useful results are now established. From de Guenin's equation (23) [4] follows directly that $\forall x_k \in X \cap Y, \beta_k^\varphi(x_k) p'_{k,x_k}(\varphi_k(x_k)) = \eta_k$. This, combined with (24), gives

$$\forall x_k \in X \cap Y, \qquad \frac{p''_{k,x_k}(\varphi_k(x_k))}{p'_{k,x_k}(\varphi_k(x_k))} d\varphi_k(x_k) = \frac{d\eta_k}{\eta_k}. \tag{25}$$

Let $x_k \in Y \backslash X$ verifying $\beta_k^\varphi(x_k) p'_{k,x_k}(0) = \eta_k$ be given. From Eqs. (24) are deduced $\eta_k + \eta_k p''_{k,x_k}(0)/p'_{k,x_k}(0)\, d\varphi_k(x_k) = \eta_k + d\eta_k$ and then $p''_{k,x_k}(0)/p'_{k,x_k}(0)\, d\varphi_k(x_k) = d\eta_k/\eta_k$. Since $\varphi(x_k) = 0$ by definition of X, a property similar to (25) holds:

$$\left.\begin{array}{r} \beta_k^\varphi(x_k) p'_{k,x_k}(0) = \eta_k \\ x_k \in Y \backslash X \end{array}\right\} \Rightarrow \frac{p''_{k,x_k}(\varphi_k(x_k))}{p'_{k,x_k}(\varphi_k(x_k))} d\varphi_k(x_k) = \frac{d\eta_k}{\eta_k}. \tag{26}$$

Since F^c is open and variation $d\eta_k$ is sufficiently small, the hypothesis $\eta_k \notin F$ yields $]\eta_k, \eta_k + d\eta_k[\cap F = \varnothing$ and $[\eta_k + d\eta_k, \eta_k[\cap F = \varnothing$. Since F contains no point of accumulation, hypothesis $\eta_k \in F$ yields $]\eta_k, \eta_k + d\eta_k[\cap F = \varnothing$ and $[\eta_k + d\eta_k, \eta_k[\cap F = \varnothing$. Thus ever holds

$$]\eta_k, \eta_k + d\eta_k[\cap F = \varnothing \qquad \text{and} \qquad [\eta_k + d\eta_k, \eta_k[\cap F = \varnothing. \tag{27}$$

In the next section, Eqs. (24), (25), and (26) will be put in the differentiated constraint $\int_E d\varphi_k(x_k)\, dx_k = d\phi_k$. It appears however that $X \subset Y$ when $d\phi_k > 0$ and $Y \subset X$ when $d\phi_k < 0$.[1] Two cases are then to be considered.

A1—Subcase $d\phi_k > 0$: In this case $X \subset Y, X \cap Y = X$ and $X \backslash Y = \varnothing$. Define:

$$\tilde{X} = \{x_k \in E / \beta_k^\varphi(x_k) \geq \eta_k / p'_{k,x_k}(0)\}.$$

So, $X \subset \tilde{X}$, and since $Y \backslash \tilde{X} \subset Y \backslash X$, the third equation of (24) yields:

$$\forall x_k \in Y \backslash \tilde{X}, d\varphi_k(x_k) = \frac{\eta_k + d\eta_k}{\beta_k^\varphi(x_k) p''_{k,x_k}(0)} - \frac{p'_{k,x_k}(0)}{p''_{k,x_k}(0)}.$$

On the other hand, equations (25) and (26) may be cast into:

$$\forall x_k \in \tilde{X}, d\varphi_k(x_k) = \frac{d\eta_k}{\eta_k} \times \frac{p'_{k,x_k}(\varphi_k(x_k))}{p''_{k,x_k}(\varphi_k(x_k))}. \tag{28}$$

Now, $d\phi_k = \int_E d\varphi_k(x_k)\, dx_k$ and $d\varphi_k(x_k) = 0$ outside $X \cup Y$ [fourth equation of (24)], and since $E = \tilde{X} \cup (Y \backslash \tilde{X}) \cup E \backslash (X \cup Y)$, we have

$$d\phi_k = \frac{d\eta_k}{\eta_k} \int_{\tilde{X}} \frac{p'_{k,x_k}(\varphi_k(x_k))}{p''_{k,x_k}(\varphi_k(x_k))} dx_k + \int_{Y \backslash \tilde{X}} \left(\frac{\eta_k + d\eta_k}{\beta_k^\varphi(x_k) p''_{k,x_k}(0)} - \frac{p'_{k,x_k}(0)}{p''_{k,x_k}(0)} \right) dx_k.$$

[1]This result is intuitively obvious. To show it, consider the function φ_{k,η_k} obtained by inverting de Guenin's equation (23) [4]. As ever seen, such function increases uniformly with η_k.

The first element of this sum is a first-order infinitesimal. Now, the cells $x_k \in Y\backslash\tilde{X}$ are those which verify bounds $\eta_k < \beta_k^\varphi(x_k)p'_{k,x_k}(0) < \eta_k + d\eta_k$. Since $]\eta_k, \eta_k + d\eta_k[\cap F = \varnothing$ from (27), there is no level $\{x_k \in E/\beta_k^\varphi(x_k)p'_{k,x_k}(0) = \epsilon\}, \epsilon \in]\eta_k, \eta_k + d\eta_k[$, with positive measure, included in $Y\backslash\tilde{X}$. Thus, the set $Y\backslash\tilde{X}$ is negligible. On the other hand, the bounds $\eta_k < \beta_k^\varphi(x_k)p'_{k,x_k}(0) < \eta_k + d\eta_k$ involve also

$$x_k \in Y\backslash\tilde{X} \Rightarrow 0 < \frac{\eta_k + d\eta_k}{\beta_k^\varphi(x_k)p''_{k,x_k}(0)} - \frac{p'_{k,x_k}(0)}{p''_{k,x_k}(0)} < \frac{d\eta_k}{\beta_k^\varphi(x_k)p''_{k,x_k}(0)}.$$

Then, the second integral in the sum is defined over a negligible set, $Y\backslash\tilde{X}$, and the integrand is a first-order infinitesimal term. The second integral is negligible, thus yielding up to a second-order infinitesimal

$$d\phi_k = \frac{d\eta_k}{\eta_k}\int_{\tilde{X}} \frac{p'_{k,x_k}(\varphi_k(x_k))}{p''_{k,x_k}(\varphi_k(x_k))}dx_k. \tag{29}$$

Using (28) together with (29), $d\varphi_k$ stands as follows:

$$\forall x_k \in \tilde{X}, \qquad d\varphi_k(x_k) = d\phi_k \frac{p'_{k,x_k}(\varphi_k(x_k))}{p''_{k,x_k}(\varphi_k(x_k))}\Bigg/\int_{\tilde{X}} \frac{p'_{k,x_k}(\varphi_k(x_k))}{p''_{k,x_k}(\varphi_k(x_k))}dx_k. \tag{30}$$

Now, the kth component of $d\mathbf{P}_{nd}(\varphi)$ appearing in Eq. (20) takes the following form:

$$\begin{aligned}\int_E \beta_k^\varphi(x_k)p'_{k,x_k}(\varphi_k(x_k))\,d\varphi_k(x_k)\,dx_k &= \int_{\tilde{X}} \beta_k^\varphi(x_k)p'_{k,x_k}(\varphi_k(x_k))\,d\varphi_k(x_k)\,dx_k \\ &\quad + \int_{Y\backslash\tilde{X}} \beta_k^\varphi(x_k)p'_{k,x_k}(\varphi_k(x_k))\,d\varphi_k(x_k)\,dx_k.\end{aligned}$$

The second integral concerns a negligible set together with a first-order infinitesimal integrand. Again, this integral can be neglected. Now, $\beta_k^\varphi(x_k)p'_{k,x_k}(\varphi_k(x_k)) = \eta_k$ whatever $x_k \in X$. Otherwise, for $x_k \in \tilde{X}\backslash X$ hold both $\varphi_k(x_k) = 0$ and $\beta_k^\varphi(x_k)p'_{k,x_k}(0) = \eta_k$. Thus $\beta_k^\varphi(x_k)$ $p'_{k,x_k}(\varphi_k(x_k)) = \eta_k$ whatever $x_k \in \tilde{X}$, so that

$$\begin{aligned}\int_E \beta_k^\varphi(x_k)p'_{k,x_k}(\varphi_k(x_k))\,d\varphi_k(x_k)\,dx_k &= \eta_k\int_{\tilde{X}} d\varphi_k(x_k)\,dx_k \\ &= \eta_k\int_{\tilde{X}} \frac{\frac{p'_{k,x_k}(\varphi_k(x_k))}{p''_{k,x_k}(\varphi_k(x_k))}d\phi_k}{\int_{\tilde{X}}\frac{p'_{k,x_k}(\varphi_k(x_k))}{p''_{k,x_k}(\varphi_k(x_k))}dx_k}dx_k \\ &= \eta_k\,d\phi_k.\end{aligned} \tag{31}$$

A2—Subcase $d\phi_k < 0$: In this case, $Y \subset X$, $X \cap Y = Y$ and $Y\backslash X = \varnothing$. We have

$$d\phi_k = \frac{d\eta_k}{\eta_k}\int_Y \frac{p'_{k,x_k}(\varphi_k(x_k))}{p''_{k,x_k}(\varphi_k(x_k))}dx_k + \int_{X\backslash Y}\left(\frac{p'_{k,x_k}(0)}{p''_{k,x_k}(0)} - \frac{\eta_k}{\beta_k^\varphi(x_k)p''_{k,x_k}(0)}\right)dx_k. \tag{32}$$

The cells $x_k \in X \backslash Y$ are those for which inequalities $\eta_k > \beta_k^{\varphi}(x_k) p'_{k,x_k}(0) \geq \eta_k + d\eta_k$ hold true. It follows that

$$x_k \in X \backslash Y \Rightarrow 0 > \frac{p'_{k,x_k}(0)}{p''_{k,x_k}(0)} - \frac{\eta_k}{\beta_k^{\varphi}(x_k) p''_{k,x_k}(0)} \geq \frac{d\eta_k}{\beta_k^{\varphi}(x_k) p''_{k,x_k}(0)}.$$

Thus, the integrand of the right integral of (32) is infinitesimal. Furthermore, $X \backslash Y$ is a negligible set, since $[\eta_k + d\eta_k, \eta_k[\cap F = \varnothing$ [refer to (27)]. Then, the right integral, $\int_{X \backslash Y} (\frac{p'_{k,x_k}(0)}{p''_{k,x_k}(0)} - \frac{\eta_k}{\beta_k^{\varphi}(x_k) p''_{k,x_k}(0)})\, dx_k$, is a second-order infinitesimal, and so

$$d\phi_k = \frac{d\eta_k}{\eta_k} \int_Y \frac{p'_{k,x_k}(\varphi_k(x_k))}{p''_{k,x_k}(\varphi_k(x_k))} dx_k. \tag{33}$$

Reminding (25) and since $Y = X \cap Y$, $d\varphi_k$ is given by

$$\forall x_k \in Y, d\varphi_k(x_k) = d\phi_k \frac{p'_{k,x_k}(\varphi_k(x_k))}{p''_{k,x_k}(\varphi_k(x_k))} \Big/ \int_Y \frac{p'_{k,x_k}(\varphi_k(x_k))}{p''_{k,x_k}(\varphi_k(x_k))} dx_k.$$

The kth component of $d\mathbf{P}_{nd}(\varphi)$ may be rewritten as a sum of two contributions, i.e.,

$$\int_E \beta_k^{\varphi}(x_k) p'_{k,x_k}(\varphi_k(x_k))\, d\varphi_k(x_k)\, dx_k = \int_Y \beta_k^{\varphi}(x_k) p'_{k,x_k}(\varphi_k(x_k))\, d\varphi_k(x_k)\, dx_k + \int_{X \backslash Y} \beta_k^{\varphi}(x_k) p'_{k,x_k}(\varphi_k(x_k))\, d\varphi_k(x_k)\, dx_k.$$

As previously, integration on $X \backslash Y$ may be neglected. Since $\beta_k^{\varphi}(x_k) p'_{k,x_k}(\varphi_k(x_k)) = \eta_k$ for $x_k \in X$, and $Y \subset X$, we can write

$$\begin{aligned} \int_E \beta_k^{\varphi}(x_k) p'_{k,x_k}(\varphi_k(x_k))\, d\varphi_k(x_k)\, dx_k &= \int_Y \eta_k\, d\varphi_k(x_k)\, dx_k \\ &= \eta_k \int_Y \frac{\frac{p'_{k,x_k}(\varphi_k(x_k))}{p''_{k,x_k}(\varphi_k(x_k))} d\phi_k}{\int_Y \frac{p'_{k,x_k}(\varphi_k(x_k))}{p''_{k,x_k}(\varphi_k(x_k))} dx_k} dx_k = \eta_k\, d\phi_k. \end{aligned} \tag{34}$$

B—Second case: $\phi_k = 0$. Numerous problems stem from the nullity of ϕ_k. First, we could remark that $\varphi_k = 0$, so that Eq. (23) makes no sense, although it is exact. Another difficulty is that we must restrict to nonnegative variations of ϕ_k and φ_k. The variation of φ_k, associated with the variation $d\phi_k$ of ϕ_k, will be denoted $\delta\varphi_k$ rather than $d\varphi_k$.[2] Moreover, $\delta\eta_k$ will denote the variation of η_k. It is then easy to show that $\delta\varphi_k$ and $\delta\eta_k$ are same order infinitesimals. These variations will be related to the minimum value:

$$\eta_k^0 = \min_{x_k \in E} (\beta_k^{\varphi}(x_k) p'_{k,x_k}(0)). \tag{35}$$

[2] In this case it is not ensured that variations of ϕ_k and φ_k are of the same order of magnitude.

From this definition follows

$$\forall x_k \in E, \qquad \beta_k^{\varphi}(x_k) \leq \frac{\eta_k^0}{p'_{k,x_k}(0)}.$$

From de Guenin's equation applied to $\eta_k^0 + \delta\eta_k$, the following implication stems (recall that $\varphi_k = 0$):

$$\begin{cases} \frac{\eta_k^0+\delta\eta_k}{p'_{k,x_k}(0)} < \beta_k^{\varphi}(x_k) \leq \frac{\eta_k^0}{p'_{k,x_k}(0)} \Rightarrow \beta_k^{\varphi}(x_k) p'_{k,x_k}(\delta\varphi_k(x_k)) = \eta_k^0 + \delta\eta_k, \\ \beta_k^{\varphi}(x_k) \leq \frac{\eta_k^0+\delta\eta_k}{p'_{k,x_k}(0)} \Rightarrow \delta\varphi_k(x_k) = 0. \end{cases}$$

The first line of this equation is of no interest, here. However, it may be noticed that $\beta_k^{\varphi}(x_k)$ $p'_{k,x_k}(0) = \eta_k^0$ up to a first order infinitesimal, whenever $\frac{\eta_k^0+\delta\eta_k}{p'_{k,x_k}(0)} < \beta_k^{\varphi}(x_k) \leq \frac{\eta_k^0}{p'_{k,x_k}(0)}$. The kth component of $d\mathbf{P}_{nd}(\varphi)$ appearing in Eq. (20) may thus be simplified:

$$\begin{aligned} \int_E \beta_k^{\varphi}(x_k) p'_{k,x_k}(\varphi_k(x_k))\, \delta\varphi_k(x_k)\, dx_k &= \int_E \beta_k^{\varphi}(x_k) p'_{k,x_k}(0)\, \delta\varphi_k(x_k)\, dx_k \\ &= \int_{\beta_k^{\varphi}(x_k) \geq \frac{\eta_k^0+\delta\eta_k}{p'_{k,x_k}(0)}} \beta_k^{\varphi}(x_k) p'_{k,x_k}(0)\, \delta\varphi_k(x_k)\, dx_k \\ &= \int_{\beta_k^{\varphi}(x_k) \geq \frac{\eta_k^0+\delta\eta_k}{p'_{k,x_k}(0)}} \eta_k^0\, \delta\varphi_k(x_k)\, dx_k \\ &= \eta_k^0 \int_E \delta\varphi_k(x_k)\, dx_k = \eta_k^0\, d\phi_k. \end{aligned} \tag{36}$$

Calculation of $d\mathbf{P}_{nd}(\phi)$: Let us define the vector $\mathbf{V}$ with T components by

$$\boxed{\begin{cases} \mathbf{V}_k = \eta_k & \text{when } \phi_k > 0 \\ \mathbf{V}_k = \eta_k^0 = \min_{x_k \in E} (\beta_k^{\varphi}(x_k) p'_{k,x_k}(0)) & \text{when } \phi_k = 0 \end{cases}} \tag{37}$$

Variation $d\mathbf{P}_{nd}(\phi)$ may be rewritten as

$$\boxed{d\mathbf{P}_{nd}(\phi) = \sum_k \int_E \beta_k^{\varphi}(x_k) p'_{k,x_k}(\varphi_k(x_k))\, d\varphi_k(x_k)\, dx_k = \mathbf{V}^t\, d\phi} \tag{38}$$

Now, let us consider $\nu \in \mathbb{R}^p$ and $\phi = \phi^0 + A^{\sim}\nu$. Suppose moreover that the resource positivity constraints are satisfied for ν. Let $d\nu$ be a vectorial infinitesimal variation of ν so that resource positivity constraints for $\nu + d\nu$ still hold. The infinitesimal variation for ϕ is thus $d\phi = A^{\sim} d\nu$, and we deduce

$$d\mathbf{P}_{nd}(\phi^0 + A^{\sim}\nu) = \mathbf{V}^t A^{\sim}\, d\nu. \tag{39}$$

4. 1. 2. Broad Lines of the Algorithm

Our algorithm involves the *gradient projection method of Rosen*. Recall that the gradient projection method is an iterative minimization method, which relies upon a suitable choice of variation direction, the gradient projection $\mathbf{d}$, which ensures the convergence (to a local minimum). This direction choice only depends on the values of gradient components and on the position relatively to the constraints borders. Rosen gradient projection method will not be detailed here; the reader may refer to [1]. Our method is converging to a global minimum for the same reasons that ensure convergence of the Brown's algorithm:

- convexity of $\mathbf{P}_{nd}$.
- convexity of the constraints domain.
- $\mathbf{P}_{nd}$ is strictly decreasing at each iteration.

In our own algorithm, the first step is to initialize ϕ^0 and $A^\sim$. Using the calculation (39) of $d\mathbf{P}_{nd}(\phi^0 + A^\sim\nu)$, Rosen method is applied so as to minimize $\mathbf{P}_{nd}(\phi^0 + A^\sim\nu)$ under constraint $\phi^0 + A^\sim\nu \geq 0$. Of course, this algorithm will encapsulate some execution of Brown method in order to compute the function $\mathbf{P}_{nd}(\phi^0 + A^\sim\nu)$ and the optimal spatial sharing φ associated with it. The complete algorithm is outlined below:

1. Compute $A^\sim$ and ϕ^0; initialize ν;
2. Set $\phi = \phi^0 + A^\sim\nu$;
3. Use Brown's algorithm to find the optimal spatial sharing φ associated with ϕ. Now, the optimal dual variable η is also computed;
4. Compute $d\mathbf{P}_{nd}(\phi^0 + A^\sim\nu)$ by applying (39);
5. Compute the gradient projection $\mathbf{d}$ according to Rosen method. This computation involves the previously defined gradient $d\mathbf{P}_{nd}(\phi^0 + A^\sim\nu)$ and the status of variable ν with respect to the constraint $\phi^0 + A^\sim\nu \geq 0$;
6. Choose a variation step Δt;
7. Update ν, i.e. $\nu := \nu + \Delta t\mathbf{d}$;
8. Return to 2 until convergence.

The choice of Δt in step 6 has not been explained so far. One theoretical choice (but not the best) for Δt, which ensures convergence, is the one that minimizes $\mathbf{P}_{nd}(\phi)$ along the direction $\mathbf{d}$. In other words, $\Delta t = \arg\min_{\theta \leq \theta_{max}} \mathbf{P}_{nd}(\phi^0 + A^\sim(\nu + \theta\mathbf{d}))$, where θ_{max} is a bound preventing to exceed the constraints. Finding this optimal Δt will require some additional execution of the Brown's algorithm.

To end with this section, it is important to remark that this theoretical algorithm is greatly underoptimized. First of all, it is actually not necessary to compute an optimal Δt to ensure practically the convergence. Otherwise, there is no necessity to have a complete convergence of Brown's algorithm in step 3. Satisfactorily, the algorithm has been tested with the use of only one cycle of Brown's algorithm. Finally, we evaluated that our algorithm's speed was proportional to Brown's one with a (rough) factor from 2 to 50 (complex examples).

À la Brown implementation: Practically, a major problem we have to face with our and also with Brown's algorithm stems from the fact that the computation of the function β_κ^φ requires a huge amount of computation time. Calculating this function needs integrating on the $(T-1)$-dimensional space E^{T-1}, for each element of E. If we consider as a time unit, u, the integration on E, the computation time of β_κ^φ is of an u^T order. Practically, the parameter u is rather large which means that this (direct) approach is clearly infeasible. In order to overcome this difficulty, the idea of Brown will be again instrumental. More precisely, the Markovian property of α can drastically reduce the computation requirements (FAB method). So, let us define the function vectors U^φ and D^φ in the following recursive way:

$$\begin{aligned} U_1^\varphi(x_1) &= 1 \quad \text{and} \quad D_T^\varphi(x_T) = 1, \\ U_{k+1}^\varphi(x_{k+1}) &= \int_E \alpha_k(y_k, x_{k+1}) p_{k,y_k}(\varphi_k(y_k)) U_k^\varphi(y_k)\, dy_k, \\ D_{k-1}^\varphi(x_{k-1}) &= \int_E \alpha_{k-1}(x_{k-1}, y_k) p_{k,y_k}(\varphi_k(y_k)) D_k^\varphi(y_k)\, dy_k. \end{aligned} \tag{40}$$

Computing U_k^φ knowing U_{k-1}^φ or computing D_k^φ knowing D_{k+1}^φ requires a time of the order of u^2. Altogether, computing all U^φ and D^φ require $2Tu^2$. Assuming U^φ and D^φ available, then we compute β^φ as a simple product:

$$\beta_k^\varphi(x_k) = U_k^\varphi(x_k) D_k^\varphi(x_k). \tag{41}$$

A refinement allows to spare even more computation time. Usually, since φ is changed, we have to compute U^φ and D^φ again. But we can remark that, when only φ_κ is changed, U_k^φ and D_l^φ stay unchanged for $k \leq \kappa$ and $l \geq \kappa$. These properties are used in order to reduce the computation of β^φ and of Brown's algorithm. This is one of the ingredients, which permits us to optimize a such massive number of variables (12,600 for some of our examples), the other argument (refer to Section 2.1) being that these variables depends on the small dimension vector η.

4. 2. Generalized de Guenin's Equations

It is now easy to apply the previous calculation of $d\mathbf{P}_{nd}(\phi)$ and establish generalized de Guenin's equations (involving conditioning on dual variable η). When optimal solution is reached, the variation $d\mathbf{P}_{nd}(\phi) = \mathbf{V}^t d\phi$ is nonnegative, for each valid variation $d\phi$, i.e., checking $Ad\phi = 0$. But there are also positivity constraints, which impose $d\phi$ satisfies $d\phi_k \geq 0$, whenever $\phi_k = 0$. Define also the projection Matrix P^ϕ by

$$\begin{cases} P_{k,k}^\phi = 1, & \text{for each } k \in \{1, \ldots, T\} \text{ verifying } \phi_k > 0, \\ P_{i,j}^\phi = 0, & \text{else.} \end{cases} \tag{42}$$

The optimality condition on $\mathbf{P}_{nd}(\phi)$ then becomes

$$\forall d\phi, \quad [(I - P^\phi) d\phi \geq 0 \quad \text{and} \quad Ad\phi = 0] \Rightarrow \mathbf{V}^t d\phi \geq 0. \tag{43}$$

To establish a necessary optimality conditions on η, a property weaker but more suitable than (43) is used:

$$\forall d\phi, \quad [(I - P^\phi) d\phi = 0 \quad \text{and} \quad Ad\phi = 0] \Rightarrow \mathbf{V}^t d\phi \geq 0.$$

Assume conditions $(I - P^\phi)d\phi = 0$ and $Ad\phi = 0$ hold true. It comes then that $\mathbf{V}^t d\phi \geq 0$. Moreover, if the two conditions are verified for $d\phi$, they are also verified for $-d\phi$, that is, $(I - P^\phi)(-d\phi) = 0$ and $A(-d\phi) = 0$. For this reason, the inequalities $\mathbf{V}^t(-d\phi) \geq 0$ and $\mathbf{V}^t d\phi \leq 0$ are simultaneously valid. Hence, we have

$$\forall d\phi, [(I - P^\phi) d\phi = 0 \quad \text{and} \quad Ad\phi = 0] \Rightarrow \mathbf{V}^t d\phi = 0.$$

This may be also rewritten:

$$\mathbf{V} \in (\text{Ker}(I - P^{\phi}) \cap \text{Ker } A)^{\perp}.$$

In other words, $\mathbf{V} \in \text{Ker}(I - P^{\phi})^{\perp} + \text{Ker } A^{\perp}$, or equivalently $\mathbf{V} \in \text{Ker } P^{\phi} + \text{Im } A^t$. It is obvious that $P^{\phi}(\text{Ker } P^{\phi} + \text{Im } A^t) = \text{Im}(P^{\phi} A^t)$, thus yielding $P^{\phi}\mathbf{V} \in \text{Im}(P^{\phi} A^t)$. In fact, referring to definition (37), the projected part of $P^{\phi}\mathbf{V}$ corresponds to the active dual variables η_k, where $\phi_k > 0$. More precisely, $P^{\phi}\mathbf{V} = P^{\phi}\eta$ and $P^{\phi}\eta \in \text{Im}(P^{\phi} A^t)$. Of course, the nonactive part of the dual variables, i.e., $(I - P^{\phi})\eta$, will stay undefined. Property 1 is then deduced.

PROPERTY 1: Let φ be an optimal solution. Define $\phi_k = \int_E \varphi_k$, for $1 \leq k \leq T$. Define by (42), the projection P^{ϕ} associated with positives indices of ϕ. For each k, verifying $\phi_k > 0$, there is an active parameter η_k so that

$$\beta_k^{\varphi}(x_k) p'_{k,x_k}(\varphi_k(x_k)) = \eta_k, \qquad \text{if } \beta_k^{\varphi}(x_k) > \eta_k / p'_{k,x_k}(0),$$

$$\varphi_k(x_k) = 0, \text{ else.}$$

These active parameters η_k are given by means of vector $P^{\phi}\eta$ satisfying to

$$\exists \mu, \qquad P^{\phi}\eta = P^{\phi} A^t \mu.$$

It is noteworthy that, for a discrete (in space) version of this problem, Property 1 is recovered from the Kuhn Tucker theorem.

5. EXTENSION TO MIXED RESOURCES

A target moving in space E is to be searched. The search being split in T periods, we denote $\alpha(\vec{x})$ the density probability of the target trajectory. The resources used for the search are of r different types, associated with the nondetection functions $(p_k^{\rho})_{1 \leq k \leq T; 1 \leq \rho \leq r}$. We denote $\varphi = (\varphi_k^{\rho})_{1 \leq k \leq T; 1 \leq \rho \leq r}$ the corresponding (local) effort functions and $\phi = (\phi_k^{\rho})_{1 \leq k \leq T; 1 \leq \rho \leq r}$ the vector of (global) efforts. These two variables are associated together as usual:

$$\forall k \in \{1, \ldots, T\}, \forall \rho \in \{1, \ldots, r\}, \qquad \int_E \varphi_k^{\rho}(x_k)\, dx_k = \phi_k^{\rho}.$$

Assuming the hypothesis of independence of searches, the following value of the nondetection probability is obtained:

$$\mathbf{P}_{nd}(\varphi) = \int_{E^T} \alpha(\vec{x}) \prod_{k=1}^{T} \left(\left(\prod_{\rho=1}^{r} p_{k,x_k}^{\rho}(\varphi_k^{\rho}(x_k)) \right) dx_k \right).$$

A linear constraint is again considered, but this time concerns also type indices ρ:

$$A\phi = \psi,$$

where A is a matrix with $T \times r$ columns and ψ is a vector.

Our aim is then to optimize φ under constraints in order to minimize $\mathbf{P}_{nd}(\varphi)$, yielding

$$\begin{array}{rl}\text{Minimize:} & \mathbf{P}_{nd}(\varphi) = \int_{E^T} \alpha(\vec{x}) \prod_{k=1}^{T} \left(\left(\prod_{\rho=1}^{r} p^{\rho}_{k,x_k}(\varphi^{\rho}_k(x_k)) \right) dx_k \right), \\ \text{under constraints:} & \varphi \geq 0, \quad \phi \geq 0, \\ & \forall k \in \{1,\dots,T\}, \forall \rho \in \{1,\dots,r\}, \quad \int_E \varphi^{\rho}_k(x_k)\, dx_k = \phi^{\rho}_k, \\ & A\phi = \psi. \end{array} \tag{44}$$

The general algorithm for solving this optimization problem is presented in Appendix A. The concept is now illustrated by two typical examples.

Multitype resources: Consider a search for a target on a space E involving two types of resource; e.g., nonrenewable resources (denoted a) and 3-period self-renewable resources (denoted b). The two types of resources are assumed to work simultaneously and independently. Assume the search duration to be T (periods). We call Φ_a and Φ_b the amounts of available resources for each type a and b. In the same way φ^a and φ^b represent the function of (local) search effort for the types a and b, and variables ϕ^a and ϕ^b the (global) search efforts respectively associated. Then, our multitype optimization can be solved by means of the following extended problem:

$$\begin{cases} r = 2, \quad p^1 = p^a, \quad p^2 = p^b, \quad \varphi^1 = \varphi^a, \quad \varphi^2 = \varphi^b, \quad \phi^1 = \phi^a, \quad \phi^2 = \phi^b, \\ A = \begin{pmatrix} A^a & 0 \\ 0 & A^b \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} \psi^a \\ \psi^b \end{pmatrix}, \end{cases}$$

where

$$\begin{aligned} A^a &= A_{R\infty} = (1 \cdots 1), \qquad \psi^a = (\Phi_a), \\ A^b &= A_{R3} = \begin{pmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ & & & \ddots & & \\ 0 & \cdots & 0 & 1 & 1 & 1 \end{pmatrix}, \qquad \text{and} \\ \psi^b &= \begin{pmatrix} \Phi_b \\ \vdots \\ \Phi_b \end{pmatrix}. \end{aligned}$$

Multimode resources: Only a single type of resource (with matrix and vector constraints A^o and ψ^o) is available. However, this resource can run in two different ways or *operating modes*. The first mode, denoted c, is characterized by elements φ^c, p^c, and variable ϕ^c. The second mode, denoted d, is characterized by elements φ^d, p^d, and variable ϕ^d. Describing the sharing of the resources between mode c and mode d, the following relation holds:

$$A^o \phi^o = \psi^o \quad \text{with } \forall k \in \{1,\dots,T\}, \quad \phi^o_k = \int_E (\varphi^c_k + \varphi^d_k)(x_k)\, dx_k = \phi^c_k + \phi^d_k.$$

This leads us to consider the following extended problem:

$$\begin{cases} r = 2, \quad p^1 = p^c, \quad p^2 = p^d, \quad \varphi^1 = \varphi^c, \quad \varphi^2 = \varphi^d, \quad \phi^1 = \phi^c, \quad \phi^2 = \phi^d, \\ A = (A^o A^o) \quad \text{and} \quad \psi = \psi^o. \end{cases}$$

6. INEQUALITY CONSTRAINTS

So far, only equality constraints have been considered (see, e.g., Section 3). In fact, the equality constraint, $A\phi = \psi$, in (15) is somewhat restrictive, especially in cases where it may be profitable to keep some resources unused so as to use them more efficiently in later periods. For this reason, it could be of interest to optimize under inequality constraint, $A\phi \leq \psi$. A second (theoretical) advantage of inequality constraint is the possibility of *overdetermined* constraint, that is, it becomes feasible to use matrix and vector A and ψ such that $AX = \psi$ has no solution. The theoretical weakness considered in Section 3.3 is thus overcome. Now, cases, where constraints $A\phi \leq \psi$ and $\phi \geq 0$ are even so incompatible, corresponds to ill-posed problems. The problem for inequality constraint is

$$\begin{array}{rl} \text{Minimize:} & \mathbf{P}_{nd}(\varphi) = \int_{E^T} \alpha(\vec{x}) \prod_{k=1}^{T} p_{k,x_k}(\varphi_k(x_k)) \prod_{k=1}^{T} dx_k, \\ \text{under constraints:} & \varphi \geq 0, \qquad \phi \geq 0, \\ & \forall k \in \{1,\ldots,T\}, \qquad \int_E \varphi_k(x_k)\, dx_k = \phi_k, \\ & A\phi \leq \psi. \end{array} \tag{45}$$

Such inequality constraints may be translated into equality ones by means of slack variables. More precisely, let us denote Θ the row number of the (constraint) matrix A; then inequality constraints $A\phi \leq \psi$ revert to considering equality constraints $(A\phi + \phi^p = \psi)$ by adding Θ slack variables $\phi_1^p, \ldots, \phi_\Theta^p$ (satisfying also to the positivity constraints $\phi_k^p \geq 0$) to each row of $A\phi = \psi$. Denoting by I^Θ the Θ-dimensional identity matrix, the preceding optimization problem becomes (details in Appendix B):

$$\begin{array}{rcl} \text{Inequality constraints} & \rightarrow & \text{Equality constraints,} \\ A\phi \leq \psi & \rightarrow & (A \quad I_\Theta) \begin{pmatrix} \phi \\ \phi^p \end{pmatrix} = \psi, \\ \phi \geq 0 & \rightarrow & \begin{pmatrix} \phi \\ \phi^p \end{pmatrix} \geq 0, \end{array} \tag{46}$$

$$\mathbf{P}_{nd}(\varphi) = \int_{E^T} \alpha(\vec{x}) \prod_{k=1}^{T} p_{k,x_k}(\varphi_k(x_k)) \prod_{k=1}^{T} dx_k$$

and

$$\left(\int_E \varphi_k(x_k)\, dx_k \right)_{1 \leq k \leq T} = \phi \text{ stay unchanged.} \tag{47}$$

Roughly the algorithm is unchanged, except the A matrix which is replaced by $(A \ \ I_\Theta)$. However, we stress that, since φ^p has no physical meaning (slack variables), the components of $\mathbf{V}$ associated with the variables ϕ^p are zeroed [calculation of $d\mathbf{P}_{nd}(\phi)$].

7. RESULTS

The space search E is a square of 30×30 cells. Target's trajectories are simulated within the following general scheme: a start position, a motion component, and (possibly) a final position.

The target starting position is represented by s, an uniform density in the 10×10 square with top-left vertex on the point (5, 5), i.e.,

$$s(x_1) = \frac{1}{100} \quad \text{if } (5,5) \leq x_1 \leq (14,14),$$
$$s(x_1) = 0 \quad \text{else.}$$

The density of the (possibly) final target location is uniform in the 10×10 square with top-left vertex on (16, 16), and is denoted f:

$$f(x_T) = \frac{1}{100} \quad \text{if } (16,16) \leq x_T \leq (25,25),$$
$$f(x_T) = 0 \quad \text{else.}$$

At each time-period the (Markovian) target motion is an uniform diffusion (toward down and right) represented by the function m on the 2D motion vector:

$$\begin{cases} m(0,0) = m(3,3) = \frac{3}{14} \quad \text{and} \quad m(2,3) = m(3,2) = \frac{2}{14}, \\ m(0,3) = m(3,0) = m(1,3) = m(3,1) = \frac{1}{14}, \\ m(x_{k+1} - x_k) = 0, \quad \text{else.} \end{cases}$$

3	0	0	1
0	0	0	1
0	0	0	2
1	1	2	3

For example, the density $(\alpha(\vec{x}))$ of a target trajectory (e.g., for a 4-time-period scenario) could take the following form:

$$\alpha(\vec{x}) = Z \times s(x_1)m(x_2 - x_1)m(x_3 - x_2)m(x_4 - x_3)f(x_4).$$

It represents a down-right diffusion diverging at the beginning from the starting square and finally converging back to the final square. The value Z represents a normalization term. The test results of the algorithm are divided into three sections. In the first one, we shall examine the effects of the form of the nondetection functions p_{k,x_k} (only uniform functions over x_k will be considered). In the second one, example with mixed resources will be presented (p_{k,x_k} nonuniform). Finally, inequality constraints are examined in the last one.

7. 1. Effects of the Detection Function p

Throughout this section we shall consider a unique modeling of the target motion and a 4-time-period search. The target trajectory distribution function is

$$\alpha(\vec{x}) = s(x_1)m(x_2 - x_1)m(x_3 - x_2)m(x_4 - x_3). \tag{48}$$

Only one type of resource is used. The nondetection function of this resource is independent of spatial location, that is, $\forall k, \forall x_k \in E, p_{k,x_k} = \pi$. We shall compare results obtained by an exponential (nondetection) function, $\pi(\varphi) = \exp(-\omega\varphi)$ with $\omega = 1$, and a nonexponential one, $\pi(\varphi) = 1/(\varphi+1)^2$. Note that the function $\log \pi$ verifies for both cases the convexity hypothesis. Exponential functions are widely used even if restricting assumptions are underlined (detection without waste; refer to Appendix D).

Exponential function: Under this assumption, the whole effort can be entirely affected to the more interesting time-period without waste. This behavior is illustrated by the following example. Here, resources are not renewable ($A = A_{R\infty}$ and $\psi = \psi_{R\infty}$; see Section 3.1), and the total effort is $\Phi = 20$. The splitting of search efforts between the consecutive periods is illustrated by Table 1. We can see that the essential of search effort is brutally put on the first periods where the density of this diffusive target is most concentrated. A similar result is obtained whatever the value of Φ.

Nonexponential function: In such case, the whole effort cannot be entirely affected at the same time period without significant waste (see [16]; refer also to Appendix D). We present three results. The first one using $A = A_{R\infty}$ and $\psi = \psi_{R\infty}$, where total effort $\Phi = 200$. The two other ones using $A = A_{R2}$ and $\psi = \psi_{R2}$ (2-period self-renewable resource; see Section 3.2), with total effort $\Phi = 100$, respectively $\Phi = 10$. Table 2 illustrates, for the three scenarios, the splitting of search effort at each time period $(\phi_1, \phi_2, \phi_3, \phi_4)$. For the sake of comparison, the optimal probability $\mathbf{P}_{nd}$ is provided. The significant resources are still concentrated on the periods with the highest target density. But the period-splitting is smoother, although the effort still tends to be concentrated on the first time periods. The splitting is more contrasted when the total amount of resource is small (second and third example: ϕ_1 and ϕ_3 increase from 73% of resources to 87%). Figure 1 presents the spatial distribution of the search effort, for the last example. We notice a *surrounding strategy* for the distribution of search efforts on the first periods.

7. 2. Mixing of Resources

From now on, all the nondetection functions will be exponential. In this section, we present a search example with multiple resources. Moreover, the nondetection functions will be depending on the space location and on the type ρ of search resource (but independent of time k), so that $p^{\rho}_{k,x_k}(\varphi) = \exp(-\omega^{\rho}_{x_k}\varphi)$. For each example, the total search duration is 7 periods. The target distribution corresponds to a diffusion from the start position:

$$\alpha(\vec{x}) = s(x_1) \prod_{k=1}^{6} m(x_{k+1} - x_k). \tag{49}$$

Table 1. Exponential function.

A	Φ	ϕ_1	ϕ_2	ϕ_3	ϕ_4
$A_{R\infty}$	20	20	0	0	0

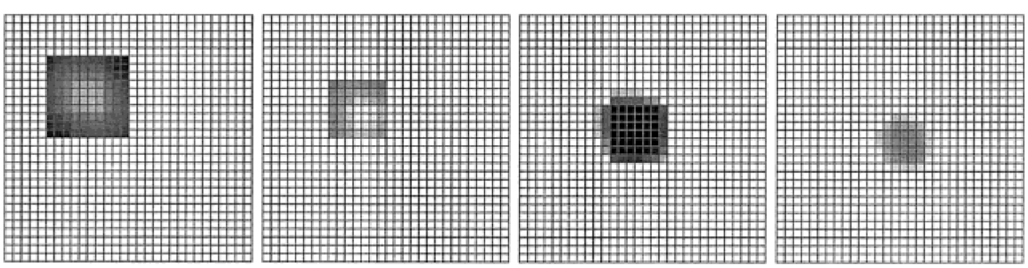

Figure 1. Self-renewable resources; $A = A_{R2}$, $\Phi = 10$; nonexponential function.

Table 2. Nonexponential function.

A	Φ	ϕ_1	ϕ_2	ϕ_3	ϕ_4	$\mathbf{P}_{nd}(\varphi)$
$A_{R\infty}$	200	112	50	25	13	7.7%
A_{R2}	100	73	27	73	27	9.1%
A_{R2}	10	8.7	1.3	8.7	1.3	71%

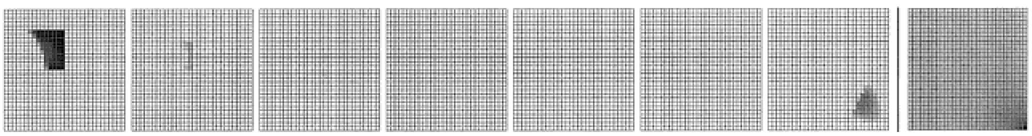

Figure 2. Type 1; nonrenewable resources; $A = A_{R\infty}$, $\Phi = 100$.

Two different resources are used. The first resource ($\rho = 1$) is a nonrenewable one with constraint matrix $A = A_{R\infty}$ and total resource $\Phi = 100$. The second resource ($\rho = 2$) is a 3-period self-renewable resource with constraint matrix $A = A_{R3}$ (see Section 3.2) and total resource $\Phi = 50$. The visibility factor of the first resource, i.e., the exponential parameter $\omega^1_{x_k}$, is decreasing from the right-down bottom of the search space E, as shown in the 8th (from left) picture of Figure 2. The visibility factor of second resource, say $\omega^2_{x_k}$, is decreasing from the left-down bottom of the search space E, as shown in Figure 3. A practical example of this situation may be detection by fixed sensors, where the search efforts correspond to the duration of the looks in a given cell (electronically steered array) and visibility factors ω_{x_k} are related to physical parameters (cell range, propagation, etc.). The optimal splitting of the two resources is presented in Table 3. Figures 2 and 3 represent respectively the spatial sharing of the first and second resources for each period (see the first seven pictures).

The second resource splitting is 50, 0, 0, 50, 0, 0, 50. This result appears quite natural since it gives the higher amount of resource. Moreover, the target spread tends to focus the search on the first period, as the gradient associated with $\omega^2_{x_k}$ reinforces the search on the central part of the diffusion. The behavior of the first resource is more surprising. The most important detections occur at the first periods (search amounts 82 and 2.8 for periods 1 and 2), but there is again detection (search amount 15 for period 7) at the end. There are two explanations of this fact. First, a conflict occurs between the target spread and the gradient associated with $\omega^1_{x_k}$. The first one tends to enforce the detection at the beginning of the movement, while the second enhances the detection when target approaches to the down-right bottom. On the other hand, the second resource spreads the splitting of the first one, since it reinforces detection occurring at the beginning, the middle, and the final periods. It is also remarkable that the search areas of the different resources are well distinct and complementary. Their locations depend on the gradient associated with $\omega^\rho_{x_k}$ (down-right for the first type and down-left for the second). Again, surrounding strategy occurs on the first periods, although it is shared between the two resources.

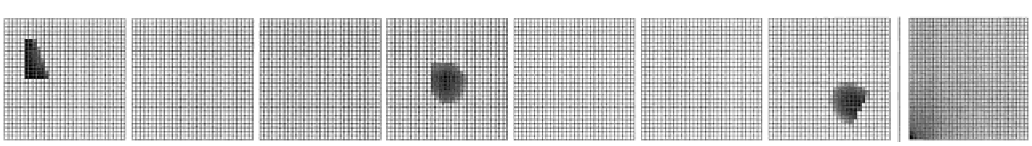

Figure 3. Type 2; self-renewable resources; $A = A_{R3}$, $\Phi = 50$.

Table 3. Mixed resources.

A	Φ	ρ	ϕ_1^ρ	ϕ_2^ρ	ϕ_3^ρ	ϕ_4^ρ	ϕ_5^ρ	ϕ_6^ρ	ϕ_7^ρ	Figure
$A_{R\infty}$	100	1	82	2.8	0.2	0	0	0	15	2
A_{R3}	50	2	50	0	0	50	0	0	50	3

Table 4. Inequality compared with equality (constraints).

A	Inequality	Φ	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	$\mathbf{P}_{nd}(\varphi)$	Figure
A_{R3}	No	50	21.1	7.8	21.1	21.1	7.8	21.1	81%	4
A_{R3}	Yes	50	32.3	10.2	7.5	7.5	10.2	32.3	79.9%	5
$A_{R\infty}$	No	100	32.3	10.2	7.5	7.5	10.2	32.3	79.9%	5
$A_{R\infty}$	Yes	100	32.3	10.2	7.5	7.5	10.2	32.3	79.9%	5

7. 3. Inequality Constraints

The four trials presented here concern a diffusion movement, on 6 periods, with a spread from the starting position followed by a convergence toward the final position:

$$\alpha(\vec{x}) = Z \times s(x_1) \left(\prod_{k=1}^{5} m(x_{k+1} - x_k) \right) f(x_6). \tag{50}$$

The nondetection function is constant in time and space ($p_{k,x_k} = \pi$). The first example concerns a single resource, which self-renews after 3 periods ($A = A_{R3}$), under equality constraints. The total amount of resource is 50. The scenario for the second example is identical, but inequality constraints are used. The third and fourth examples involve a single non renewable resource ($A = A_{R\infty}$), which total amount equals 100. Equality and inequality cases are both tested. The four problems are then quantitatively equivalent. Results are given in Table 4 and are spatially presented in Figures 4 and 5.

First, it is remarkable that the solutions are symmetric ($\phi_1 = \phi_6, \phi_2 = \phi_5$, and $\phi_3 = \phi_4$). This is not surprising since functions f and s are mutually symmetric and the constraint matrices

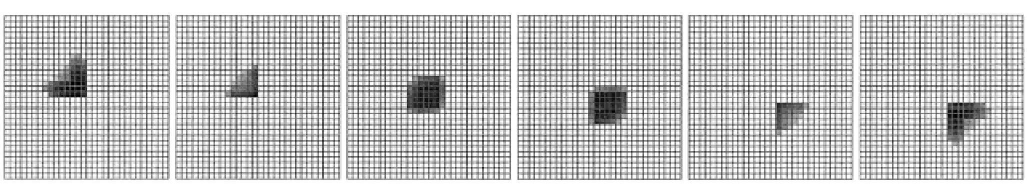

Figure 4. Self-renewable resource without inequality constraints; $A = A_{R3}$, $\Phi = 50$.

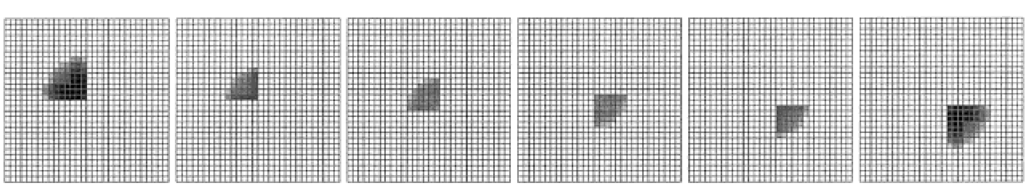

Figure 5. Self-renewable resource with inequality constraints; $A = A_{R3}$, $\Phi = 50$.

Table 5. Inequality compared with equality: Large target spread.

A	Inequality	Φ	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	$\mathbf{P}_{nd}(\varphi)$
A_{R3}	No	50	45	0	5	45	0	5	75.7%
A_{R3}	Yes	50	50	0	0	0	0	50	60.7%

and vectors preserve this symmetry. Thus, the optimal solution (or the set of optimal solutions) is symmetric. The symmetry of the problem equalizes the degree of freedom of the three last examples, so that the $\mathbf{P}_{nd}(\varphi)$ values are identical. The splitting (32.3, 10.2, 7.5, 7.5, 10.2, 32.3) is coherent with the nature of the movement (a spread followed by a convergence). In the first example, sums of three consecutive ϕ_k must be equal to 50, because of equality constraint with A_{R3}. In this case, the full optimal time sharing is forbidden, since $10.2 + 7.5 + 7.5 < 50$. Thus, the first example is forced to a suboptimal solution, where a significant amount of resources (21.1) is put on the middle of the movement (periods of maximal target diffusion). Otherwise, surrounding appears for each examples at the beginning period—as well at the final period! Such final surrounding is somewhat upsetting, but it corresponds to the wait for the far incoming trajectories, when the final converging movement occurs.

The earlier example shows that equality constraints is almost as good as inequality constraints (81% versus 79.9%). But that is true only when, as in the example, the target spread is very weak. We consider now a final example with a (very) diffusive target. The target distribution includes start and final positions. Probabilities on intermediate periods are uniform, and all periods are independent:

$$\alpha(\vec{x}) = 900^{-4} \times s(x_1)f(x_6). \tag{51}$$

Optimal solutions are presented in Table 5 (the solution asymmetry results from the nonunicity of the solution). The more striking point of these results is the great difference between the two probabilities (75.7% versus 60.7%). In term of detection probability, it gives 24.3% versus 39.3%. Improvement is thus considerable.

8. CONCLUSION

Our aim was to solve the problem of spatial and temporal splitting of (possibly) renewable resources. In order to develop feasible optimization methods, the formalism and the algorithm of Brown–de Guenin have played again a central role, allowing us to obtain a whole variety of algorithms, solving problems of increasing difficulty. These algorithms are robust (convergence ensured) and fast since computation requirements are of the same order as Brown's. Moreover, they seem sufficiently general to handle numerous problems of sensor and resource managements arising in complex systems of detection (e.g. sonar, radar, infrared), involving various types of sensors and operating modes. These points have been considered in a general setting. For specific applications, more work has to be done; however our approach seems sufficiently open and versatile to deal with numerous practical search problems.

APPENDIX A
ALGORITHM FOR SEARCH INVOLVING MIXED RESOURCES

In (44), the constraint $A\phi = \psi$ yields the equivalent parameterization $\phi = \phi^0 + A^{\sim}\nu$. The optimization method will combine this parameterization with the differential behavior of $\mathbf{P}_{nd}(\phi) = \min_{\varphi:\forall k \forall \rho \int \varphi_k^\rho = \phi_k^\rho} \mathbf{P}_{nd}(\varphi)$ to make an iterative approximation of the minimum. The differential of $\mathbf{P}_{nd}(\phi)$ is simply expressed by means of the associated variation vector $\mathbf{V}$, i.e., $d\mathbf{P}_{nd}(\phi) = \mathbf{V}^t d\phi$. The new algorithm follows the same guidelines as before; alternatively choosing a suitable variation $\Delta\phi$ of ϕ according to $d\mathbf{P}_{nd}(\phi)$ and to the parameterization, and then running a Brown's process for function φ. FAB principle will still be used for computing integrals. The presentation of following results is voluntary concise (details are omitted).

Optimality conditions: Let κ and ϱ be two particular value of k and ρ indexes. Defining

$$\beta^{\varphi}_{\kappa,\varrho}(x_\kappa) = \left(\prod_{1\le\rho\le r}^{\rho\neq\varrho} p^{\rho}_{\kappa,x_\kappa}(\varphi^{\rho}_{\kappa}(x_\kappa)) \right) \int_{E^{T-1}} \alpha(\vec{x}) \prod_{1\le k\le T}^{k\neq\kappa} \left(\left(\prod_{\rho=1}^{r} p^{\rho}_{k,x_k}(\varphi^{\rho}_{k}(x_k)) \right) dx_k \right), \tag{52}$$

we obtain

$$\mathbf{P}_{nd}(\varphi) = \int_E \beta^{\varphi}_{\kappa,\varrho}(x_\kappa) p^{\varrho}_{\kappa,x_\kappa}(\varphi^{\varrho}_{\kappa}(x_\kappa))\, dx_\kappa. \tag{53}$$

When the search efforts are fixed for all indexes (k, ρ) except for the index (κ, ϱ), the problem is in the de Guenin's optimization scheme i.e.:

$$\begin{array}{ll} \text{Minimize:} & P_{nd}(\varphi^{\varrho}_{\kappa}) = \int_E \beta^{\varphi}_{\kappa,\varrho}(x_\kappa) p^{\varrho}_{\kappa,x_\kappa}(\varphi^{\varrho}_{\kappa}(x_\kappa))\, dx_\kappa, \\ \text{subject to:} & \int_E \varphi^{\varrho}_{\kappa}(x_\kappa)\, dx_\kappa = \phi^{\varrho}_{\kappa} \quad \text{and} \quad \varphi^{\varrho}_{\kappa} \ge 0. \end{array} \tag{54}$$

This optimization yields the following conditions of de Guenin:

$$\begin{cases} \beta^{\varphi}_{\kappa,\varrho}(x_\kappa) p^{\varrho'}_{\kappa,x_\kappa}(\varphi^{\varrho}_{\kappa}(x_\kappa)) = \eta^{\varrho}_{\kappa} & \text{if } \beta^{\varphi}_{\kappa,\varrho}(x_\kappa) > \eta^{\varrho}_{\kappa} / p^{\varrho'}_{\kappa,x_\kappa}(0), \\ \varphi^{\varrho}_{\kappa}(x_\kappa) = 0 & \text{else.} \end{cases} \tag{55}$$

Using notation $\eta^0_{k,\rho} = \min_{x_k \in E} \beta^{\varphi}_{k,\rho}(x_k) p^{\rho'}_{k,x_k}(0)$, the vector $\mathbf{V}$ is now defined by

$$\begin{cases} \mathbf{V}^{\varrho}_{k} = \eta^{\rho}_{k} & \text{when } \phi^{\rho}_{k} > 0, \\ \mathbf{V}^{\rho}_{k} = \eta^{0}_{k,\rho} & \text{when } \phi^{\rho}_{k} = 0. \end{cases}$$

APPENDIX B
INEQUALITY CONSTRAINTS

As previously seen and adding slack variables, inequality constrained problems revert to consider the following one (with equality constraints):

$$\begin{array}{ll} \text{Minimize:} & \mathbf{P}_{nd}(\varphi) = \int_{E^T} \alpha(\vec{x}) \prod_{k=1}^{T} p_{k,x_k}(\varphi_k(x_k)) \prod_{k=1}^{T} dx_k, \\ \text{subject to:} & \forall k \in \{1 \cdots T\}, \quad \int_E \varphi_k(x_k)\, dx_k = \phi_k, \\ & (AI_\Theta) \begin{pmatrix} \phi \\ \phi^p \end{pmatrix} = \psi \quad \text{and} \quad \begin{pmatrix} \phi \\ \phi^p \end{pmatrix} \ge 0. \end{array} \tag{56}$$

The algorithm is the same as usually, but the fictitious periods associated to ϕ^p are not directly considered. In the Brown's process, only the function φ is obtained by means of the de Guenin algorithm (a function φ^p should have no sense). The other difference comes from relaxing the parameterization of ϕ, that is, $\phi = \phi^0 + (AI_\Theta)^{\sim}\nu$. The calculation of the associated variation vector $\mathbf{V}$ is changed in consequence, yielding

$$\begin{cases} \mathbf{V}_k = \eta_k & \text{when } \phi_k > 0 \quad \text{and} \quad 1 \le k \le T, \\ \mathbf{V}_k = \eta^0_\kappa & \text{when } \phi_k = 0 \quad \text{and} \quad 1 \le k \le T, \\ \mathbf{V}_k = 0 & \text{when } T < k \le T + \Theta. \end{cases} \tag{57}$$

APPENDIX C
LOCAL SEARCH CONSTRAINTS

In this section, we outline an extension of our approach when we have local constraints for each cell and weighting coefficient in the resource use. Weighting does not change the problem fundamentally, but adding local constraints modify the basic de Guenin's algorithm as well as the associated variation vector $\mathbf{V}$. Only here lie some difficulties. The new problem (equality constraint version, easily extensible to inequality case) is set in the following terms:

$$\text{Minimize:} \quad \mathbf{P}_{nd}(\varphi) = \int_{E^T} \alpha(\vec{x}) \prod_{k=1}^T p_{k,x_k}(\varphi_k(x_k)) \prod_{k=1}^T dx_k,$$

$$\text{under constraints:} \quad \forall k \in \{1, \ldots, T\}, \quad \forall x_k \in E, \varphi_k^1(x_k) \le \varphi_k(x_k) \le \varphi_k^2(x_k), \tag{58}$$

$$\forall k \in \{1, \ldots, T\}, \quad \int_E c_k(x_k)\varphi_k(x_k)\, dx_k = \phi_k, \tag{59}$$

$$A\phi = \psi. \tag{60}$$

Functions φ^1 and φ^2 are constants of the problem, which define lower and upper local bounds for each cell. It is usual to choose φ^1 nonnegative. Function c is a weighting function with positive values. It is noteworthy that the constraint $\phi \ge 0$ are changed into

$$\forall k \in \{1, \ldots, T\}, \quad \int_E \varphi_k^1(x_k)\, dx_k \le \phi_k \le \int_E \varphi_k^2(x_k)\, dx_k. \tag{61}$$

Here are derived (without proof) new de Guenin equations, and the new calculus of vector $\mathbf{V}$, which are the main ingredients of our algorithm. For $\kappa \in \{1 \ldots T\}$, de Guenin equations stand as follows:

$$\begin{cases} \beta_\kappa^\varphi(x_\kappa) p'_{\kappa,x_\kappa}(\varphi_\kappa(x_\kappa)) = \eta_\kappa c_\kappa(x_\kappa) & \text{if } \frac{\eta_\kappa c_\kappa(x_\kappa)}{p'_{\kappa,x_\kappa}(\varphi_\kappa^1(x_\kappa))} < \beta_\kappa^\varphi(x_\kappa) < \frac{\eta_\kappa c_\kappa(x_\kappa)}{p'_{\kappa,x_\kappa}(\varphi_\kappa^2(x_\kappa))}, \\ \varphi_\kappa(x_\kappa) = \varphi_\kappa^1(x_\kappa) & \text{if } \beta_\kappa^\varphi(x_\kappa) \le \eta_\kappa c_\kappa(x_\kappa)/p'_{\kappa,x_\kappa}(\varphi_\kappa^1(x_\kappa)), \\ \varphi_\kappa(x_\kappa) = \varphi_\kappa^2(x_\kappa) & \text{if } \beta_\kappa^\varphi(x_\kappa) \ge \eta_\kappa c_\kappa(x_\kappa)/p'_{\kappa,x_\kappa}(\varphi_\kappa^2(x_\kappa)), \end{cases} \tag{62}$$

where

$$\beta_\kappa^\varphi(x_\kappa) = \int_{E^{T-1}} \alpha(\vec{x}) \prod_{1 \le k \le T}^{k \ne \kappa} (p_{k,x_k}(\varphi_k(x_k))\, dx_k).$$

Vector $\mathbf{V}$ is then defined by

$$\begin{cases} \mathbf{V}_k = \eta_k, \\ \mathbf{V}_k = \eta_k^1 = \min_{x_\kappa \in E} (\beta_k^\varphi(x_\kappa) p'_{k,x_\kappa}(\varphi_k^1(x_\kappa))) & \text{when } \phi_k = \int_E \varphi_k^1(x_\kappa)\, dx_\kappa, \\ \mathbf{V}_k = \eta_k^2 = \max_{x_\kappa \in E} (\beta_k^\varphi(x_\kappa) p'_{k,x_\kappa}(\varphi_k^2(x_\kappa))) & \text{when } \phi_k = \int_E \varphi_k^2(x_\kappa)\, dx_\kappa. \end{cases} \tag{63}$$

The whole algorithm then runs as it has been described in the main part of this contribution.

APPENDIX D
CONVEXITY OF $\log P_{k,x_k}$

When search efforts vary from φ to $\varphi + d\varphi$, the nondetection probability may be rewritten:

$$p_{k,x_k}(\varphi + d\varphi) = p_{k,x_k}(\varphi) p_{k,x_k}(d\varphi|\varphi),$$

where $p_{k,x_k}(d\varphi|\varphi)$ represents the elementary probability of nondetection for a new effort $d\varphi$, knowing that φ resources have already been in use. It is assumed in this paper that $p_{k,x_k}(d\varphi|\varphi)$ is constant or increases with φ. The last case means that resources concentration lowers the detection power of these resources: **detection holds with waste.** On the other hand, the first case means that the detection power of the resources does not depend on their concentration: **detection holds without waste.** This hypothesis is commonly used in the literature. Now, writing $p_{k,x_k}(d\varphi|\varphi) = 1 - \omega_{k,x_k}(\varphi) d\varphi$, the following is obtained:

$$\frac{dp_{k,x_k}}{p_{k,x_k}} = -\omega_{k,x_k}(\varphi)\, d\varphi.$$

It follows $\frac{d \log p_{k,x_k}}{d\varphi} = -\omega_{k,x_k}(\varphi)$. Increaseness hypothesis made on $p_{k,x_k}(d\varphi|\varphi)$ yields the decreaseness of $\omega_{k,x_k}(\varphi)$. Then the convexity of $\log p_{k,x_k}$ holds.

Denote $W_{k,x_k} = \log p_{k,x_k}$. Then, the elementary nondetection probability for a trajectory $\vec{x}$ is given by

$$\prod_{k=1}^T p_{k,x_k}(\varphi_k(x_k)) = \exp\left(\sum_{k=1}^T W_{k,x_k}(\varphi_k(x_k))\right).$$

The convexity of this product and then the convexity of the problem is deduced, since exp is increasing and convex.

ACKNOWLEDGMENTS

The referees and the associate editor are gratefully acknowledged for their helpful advice and motivating comments, in particular for suggesting extension 3.1.

REFERENCES

[1] M.S. Bazaraa, H.D. Sherali, and C.M. Shetty, Nonlinear programming, theory and algorithms, Wiley, New York, 1993.

[2] S.J. Benkovski, M.G. Monticino, and J.R. Weisinger, A survey of the search theory literature, Nav Res Logistics 38 (1991), 469–491.

[3] S.S. Brown, Optimal Search for a Moving Target in Discrete Time and Space, Oper Res 28 (1980), 1275–1289.

[4] J. de Guenin, Optimum distribution of effort: An extension of the Koopman basic theory, Oper Res 9 (1961), 1–7.

[5] R. Hohzaki and K. Iida, A Concave Maximization Problem with Double Layers of Constraints on the total amount of resources, J Oper Res Soc Jpn 43(1) (2000).

[6] T. Ibaraki and N. Katoh, Resource allocation problems: Algorithmic Approaches, MIT Press, Cambridge, MA, 1988.

[7] K. Iida, Studies on the optimal search plan, Lecture Notes in Statistics, Vol. 70, Springer-Verlag, New York, 1992.

[8] B.O. Koopman, Search and screening: General principle with historical applications, MORS Heritage Series, Alexandria, VA, 1999.

[9] Operations Analysis Study Group (U.S. Naval Academy), Naval operations analysis, 2nd edition, Naval Institute Press, Annapolis, MD, 1977.

[10] J.S. Przemieniecki, Mathematical methods in defense analyses, 2nd edition, American Institute of Aeronautics and Astronautics (AIAA), Washington, DC, 1981.

[11] H.R. Richardson, Search theory, Enc Stat Sci 8 (1988), 314–321.

[12] L.D. Stone, What's happened in search theory since the 1975 Lanchester Prize? Oper Res 37(3) (1989), 501–506.

[13] L.D. Stone, Theory of optimal search, 2nd edition, Operations Research Society of America, Arlington, VA, 1989.

[14] L.D. Stone and H.R. Richardson, Search for targets with conditionally deterministic motion, SIAM J Appl Math 27(2) (1974), 239–255.

[15] W.R. Stromquist and L.D. Stone, Constrained optimization of functionals with search theory applications, Math Oper Res 6(4) (1981), 518–519.

[16] D.H. Wagner, W.C. Mylander, and T.J. Sanders (Editors), Naval operations analysis, 3rd edition, Naval Institute Press, Annapolis, MD, 1999, Chaps. 5 and 8.

[17] A.R. Washburn, Search for a moving target: The FAB algorithm, Oper Res 31 (1983), 739–751.

[18] A.R. Washburn, Search and detection, 2nd edition, Operations Research Society of America, Arlington, VA, 1989.

[19] A.R. Washburn, Finite methods for a nonlinear allocation problem, J Optim Theory Appl 85(3) (1995), 705–726.