Allocation of Search Effort to Optimize Information

J.-P. Le Cadre
IRISA/CNRS
Campus de Beaulieu, 35 042, Rennes, France
e-mail :lecadre@irisa.fr .

Abstract – *Search theory is the discipline that studies the problem of how best to search for an object when the amount of searching efforts is limited and only probabilities of the object's possible position are given. Then, the problem is to find the optimal distribution of this total effort that maximizes the probability of detection. Although the general formalism of search theory will be used subsequently, we shall consider now a radically different problem. More precisely, the problem under consideration is to maximize the expected amount of information about the location of the target by optimally allocating a given search effort. For example, in a reconnaissance problem the aim is not to discover but to locate as precisely as possible (information search).*

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1 Introduction

Search theory is the discipline that treats the problem of how best to search for an object when the amount of searching efforts is limited and only probabilities of the object's possible position are given. Search theory came into being during World War II with the work of B.O. Koopman and his colleagues. Since that time, search theory has grown to be a well-recognized discipline within the field of operations research. An important literature has been devoted to this subject; the interested reader may consult an extensive survey [2], introductive texts [4] and specialized books [5], [6].

A search theory problem is characterized by three pieces of data: (i) the probabilities of the searched object (the "target") being in various possible locations; (ii) the local *detection probability* that a particular amount of local search effort could detect the target; (iii) the total amount of searching effort available. The problem is to find the *optimal* distribution of this total effort that maximizes the probability of detection.

The rapid growth of the search theory literature is chronicled in reference [2]. For instance, the last item (search games) is the primary focus of recent researches, including numerous sub-domains such as : mobile evaders, avoiding target, ambush games, inspection games and tactical games. For moving target problems, decisive progress have been made in developing search strategies that maximize the probability of detecting the (moving) target within a fixed amount of time. In particular, Brown [7] and Washburn [8] have proposed an iterative algorithm in which the motion space and the time frame have been discretized, and the search effort available in each period is infinitely divisible between the grid cells of the target motion space. In this approach, the search effort available in each period is bounded above by a constant that does not depend on the allocations made during any other periods.

However, although the general formalism of search theory will be used subsequently, we shall study a radically different problem. More precisely, the problem under consideration is to maximize the expected amount of information about the location of the target by optimally allocating a given search effort [9]. For example, in a reconnaissance problem the aim is not to discover but to locate as precisely as possible (information search).

2 The one-sided information search problem

This section is devoted to the one-sided problem; which means that only the optimization of the searcher policy is considered. Simultaneous optimization of both searcher and target policies will be the object of section 3 (two-sided search).

2.1 Definitions and basic problem formulation

Assume, at first, that an object is hidden somewhere in a space E, divided in n cells. Prior distribution of the obect is given by a vector $\mathbf{P}$ of prior probabilities p_i ; i.e. : $\mathbf{P} = (p_1, p_2, \cdots, p_n)$, along with $p_i \geq 0 \ \forall i$, $\sum_{i=1}^n \ p_i = 1$. The searcher aim is to maximize its information about possible location of the object by allocating a fixed search effort Φ. The search policy itself is represented by an n-dimensional effort $\mathbf{X} = (x_1, x_2, \cdots, x_n)$, obeying also to the following constraints : $x_i \geq 0 \ \forall i$, $\sum_{i=1}^n \ x_i = \Phi$. Considering the Nakai's formalism, we define the following information functional, denoted $I\left(\mathbf{X}, \mathbf{P}\right)$:

$$I\left(\mathbf{X}, \mathbf{P}\right) = H(\mathbf{P}) - \left[D_{\mathbf{X}} \, H_{\text{det}} + (1 - D_{\mathbf{X}}) \, H(T_{\mathbf{X}} \mathbf{P}) \ \right] . \tag{1}$$

In (1), the various terms have the following meanings :

- $H(\mathbf{P})$ is the prior entropy ($H(\mathbf{P}) \stackrel{\Delta}{=} -\sum_i^n p_i \ln p_i$),
- $D_{\mathbf{X}}$ is the probability to detect (and locate) the object by using the search policy $\mathbf{X}$,
- H_{det} is the entropy about a detected target; i.e. equal to 0 [1] ,
- $T_{\mathbf{X}}\mathbf{P} = (\ (T_{\mathbf{X}}\mathbf{P})_1, \cdots, T_{\mathbf{X}}\mathbf{P})_n \ )$ is the n-dimensional vector of the posterior distribution of the object, given a prior distribution $\mathbf{P}$ assuming it is non detected by a search policy $\mathbf{X}$,
- $H(T_{\mathbf{X}}\mathbf{P})$ is its associated entropy.

Thus in (1), the (right) term inside brackets represents the difference between the prior entropy and the posterior entropy, that is the expected amount of information gained by the policy $\mathbf{X}$.

The exponential assumption is very general and is obtained by the following elementary reasoning [1]. Consider now that the search effort is represented by time (t) and let us denote $q(t)$ ($q(t) = 1 - p(t)$) the probability of non-detection. Denoting w as the "instantaneous" probability of detection, the increment in probability of detection associated with the time increment dt will be $w \ dt$, so that:

$$\begin{array}{l} q(t + dt) = q(t) \ (1 - w \ dt) \\ \text{or :} \\ \frac{d}{dt} q(t) = -w \ q(t) \ \text{ and: } p(t) = 1 - e^{-w\,t} \ . \end{array} \tag{2}$$

A more general presentation of the exponential density (for the probability of detection) can be found in [4]. For instance, elementary calculations yield (with the notations of [4]) :

$$P(\text{det}) = 1 - e^{-w\,L/A} \ , \tag{3}$$

where A denotes the area of the region containing the target, L the length of the search segment and w the visibility parameter (here the sweep width). The term $w\,L/A$ then represents the elementary area coverage.

[1] For a *detected* target its entropy is zero, since no uncertainty remains about source location.

Let us consider now this exponential assumption, then we have :

$$\begin{array}{l} D_{\mathbf{X}} = 1 - \displaystyle\sum_{i=1}^n p_i \ \exp(-w_i \, x_i) \ , \\ (w_i \ \text{ : visibility factor on cell } i) \ , \end{array} \tag{4}$$

$$\left(T_{\mathbf{X}}\mathbf{P}\right)_i = \frac{p_i \exp(-w_i \ x_i)}{\displaystyle\sum_{j=1}^n p_j \exp(-w_j \ x_j)} \ , \ \forall i = 1, \cdots, n.$$

We thus have to deal with the following constrained optimization problem ($f(\mathbf{X}) \stackrel{\Delta}{=} -\mathbf{I}(\mathbf{X}, \mathbf{P})$:

$$\mathcal{P} \left\{ \begin{array}{l} \displaystyle\min_{\mathbf{X}} f(\mathbf{X}) \quad \text{with :} \\ f(\mathbf{X}) = \displaystyle\sum_{i=1}^n p_i \exp(-w_i \ x_i) \ \ln \left[\frac{\sum_{k=1}^n p_k \exp(-w_k \ x_k)}{p_i \exp(-w_i \ x_i)} \right] \ , \\ \text{under the resource constraints :} \\ \displaystyle\sum_{i=1}^n x_i = \Phi \ , \ x_i \geq 0 \quad \forall i \in \{1, \cdots, n\} \ . \end{array} \right. \tag{5}$$

From an algorithmic point of view, the main difference with the "classical" search problem is that now the objective fuctional (i.e. $f(\mathbf{X})$) is not separated relatively to the optimization variables x_i. Thus, even in this elementary framework (stationary target), the difficulty of the optimization problem grows significantly. At a first time, it is worth considering the elementary property of $f(\mathbf{X})$. Noting that the functional of interest is :

$$\begin{array}{l} f(\mathbf{X}) = \displaystyle\sum_{i=1}^n p_i \exp(-w_i \ x_i) \ \ln(\ t_i(\mathbf{X}) \ ) \ , \\ \text{where :} \\ t_i(\mathbf{X}) = \dfrac{\sum_{k=1}^n p_k \exp(-w_k \ x_k)}{p_i \exp(-w_i \ x_i)} \ , \end{array} \tag{6}$$

we obtain successively (Proofs are given in Appendix A):

$$\left| \begin{array}{l} \frac{\partial}{\partial x_i} f(\mathbf{X}) = -p_i \ w_i \ \exp(-w_i \ x_i) \ \ln(t_i(\mathbf{X}) \ ) \ , \\ \frac{\partial^2}{\partial x_i^2} f(\mathbf{X}) = p_i \ w_i^2 \ \exp(-w_i \ x_i) \\ \qquad \times \left[\ln(t_i(\mathbf{X}) \ ) + (t_i(\mathbf{X}) \ )^{-1} - 1 \right] \ , \\ \frac{\partial^2}{\partial x_i \, \partial x_j} f(\mathbf{X}) = p_i \ w_i w_j \ \exp(-w_i \ x_i) \ (t_j(\mathbf{X}) \ )^{-1} \ , \ i \neq j \ . \end{array} \right. \tag{7}$$

Consider for a while the function $g(t) = t^{-1} + \ln t - 1$. Since $g(1) = 0$ and $g'(t) = (t-1)/t^2 \ \geq 0$ for $t \geq 1$, we know that $g(t) \geq 0$ whatever the value t ($t \geq 1$). Therefore $\frac{\partial^2}{\partial x_i^2} f(\mathbf{X}) \geq 0$ is positive whatever the x_i are ($x_i \geq 0$).

Let $\mathcal{H}$ the Hessian matrix associated with $f(\mathbf{X})$ and $\mathbf{Y}$ a generic vector of $\mathbb{R}^n$, from the previous calculation we deduce that :

$$\mathbf{Y}^t \mathcal{H} \mathbf{Y} = \sum_{i=1}^{n} p_i \exp(-w_i x_i) \ g_i(\mathbf{X}, \mathbf{Y}) \ ,$$

where :

$$g_i(\mathbf{X}, \mathbf{Y}) = (w_i y_i)^2 \left[\ln(t_i(\mathbf{X})) + t_i^{-1}(\mathbf{X}) - 1\right] + 2 \sum_{j \neq i} (w_i y_i)(w_j y_j) t_j^{-1}(\mathbf{X}) \ . \tag{8}$$

In general, the sign of the quadratic form $\mathbf{Y}^t \mathcal{H} \mathbf{Y}$ is not necessarily positive and this may constitue a drawback for an iterative optimization. However, a more precise study allows us to conclude that local minima are rare.

2.2 Optimizing the policy

A condition for a policy $\mathbf{X}_o = (x_{1,o}, \cdots, x_{n.o})$ to be optimal is obtained by means of the Kuhn-Tucker necessary conditions, i.e. :

$$\begin{cases} L_i(\mathbf{X}_o) = \mu \quad \text{if} : x_{i,o} > 0 \\ L_i(\mathbf{X}_o) < \mu \ \ \text{if} : x_{i,o} = 0 \ , \\ \text{where :} \\ L_i(\mathbf{X}) = \frac{\partial}{\partial x_i} f(\mathbf{X}). \end{cases} \tag{9}$$

It is not hard to prove [11] that $L_i(\mathbf{X})$ is decreasing with respect to each variable x_k ($k = 1, \cdots, n$). A solution has been proposed by Nakai based on this property [11]. However, the information functional is non separable relatively to the variables x_k ($k = 1, \cdots, n$) and this renders this appraoch unfeasible even for a moderate cell number n. We thus consider the following algorithm :

- **Initialization :**
  $\mathbf{X}_0$ such that $x_o(i) \geq 0$, $i = 1, \cdots, n$ and $\sum_{i=1}^{n} x_o(i) = \Phi$.

- **Master programm :**
  Solve the following *linear* programm :

$$\begin{cases} \max \ z \ , \\ \text{such that} : z \leq f(\mathbf{X}_j) - \sum_{i=1}^{n} (u_i x_{i,j}) + v h(\mathbf{X}_j) \\ \qquad \text{for } j = 0, \cdots, k-1 \ , \\ u_1 \geq 0, \cdots, u_n \geq 0 \ , \\ h(\mathbf{X}_j) = \sum_{i=1}^{n} x_{i,j} - \Phi \ . \end{cases} \tag{10}$$

  For this linear programm the unknowns are z, the *inequality* Lagrange multipliers $\mathbf{u} = (u_1, \cdots, u_n)$ and the constraint parameter v. Note that all the u_i are positive, while the sign of v is undetermined (equality constraint). It may be solved quite efficiently by a Simplex algorithm even for a large value of n.

- **Second Step :**
  Admitting that the Master programm has been solved, we obtain a new vector $\mathbf{u}_k$, as well as a new value v_k. The index k corresponds to the general iteration index. Consider the following *unconstrained* problem :

$$\min_{\mathbf{X}} \ f(\mathbf{X}) - \sum_{i=1}^{n} (u_{i,k} x_i) + v_k \ h(\mathbf{X}) \ . \tag{11}$$

  Practically, this optimization step may be solved by a standard algorithm like DFP or BFGS. After convergence of this algorithm, a new vector $\mathbf{X}_k$ is obtained and added to the list $[(\mathbf{X}_1, \cdots, \mathbf{X}_{k-1}), \mathbf{X}_k]$. Return to the Master Programm.

Actually, the choice of convenient stepsizes for the iterative algorithm involved in the Second Step may be quite critical, especially if n is large. For that purpose, the Goldstein rule performs quite satisfactorily. Denoting $q(t)$ the functional to be minimized (see Second Step) along a descent direction, this rule takes the following form:

Choose 2 coefficients m_1 and m_2 with
$0 < m_1 < 1/2 < m_2 < 1$, perform the following test :
a) if $m_2 \ q'(0) \leq (q(t) - q(0)\ ) \leq m_1 \ q'(0) \rightarrow$
stop the stepsize search ,
b) if $m_1 \ q'(0) < (q(t) - q(0)\ ) \rightarrow$ decrease the stepsize ,
c) if $(q(t) - q(0)\ ) < m_2 \ q'(0) \rightarrow$ increase the stepsize . (12)

The whole algorithm performs quite satisfactorily, even for rather large values of the parameter n (up to 400). For larger values of n, specific algorithms are more relevant [12] [13] (interior-point algorithms).

2.3 Some extensions

A natural extensions consist in considering the use of multiple detection resources. Let us denote $\mathbf{X}^1, \cdots, \mathbf{X}^m$ the search policies associated with resources 1 to m. The visibility factor $\{w_i^k\}$ represents the visibility factor associated with the cell i and resource k. They are related to operational considerations. Budget constraints take also the following forms :

$$\mathcal{B}\begin{cases} \sum_{i=1}^{n} x_i^1 + \cdots + \sum_{i=1}^{n} x_i^m = \Phi \\ \text{multimode resources , or :} \\ \sum_{i=1}^{n} x_i^1 = \Phi_1, \cdots, \sum_{i=1}^{n} x_i^m = \Phi_m \ , \\ \text{noncooperative resources .} \end{cases} \tag{13}$$

More generally, we can consider budget constraints of the form [14]:

$$\mathcal{A}\mathcal{X} = \mathcal{M}(\Phi_1, \cdots, \Phi_m) \ , \tag{14}$$

where the $\mathcal{A}$ and $\mathcal{M}$ matrices have compatible dimensions and represents operational requirements (e.g. total amount of a given resource, choice of mode, etc.). The information functional is defined as previously, i.e. :

$$I\left(\mathbf{X},\mathbf{P}\right) = H(\mathbf{P}) - \left[D_{\mathbf{X}}\, H_{\text{det}} + (1-D_{\mathbf{X}})\, H(T_{\mathbf{X}}\mathbf{P}) \ \right] \ ,$$

where, in the case of independent detections :

$$\begin{array}{l} (1-D_{\mathbf{X}}) = \sum_{j=1}^{n} p_j \prod_{k=1}^{m} \exp(-w_j^k \ x_j^k) \ , \\ (T_{\mathbf{X}}\mathbf{P})_i = \dfrac{p_i \prod_{k=1}^{m} \exp(-w_i^k \ x_i^k)}{\sum_{j=1}^{n} p_j \prod_{k=1}^{m} \exp(-w_j^k \ x_j^k)} \ , \ \forall i = 1, \cdots, n. \end{array} \tag{15}$$

So that we have to deal with the optimization of the following functional :

$$\begin{array}{l} f(\mathbf{X}) = \sum_{i=1}^{n} \left[p_i \prod_{k=1}^{m} \exp(-w_i^k \ x_i^k) \right] \ \ln(t_i(\mathbf{X}) \ ), \\ \text{where :} \\ t_i(\mathbf{X}) = \dfrac{\sum_{j=1}^{n} p_j \prod_{k=1}^{m} \exp(-w_j^k \ x_j^k)}{p_i \prod_{k=1}^{m} \exp(-w_i^k \ x_i^k)} \ . \end{array} \tag{16}$$

Once again, this definition is quite arbitrary. For instance, the above expression of the non-detection probability means that the target is said undetected if it has not been detected by any detection device, associated with resources $\{\Phi_1, \cdots, \Phi_m\}$. This corresponds to an "and" rule for non-detection. Other definitions are classical. For example, it is possible to consider that the target is said detected if it is detected by, at least, k elementary resources, i.e. :

$$\begin{array}{l} P_{td} = \sum_{i=k}^{n} \left\{ \left(\sum_{p=0}^{i-k} (-1)^p \ C(i,p) \right) \ \left(\sum_{C_{i,n}} \left[\prod_j Pd_j \right] \right) \right\} \ , \\ \text{where :} \\ C(i,p) = \frac{i!}{p!\,(i-p)!} \ . \end{array} \tag{17}$$

In (17), the term $\sum_{C_{i,n}} \left[\prod_j Pd_j\right]$ is the sum of all the possible products of i elementary detections that can be formed from the whole elementary detections. In all the cases, the optimization problem must be solved by a general algorithm (see section 2.2). Multiple resource constraints (14) add some difiifficulty, however quite moderate.

A straightforward extension to moving target is to consider a conditionnally deterministic assumption [10] [15] about the target motion. Note that this problem is basically defined in a multiperiod framework. We have then to deal with the following (primal) optimization problem :

$$\mathcal{P}\begin{cases} \max \sum_{\theta\,\in\,\Theta} \mathcal{I}\left(x_{1,\theta}, x_{2,\theta}, \cdots, x_{p,\theta}\right) \ , \\ \text{where :} \\ \mathcal{I}\left(x_{1,\theta}, x_{2,\theta}, \cdots, x_{p,\theta}\right) \triangleq \\ \quad I\left(p(x_{1,\theta}),\, p(x_{2,\theta}), \cdots,\, p(x_{p,\theta})\,\right) \ , \\ \text{under the resource constraints :} \\ \sum_{\theta\,\in\,\Theta} \ x_{1,\theta} \ + x_{2,\theta} \cdots + x_{p,\theta} = \Phi \ , \\ x_{1,\theta} \geq 0 \ , x_{2,\theta} \geq 0 \ , \cdots, x_{p,\theta} \geq 0 \ , \ \forall\, \theta \in \Theta \ . \end{cases} \tag{18}$$

In (18), $x_{k,\theta}$ represents a search effort, affected to the cell indexed by the parameter θ, at the search period k. The index k takes its values in the subset $\{1, \cdots, n\}$. The parameter θ takes its values in a multidimensional space or set (denoted Θ), characterizing the target trajectory (e.g. initial position and velocity) and the p-dimensional vector $\mathbf{X}_\theta \triangleq (x_{1,\theta}, x_{2,\theta}, \cdots, x_{p,\theta})^*$ represents the effort vector associated with the target trajectory (or track) indexed by θ. Furthermore, $p(x_{k,\theta})$ is the elementary probability of detection in the cell (k,θ), for a search effort $x_{k,\theta}$; while I is a given differentiable function. The following simple remarks are then fundamental :

- the functional $\mathcal{I}\left(x_{1,\theta}, \cdots, x_{n,\theta}\right)$ is a differentiable functional of the variables $x_{k,\theta}$,
- the constraints are qualified since they are linear,

Again, the previous framework can be used to optimize the information functional. For the sake of brevity, it is omitted here since it is essentially similar to section 2.2 . A natural extension is to consider Markovian targets; i.e. targets whose movements have the Markov property. The classical optimization framework we used previously is here useless due to the complexity of elementary events. Instead, we shall use Brown's approach [7] [8], where a sequence of search plans is generated incrementally.

The target is moving among a finite number of cells and its path is decribed by $\omega = (\omega_1, \omega_2, \cdots, \omega_T) \in C^T$. The searcher starts with a function $g \ : \ C^T \rightarrow [0,1]$, where $g(\omega)$ is the probability that the target takes the path ω. During the t-th period, the searcher has Φ_t units of search efforts which he may divide between the cells of the t-th period in arbitrary proportions. Thus, the search effort distribution at time t may be described by a vector X_t, with components $x(c,t)$, giving the search effort placed in cell c at time t. We assume that the searches at distinct time periods are statistically independent, so that the probability that a target with path ω be undetected is :

$$1 - D_{\mathbf{X}} = \sum_{\omega \in \Omega} g(\omega) \exp\left[-\sum_{t=1}^{T} w_{\omega_t,t}\, x_{\omega_t,t}\right] ,$$

and :

$$T_{\mathbf{X}}(\mathbf{P}) = \frac{g(\omega)\exp\left[-\sum_{t=1}^{T} w_{\omega_t,t}\, x_{\omega_t,t}\right]}{\sum_{\omega \in \Omega} g(\omega)\exp\left[-\sum_{t=1}^{T} w_{\omega_t,t}\, x_{\omega_t,t}\right]} . \tag{19}$$

We have now to consider the following information functional :

$$f(\mathbf{X}) = \sum_{\omega \in \Omega} g(\omega) \exp\left[-\sum_{t=1}^{T} w_{\omega_t,t}\, x_{\omega_t,t}\right] \ \ln[t_\omega(\mathbf{X})] ,$$

where :

$$t_\omega(\mathbf{X}) = \frac{\sum_{\omega \in \Omega} g(\omega)\exp\left[-\sum_{t=1}^{T} w_{\omega_t,t}\, x_{\omega_t,t}\right]}{g(\omega)\exp\left[-\sum_{t=1}^{T} w_{\omega_t,t}\, x_{\omega_t,t}\right]} , \tag{20}$$

and our problem is to minimize $f(\mathbf{X})$ with the constraints $\{x(c,t) \geq 0 \ , \ \forall t, \forall c\}$ and $\sum_{c_t \in C_t} x(c_t,t) \leq \Phi_t$. Then, it is worth considering the following "factorization" of $f(\mathbf{X})$:

$$f(\mathbf{X}) = \sum_{c} \left[\sum_{\omega \in \Omega:\, \omega_t = c} g(\omega) \exp\left(-\sum_{t=1}^{T} w_{\omega_t,t}\, x_{\omega_t,t}\right) \ln(t_\omega(\mathbf{X}))\right] \tag{21}$$

Now, thanks to the Markov prperty, we have :

$$g(\omega) = g_1(\omega_1) t(\omega_1,\omega_2) \cdots t(\omega_{T-1},\omega_T) g_T(\omega_T) , \tag{22}$$

where $t(\omega_t, \omega_{(t+1)})$ is the probability of transition fronm the cell ω_t to the cell $\omega_{(t+1)}$, and $g_1(\omega_1)$, $g_T(\omega_T)$ represent priors about target trajectory. An instrumental idea consists then in considering that all the search plan is fixed, *except* for period τ (denote this plan $\tilde{\mathbf{X}}_\tau$, and to consider the following definitions of Forward and Backward (denoted F and B) quantities :

$$\begin{aligned}
F(c, \tilde{\mathbf{X}}_\tau) &= \sum_{\omega_t = c} g_1(\omega_1) t(\omega_1,\omega_2) \cdots t(\omega_{\tau-1}, c) , \\
&\times \exp\left(-\sum_{t=1}^{\tau-1} w_t(\omega_t) x_{\omega_t,t}\right) , \\
B(c, \tilde{\mathbf{X}}_\tau) &= \sum_{\omega_t = c} g_T(\omega_T) t(\omega_T,\omega_{T-1}) \cdots t(\omega_{\tau+1}, c) \\
&\times \exp\left(-\sum_{\tau+1}^{T} w_t(\omega_t) x_{\omega_t,t}\right) .
\end{aligned} \tag{23}$$

The functional [2] $f(\mathbf{X}_\tau)$ takes then the following form :

$$\begin{aligned}
f(\mathbf{X}_\tau) &= \sum_{c \in C_\tau} F(c, \tilde{\mathbf{X}}_\tau) B(c, \tilde{\mathbf{X}}_\tau) \ \exp(-w_{c,\tau} x_{c,\tau}) , \\
&\times \ln\left[\sum_{c \in C_\tau} F(c, \tilde{\mathbf{X}}_\tau) B(c, \tilde{\mathbf{X}}_\tau) \ \exp(-w_{c,\tau} x_{c,\tau})\right] , \\
&- \sum_{c \in C_\tau} F(c, \tilde{\mathbf{X}}_\tau) B(c, \tilde{\mathbf{X}}_\tau) , \\
&\times \ln\left[F(c, \tilde{\mathbf{X}}_\tau) B(c, \tilde{\mathbf{X}}_\tau) \ \exp(-w_{c,\tau} x_{c,\tau})\right] .
\end{aligned} \tag{24}$$

Assuming that the quantities $F(c, \tilde{\mathbf{X}}_\tau)$ and $B(c, \tilde{\mathbf{X}}_\tau)$ are known, the vector $\tilde{\mathbf{X}}_\tau$ is determined by the algorithm p-resented in section 2.2. . Note that the above quantities represent the probabilities that a target whose trajectory crsos the cell c at the time-period τ be undetected, both before τ and after τ The only remaining problem is the calculation of $F(c, \tilde{\mathbf{X}}_\tau)$ and $B(c, \tilde{\mathbf{X}}_\tau)$. This is achieved by means of Forward and Backward recursions, i.e. :

$$\begin{aligned}
F(c, \tilde{\mathbf{X}}_t) &= \sum_{d \in C} F(d, \tilde{\mathbf{X}}_{t-1}) \exp(-w_{d,t-1} x_{d,t-1}) t(d,c) , \\
B(c, \tilde{\mathbf{X}}_t) &= \sum_{d \in C} F(d, \tilde{\mathbf{X}}_{t+1}) \exp(-w_{d,t+1} x_{d,t+1}) t(d,c) .
\end{aligned} \tag{25}$$

The description of the algorithm is now complete. Despite the huge number of paths under consideration, we have only to solve, at each time period, a stationary problem. The only price to pay is to perform multiple iterations of the whole algorithm. Thus, it is *quite feasible* even if, practically, both the variable number and the number of paths may be tremenduously high. This is due to the sequential nature of the Brown's algorithm.

3 Two-sided search

Up to now, our efforts have been exclusively devoted to the one-sided search, which means that decisions are

[2] The time index tau means that *only* the search plan associated wit tau is considered.

only made by the searcher. For the two-sided search, game theory is the natural framework. If the objective functional is the probability of detection then the two-sided search problem has an explicit and simple solution [11]. The simple nature of the solution is certainly due to the separable nature of the optimizations, which means that all the variables (i.e. the search efforts and the target priors) play separable roles. Furthermore, notice that the optimal searcher and target strategies are proportional. Quite intuitively, this strategy is such that the product $x_i^* p_i^*$ remains constant.

For the information search, the detection functional is replaced by the information functional $\mathcal{I}(\mathbf{P},\mathbf{X}) = (1 - D_{\mathbf{X}}(\mathbf{P})\ )\ H(T_{\mathbf{X}}(\mathbf{P})$. The elementary problem consists to find the vectors $\mathbf{P}^*$ and $\mathbf{X}^*$, solutions of the following min-max problem :

$$\begin{aligned}&\mathcal{I}(\mathbf{P}^*,\mathbf{X}) \le \mathcal{I}(\mathbf{P}^*,\mathbf{X}^*) \le \mathcal{I}(\mathbf{P},\mathbf{X}^*)\ ,\\ &\forall(\mathbf{X},\mathbf{P})\ :\ \sum_i p_i = 1\ ,\ \sum_i x_i = \Phi\ .\end{aligned} \tag{26}$$

Alternatively, we see that the couple $\{\mathbf{P}^*,\mathbf{X}^*\}$ is a saddle point for the functional $\mathcal{I}(\mathbf{P},\mathbf{X})$. Assuming that no local minimum of $\mathcal{I}(\mathbf{X})$ does exist, existence and unicity of this saddle point may be proved.The following KKT conditions are easily derived (see [11]) and give some insights about optimal policies. Note that it involves *two positive* Lagrange multipliers (ξ and μ).

$$\begin{aligned}&K_i(\mathbf{P}^*,\mathbf{X}^*) : \begin{cases} =\xi & \text{if} \quad p_i^*=0\\ <\xi & \text{if} \quad p_i^*>0\end{cases}\\ &p_i^* w_i\ K_i(\mathbf{P}^*,\mathbf{X}^*) : \begin{cases} =\mu & \text{if} \quad x_i^*>0\\ <\mu & \text{if} \quad x_i^*=0\end{cases}\\ &\text{where :}\\ &K_i(\mathbf{P},\mathbf{X}) = \exp(-w_i x_i)\ln\left[t_i(\mathbf{X},\mathbf{P})\right]\ .\end{aligned} \tag{27}$$

Inequalities in (27) are deduced from the decreasing property (as a function of p_i, and as a function of x_i) of $K_i(\mathbf{P},\mathbf{X})$. Even if these conditions are quite general, they are not truly enlightening. However, the following elementary consequences of these conditions [11] allow us to develop an algorithmic approach.

1. If $p_i^*=0$, then $x_i^*=0$.
2. If $p_i^* > 0$, then two cases must be considered according to the value of x_i^*. If $x_i^*>0$, then we have $p_i^*\ w_i = \frac{\mu}{\xi}$. If $x_i^*=0$, then $\ln\left[\sum_{k=1}^n p_k^*\exp(-w_k x_k^*)\right] = \xi\, p_i^*$, so that p_i^* is independent of the index i.

So, p_i^* is either equal to $\alpha\, w_i^{-1}$ (if $x_i^*>0$), or to a constant β (if $x_i^*=0$). Notice that β may be equal to zero. This gives us [11]the general form of the vectors $\mathbf{P}^*$ and $\mathbf{X}^*$:

$$\begin{aligned}&\text{Assume (without loss of generality) that :}\\ &w_1 \ge w_2 \cdots w_n \quad ,\ \text{then :}\\ &\mathbf{P}^* = \left(\alpha\, w_l^{-1},\cdots,\alpha\, w_1^{-1},\beta,\cdots,\beta\right)\ ,\\ &\mathbf{X}^* = \left(x_1^*,\cdots,x_l^*,0,\cdots,0\right)\ ,\\ &\text{with the constraints :}\\ &\sum_{i=1}^l x_i^* = \Phi\ \text{, and : } (n-l)\beta + \alpha\sum_{i=1}^l W_i^{-1} = 1\ .\end{aligned} \tag{28}$$

Thus, for $\mathbf{P}^*$ there is only one undertermined parameter. Thus, the algorithm consists practically in using the algorithm of section 2.2 in conjunction with (28). This two-sided formulation may be extended to the multiple resource case and to Markovian targets (see section 2.3), even if it becomes considerably more intricated. In fact, this requires to analyze a great variety of situations.

4 Results

The aim of this section is to provide some examples illustrating the behavior of the algorithm we have developed for optimizing the information search. The prior about target location (parametrized by a couple $\{x,y\}$) is given by the following formula :

$$\begin{aligned} p(x,y) &= \exp\left(-\frac{1}{2}\left[(x-1)^2+(y-1)^2\right]\right)\ ; \\ &\quad + \exp\left(-\frac{1}{2}\left[(x-5)^2+(y-5)^2\right]\right)\ ,\end{aligned} \tag{29}$$

illustrated by fig. 1, while the value of the visibility factor is given by the formula $w(x,y) = 1/(\sqrt{x^2+y^2})$. These continuous values are discretized on a 10×10 grid. The total number of cells is thus equal to 100.

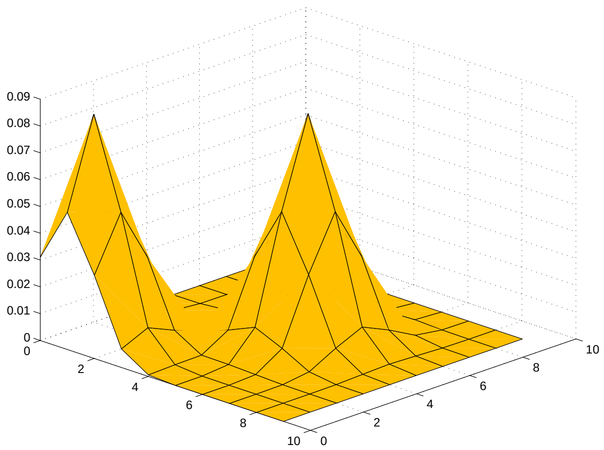

Figure 1: Values of the prior $p(x,y)$

We present then (see fig.4) the values of the information search efforts; for the prior given in fig. 1 and the visibility factor given by the above formula. The value of the total search effort Φ is $\Phi = 60$. We use the cutting plane algorithm presented in section 2.2. Convergence needs some iterations of the master programm, so that the computation time is about 500 sec. in this case. The constraints are satisfied since $\sum_{i=1}^{100} x_{i,o} = 59.43$, all the $x_{i,o}$ are positive; while the value of the information functional $\mathcal{I}$ is equal to 1.788 (for $\Phi = 30$, we found $\mathcal{I} = 2.211$). The algorithm has been initialized by a vector filled of very small values.

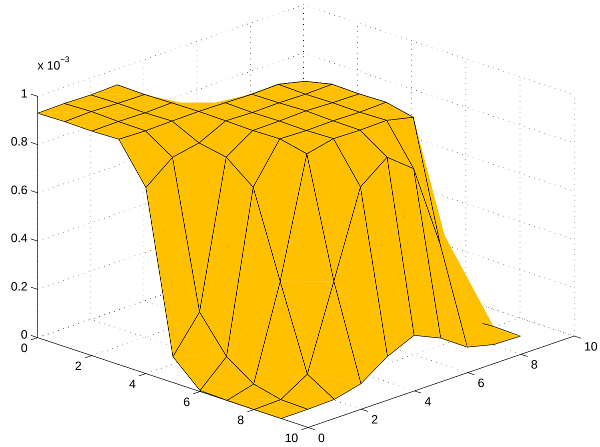

Figure 3: KKT conditions, values of $L_i(\mathbf{X}_0)$.

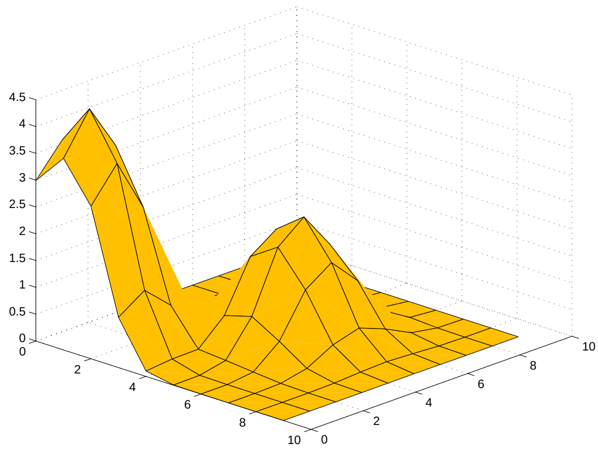

Figure 2: Values of the information search efforts, $\Phi = 60$.

KKT conditions are illustrated by fig.3, in this case the visibility factor is constant ($w(x,y) = 1\,, \forall (x,y)$). The values of the partial derivatives of the Lagrangian (i.e. $L_i(\mathbf{X}_0)$ are plotted as a function of the cell index ($\mathbf{X}_o$ value of the information searc vector at convergence). We see that $L_i(\mathbf{X}_0)$ are (almost) equal together when the search efforts are strictly positive and lower for zero values of the efforts. Finally, we present the values of the information functional for optimized values of the search efforts versus the total amount of search effort. The prior is as above but, this time, the visibility function is more complicated ($w(x,y) = \cos^2(\sqrt{x^2+y^2}) \exp(-0.2\sqrt{x^2+y^2})$). The total amount of search effort Φ is ranging from 0 to 500). We can notice that the information functional decreases fastly at first and then that the decrease becomes quite slower.

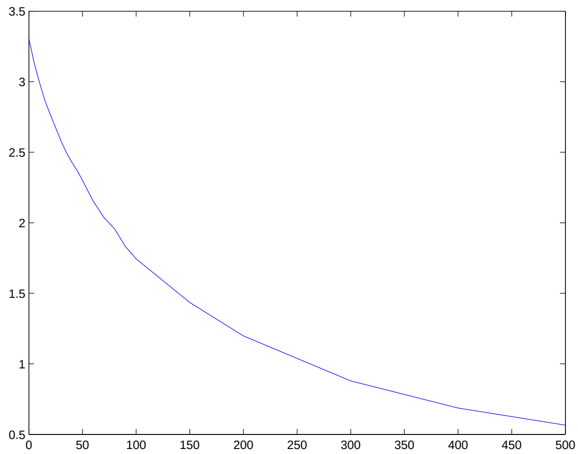

Figure 4: Values of the Information functional $f(\mathbf{X}_0)$ versus the total amount Φ.

5 Conclusion

The aim of this paper is to study the management of the information search. Despite great similarities with the "classical" search problem, it involves rather intricate optimization problems, for which algorithmic solutions have been detailed and tested. For the eseential, difficulties arise from the non-separability of the information functional. As it is usual in search theory, once the elementary information search problem (monoperiod, monoresource) has been solved, then numerous extensions (multiple types of resources, two-sided, Markovian target) can be treated. However, the algorithmic complexity of the elementary step greatly increase the difficulty of the extensions.

A Appendix

Let us now detail the calculation of $\frac{\partial}{\partial x_j} f(\mathbf{X})$:

$$\frac{\partial}{\partial x_j} f(\mathbf{X}) = -p_j w_j \exp(-w_j x_j) \ln(t_j(\mathbf{X})) \;, \\ +\sum_i^n p_i w_i \exp(-w_i x_i) \frac{\partial}{\partial x_j} \ln(t_i(\mathbf{X})) \;. \tag{30}$$

It remains to calculate the term $\frac{\partial}{\partial x_j} \ln(t_j(\mathbf{X}))$, we have :

$$\frac{\partial}{\partial x_j} \ln(t_j(\mathbf{X})) = -\frac{w_j p_j \exp(-w_j x_j)}{\sum_{k=1}^n p_k \exp(-w_k x_k)} + w_i \; \delta_{i,j} \;, \tag{31}$$

where $\delta_{i,j}$ stands for the Kronecker symbol (i.e. $\delta_{i,j} = 1$ if $i = j$; 0 else). Therefore :

$$\sum_i^n p_i w_i \exp(-w_i x_i) \frac{\partial}{\partial x_j} \ln(t_i(\mathbf{X})) \\ = -w_j p_j \exp(-w_j x_j) \frac{\sum_{i=1}^n p_i \exp(-w_i x_i}{\sum_{k=1}^n p_k \exp(-w_k x_k)} \\ +\sum_{i=1}^n [w_i \delta_{i,j} p_i \exp(-w_i x_i] = 0 \;. \tag{32}$$

ending the proof.

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