Optimal Distribution of Continuous Search Effort for Detection of a Target in a Min-Max-Game Context

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Abstract – *Analytical resolution of search theory problems, as formalized by B.O. Koopman, may be applied with some model extension to various resource management and data fusion issues. Such method is based on a probabilistic prior about the target. Even so, this approximation forbids any reactive behavior of the target. As a preliminary step towards reactive target study stands the problem of resource placement under a min-max game context. This paper is related to Nakai's work about the game placement of resources for the detection of a stationary target. However, this initial problem is extended by adding new and more general constraints, allowing a more subtle modeling of the target and resource behaviors.*

Keywords: Sensor & Resource Management, target detection, Search game, Search theory, Resource allocation.

Notations

- $\varphi(x)$: Search effort,
- ϕ_o: Total amount of search effort,
- $\alpha(x)$: Probabilistic target distribution,
- A_o: Total target probability,
- $p_x(\varphi(x))$: Conditional detection probability.

1 Introduction

The initial framework of Search Theory [3][1][2], introduced by B.O. Koopman and his colleagues, sets the general problem of the detection of a target in a space, in view of optimizating the detection resources. A thorough extension of the prior formalism has been made by Brown towards the detection at several periods of search [4][5]. These simple but meaningful formalism were also luckily applied to various resource management and data fusion issues [6]. But, in all these problems, a probabilistic prior on the target was required. In addition, in case of moving target problems, a Markovian hypothesis is necessary for algorithmic reasons. While this formalism is sufficient for almost "passive" targets, it is useless when a target has a complex (and realistic) move. In a military context especially, the behavior of the "interesting" targets is not neutral and cannot be modeled by a simple probabilistic prior. A conceivable way for enhancing the prior on the target in a manner that involves more properly the complexity or the reactiveness of the target, is to consider a min-max game version of Koopman optimization problems. Nakai presented and solved in [7] a game with placement of resources for the detection of a stationary target. In this work the constraints on game were given by the available placement of target and detection resources. Thus, constraints were defined at the *pure strategy* level. The purpose of this paper is to present an extension of Nakai's game by addition of new constraints defined on the set of available mixed strategies. In other words, constraints are now defined at the *mixed strategy* level. Before explaining properly the extended problem, we intend to give in this introduction a short description of Nakai's game.

Definitions: The searcher want to detect a target positioned in a spatial search space E. To perform this detection, the searcher has available a total amount of (detection) resources ϕ_o. Theses resources may be shared between each cell x of the search space E. Detection on cell x is a known function of the search effort put on x. For $x \in E$, the variable $\varphi(x)$ denotes the local amount of resources placed on cell x. A constraint naturally holds for the resources :

$$\int_E \varphi(x)\,dx \leq \phi_o\ .$$

Since detection is better when the whole resources are used, the previous constraint may be replaced by the more mathematically suitable one:

$$\int_E \varphi(x)\,dx = \phi_o\ . \tag{1}$$

The set of valid sharing functions φ is thus defined by:

$$\mathcal{R}(\phi) = \left\{ \varphi \in {\mathbb{R}^+}^E \Big/ \int_E \varphi = \phi_o \right\} .$$

When local resource $\varphi(x)$ is used on cell x and target is located on x, the probability of non detection is given by value $p_x(\varphi(x))$, a conditional probability. This probability may depend upon x, since practically visibility and resource efficiency vary with the concerned cell. For x fixed, p_x decreases with the effort used and $p'_x < 0$. The detection follows the rule of decreasing return, so that p'_x increases strictly with φ. On the other hand, the target have the choice between available positions $\mathbb{T} \subset E$. Then, a game occurs between the searcher and the target. The searcher attempts to minimize the probability of non detection by optimizing the search resource sharing φ, while the target aim is to maximize the probability of non detection by choosing his position. The value of the game is given by $p_\Theta(\varphi(\Theta))$, for a target strategy Θ and a searcher strategy φ. This problem was solved by Nakai [7]. Since p is convex, it appears that the game is convex. Thus, there is a mixed optimal strategy for the target and a pure optimal strategy for the searcher. A mixed strategy for the target is given by a density probability α on the target position, with property $\alpha(E \setminus \mathbb{T}) = 0$. We denote:

$$\mathcal{P}(\mathbb{T}) = \left\{ \alpha \in {\mathbb{R}^+}^E \Big/ \alpha(E \setminus \mathbb{T}) = 0 \text{ and } \int_E \alpha = 1 \right\} .$$

the set of such probabilities. For strategies (α, φ), the value of the game is then given by the average :

$$P_{nd}(\alpha, \varphi) = \int_E \alpha(x) p_x(\varphi(x))\, dx .$$

An optimal (min-max) couple of strategies (α_o, φ_o) is also defined by :

$$\begin{cases} \alpha_o = \arg \max\limits_{\alpha \in \mathcal{P}(\mathbb{T})} \min\limits_{\varphi \in \mathcal{R}(\phi_o)} \displaystyle\int_E \alpha(x) p_x(\varphi(x))\, dx , \\ \\ \varphi_o = \arg \min\limits_{\varphi \in \mathcal{R}(\phi_o)} \max\limits_{\alpha \in \mathcal{P}(\mathbb{T})} \displaystyle\int_E \alpha(x) p_x(\varphi(x))\, dx . \end{cases}$$

Two optimality conditions are obtained, by differentiation around the optimal strategies :

$$\begin{cases} \alpha_o(x)\, p'_x(\varphi_o(x)) = \eta , \text{ when } \alpha_o(x) > \dfrac{\eta}{p'_x(0)} \\ \\ \varphi_o(x) = 0 , \text{ else} \end{cases} \tag{2}$$

and

$$\exists \lambda \in \mathbb{R}^+,\ \alpha_o(x) > 0 \Longrightarrow p_x(\varphi_o(x)) = \lambda . \tag{3}$$

We can recognize in (2) the classical optimality equation of de Guenin. By use of these equations, a mathematical solution of the problem is built. The first step is to verify the obviously intuitive result:

$$x \in \mathbb{T} \Longleftrightarrow \Big[\alpha_o(x) > 0 \text{ and } \varphi_o(x) > 0 \Big] . \tag{4}$$

Then, the combination of equations (1) and (3) yields $\int_\mathbb{T} p_x^{-1}(\lambda)\, dx = \phi_o$. Defining the function $\mathbb{P}$ by:

$$\mathbb{P}^{-1}(\lambda) = \int_\mathbb{T} p_x^{-1}(\lambda)\, dx$$

yields $\lambda = \mathbb{P}(\phi_o)$. Results $\alpha_o(x) = \eta / p'_x(p_x^{-1}(\mathbb{P}(\phi_o)))$ and $\varphi_o(x) = p_x^{-1}(\mathbb{P}(\phi_o))$ are finally derived from equations (2), (3) and (4). Since α_o is a probability density, it follows that $\int_\mathbb{T} \alpha_o(x)\, dx = 1$. This property permits to find the dual variable η. After simplification, the simple formula $\eta = \mathbb{P}'(\phi_o)$ is obtained. Finally, the min-max optimal strategies (α_o, φ_o) are simply given by:

$$\boxed{\forall x \in \mathbb{T},\ \begin{cases} \alpha_o(x) = (p_x^{-1} \circ \mathbb{P})'(\phi_o) \\ \\ \varphi_o(x) = (p_x^{-1} \circ \mathbb{P})(\phi_o) \end{cases}} \tag{5}$$

The Nakai game problem thus admits a mathematical solution. In fact, the game remodeling of the search problem yields some complexity simplification. In comparison, it is noteworthy that the equivalent Koopman search problem (i.e. with a probabilistic prior on the target) is not analytically solvable in general. This problem, known as de Guenin's search problem, involves a new constant of the problem, say α_p the known probabilistic prior on the target position. Game aspects disappear and de Guenin's problem is a simple optimization:

$$\varphi_o = \arg \min_{\varphi \in \mathcal{R}(\phi_o)} \int_E \alpha_p(x) p_x(\varphi(x))\, dx .$$

The fast de Guenin's algorithm relies on a bisectional method for computing the optimal solution. However, there is no general mathematical solution. In the next section, an extension of Nakai problem will be considered. It is a min-max game, where constraints are given on the target mixed strategies. It will be shown that such problem is a generalization of both Nakai game and de Guenin's problem, but is much more complex than these two parent problems. In particular, no equivalent of crucial property (2) holds anymore. New properties will be established to handle these difficulties and an *original* algorithm will be presented.

2 Bounding constraints

In Nakai game, the prior on target is given by the set of available target positions. This hypothesis constitutes a prior more general and more flexible than a probabilistic density on target position, in particular for modeling uncertain targets. Nevertheless, it does not allow sufficient refinement, for modeling target behavior. For example, when the detection occurs after a preliminary target move, it is wise to handle target motion modeling. Itself depending on the target reactiveness capabilities, it follows that some final positions are more probable than other. To model this fact, we will simply introduce an up and down bounding on the probability associated with the target mixed strategy.

Similarly, it is also possible to define an up and down bounding on the resources sharing functions. Doing so involves a symmetrization of our problem. However, such bounding constraints on resources have a physical meaning. It implies a minimum and a maximum of resource affectation on each cell of the space search. Definitions have now to be clarified.

Definition: The placement of the target and the search are accomplished on a space E. Each element $x \in E$ is called a cell. The target mixed strategy is represented by a density function α defined on E. Function α is a variable of the problem. The summation of α on E is known and is denoted A_o. The following constraint then holds:

$$\int_E \alpha(x)\,dx = A_o\;.$$

Since α is a density probability, A_o generally equals 1. Two bounding functions α_1 and α_2 with property $\alpha_1 \leq \alpha_2$ are given. These functions are constants of the problem and yield a bounding constraint on the mixed target strategy:

$$\alpha_1 \leq \alpha \leq \alpha_2\;.$$

The searcher pure strategy is represented by a resource sharing function φ defined on E. Function φ is also a variable of the problem. The total amount of resources ϕ_o is fixed, so that :

$$\int_E \varphi(x)\,dx = \phi_o\;.$$

Also given are two bounding functions φ_1 and φ_2 with property $\varphi_1 \leq \varphi_2$. These functions are constants of the problem and yield a bounding constraint on the pure search strategy:

$$\varphi_1 \leq \varphi \leq \varphi_2\;.$$

For each cell x, a decreasing and convex non detection function p_x is defined. The value $p_x(\varphi(x))$ represents the conditional probability of non detection, when the target is located on cell x. The value of game for a couple of strategies (α, φ) is given by the averaged probability of non detection:

$$P_{nd}(\alpha, \varphi) = \int_E \alpha(x) p_x(\varphi(x))\,dx\;.$$

Again, since the game is convex, there is a couple of optimal strategies involving a mixed strategy for the target and a pure strategy for the searcher. The associated minmax optimization problem stands as follow:

Find:

$$\alpha_o = \arg\max_\alpha \min_\varphi \int_E \alpha(x) p_x(\varphi(x))\,dx$$

and

$$\varphi_o = \arg\min_\varphi \max_\alpha \int_E \alpha(x) p_x(\varphi(x))\,dx\;,$$

under constraints:

$$\int_E \alpha(x)\,dx = A_o\;,\; \int_E \varphi(x)\,dx = \phi_o\;,$$
$$\forall x \in E,\; \alpha_1(x) \leq \alpha(x) \leq \alpha_2(x)\;,$$
$$\forall x \in E,\; \varphi_1(x) \leq \varphi(x) \leq \varphi_2(x)\;.$$

3 Optimality equations

Considering an optimal couple of strategies (α_o, φ_o) as a saddle point for the game value $P_{nd}(\alpha, \varphi)$, two optimality equations are obtained by variational means.

de Guenin's equation: Since (α_o, φ_o) is a saddle point, it appears that:

$$\varphi_o \in \arg\min_\varphi P_{nd}(\alpha_o, \varphi)\;.$$

Constraints $\varphi_1 \leq \varphi \leq \varphi_2$ apply to the minimization. A result very similar to classical de Guenin's equation is thus obtained. More precisely, let $a \in E$ and $b \in E$ verifying $\varphi_o(a) > \varphi_1(a)$ and $\varphi_o(b) < \varphi_2(b)$. Let $dt > 0$ be a positive infinitesimal variation, and define a new sharing function $\tilde{\varphi}$ by:

$$\begin{cases} \tilde{\varphi}(a) = \varphi_o(a) - dt \text{ and } \tilde{\varphi}(b) = \varphi_o(b) + dt\;, \\ \tilde{\varphi}(x) = \varphi_o(x) \text{ for } x \neq a, b\;. \end{cases}$$

By definition of a and b,constraint $\int_E \tilde{\varphi}(x)\,dx = \phi_o$ is also satisfied by the function $\tilde{\varphi}$. Thus, since φ_o is a minimizer, holds $P_{nd}(\alpha_o, \varphi_o) \leq P_{nd}(\alpha_o, \tilde{\varphi})$. Since $dt > 0$, the following inequality is obtained after simplification :

$$\alpha_o(a) p'_a(\varphi_o(a)) \leq \alpha_o(b) p'_b(\varphi_o(b))\;.$$

It is easy, then, to derive a weak optimality condition, i.e. the existence of a (negative) dual variable η such that:

$$\begin{cases} \varphi_1(x) < \varphi_o(x) < \varphi_2(x) \Rightarrow \alpha_o(x) p'_x(\varphi_o(x)) = \eta\;, \\ \varphi_o(x) = \varphi_1(x) \text{ or } \varphi_2(x) \text{ else}\;. \end{cases} \tag{6}$$

But this property is somewhat insufficient or badly formulated for really defining φ_o. A more precise property will be proven. However, it requires a further (but not restrictive) assumptions. First assumption is $\varphi_1 < \varphi_2$. This assumption is absolutely not restrictive, since for cells x verifying $\varphi_1(x) = \varphi_2(x)$, the value $\varphi_o(x)$ is defined by $\varphi_o(x) = \varphi_1(x) = \varphi_2(x)$. So, it is of no consequence not to consider these cases. We state also, that $\exists x,\ \varphi_o(x) > \varphi_1(x)$ and $\exists x,\ \varphi_o(x) < \varphi_2(x)$. This case is also no more restrictive, since otherwise, we would have $\forall x,\ \varphi_o(x) = \varphi_1(x)$ or $\forall x,\ \varphi_o(x) = \varphi_2(x)$, which are exactly equivalent to property $\phi_o = \int_E \varphi_1(x)\,dx$ or $\phi_o = \int_E \varphi_2(x)\,dx$ respectively. These specific cases are also directly checked, if necessary. Then, if all these assumptions are in use, the following property holds:

Proposition 1 *There exists a negative scalar η for which, the following alternative holds true for all $x \in E$:*

$$\begin{cases} \varphi_1(x) < \varphi_o(x) < \varphi_2(x) \Longrightarrow \alpha_o(x) p'_x(\varphi_o(x)) = \eta\ , \\ \varphi_o(x) = \varphi_1(x) \textit{ or } \varphi_2(x) \textit{ else}\ , \end{cases}$$

in accordance with the following discriminating equations:

$$\begin{cases} \alpha_o(x) > \dfrac{\eta}{p'_x(\varphi_1(x))} \Longrightarrow \varphi_o(x) > \varphi_1(x) \\ \alpha_o(x) < \dfrac{\eta}{p'_x(\varphi_2(x))} \Longrightarrow \varphi_o(x) < \varphi_2(x) \end{cases} \tag{7}$$

Proof:

case a : For this case, the existence of a cell $b \in E$ such that $\varphi_1(b) < \varphi_o(b) < \varphi_2(b)$ is assumed. Let $a \in E$ be a cell such that $\varphi_o(a) = \varphi_1(a)$ and $\alpha_o(a) > \frac{\eta}{p'_a(\varphi_1(a))}$. Let $dt > 0$ be a positive variation. Construct a perturbation $\tilde{\varphi}$ of φ_o defined by:

$$\begin{cases} \tilde{\varphi}(a) = \varphi_o(a) + dt \text{ and } \tilde{\varphi}(b) = \varphi_o(b) - dt\ , \\ \tilde{\varphi}(x) = \varphi_o(x) \text{ for } x \neq a, b\ . \end{cases}$$

Function $\tilde{\varphi}$ also satisfies to constraint $\int_E \tilde{\varphi}(x)\,dx = \phi_o$. Now, φ_o is a minimizer and $P_{nd}(\alpha_o, \varphi_o) \le P_{nd}(\alpha_o, \tilde{\varphi})$. Equation $\alpha_o(a) p'_a(\varphi_o(a)) \ge \alpha_o(b) p'_b(\varphi_o(b))$ is obtained after simplifications. Now, from hypothesis on b and equation (6), we have $\alpha(b) p'_b(\varphi_o(b)) = \eta$. A combination of the two previous results yields $\alpha_o(a) \le \frac{\eta}{p'_a(\varphi_o(a))}$. This contradicts assumption on a. We have just refuted the existence of $x \in E$ such that $\varphi_o(x) = \varphi_1(x)$ and $\alpha_o(x) > \frac{\eta}{p'_x(\varphi_1(x))}$. Similarly, there is no $x \in E$ such that $\varphi_o(x) = \varphi_2(x)$ and $\alpha_o(x) < \frac{\eta}{p'_x(\varphi_2(x))}$. Thus, equations (7) are proven, whenever the existence of b is assumed.

case b : Assume now, there is no cell $b \in E$, such that $\varphi_1(b) < \varphi_o(b) < \varphi_2(b)$. Then, variable η is not given by de Guenin's equation, and have to be built. Since $\exists x,\ \varphi_o(x) < \varphi_2(x)$ and $\exists x,\ \varphi_o(x) > \varphi_1(x)$, There is a cell a so that $\varphi_o(a) = \varphi_1(a)$ and a cell b so that $\varphi_o(b) = \varphi_2(b)$. Consider variation $dt > 0$ and a perturbation $\tilde{\varphi}$ of φ_o defined by:

$$\begin{cases} \tilde{\varphi}(a) = \varphi_o(a) + dt \text{ and } \tilde{\varphi}(b) = \varphi_o(b) - dt\ , \\ \tilde{\varphi}(x) = \varphi_o(x) \text{ for } x \neq a, b\ . \end{cases}$$

Function $\tilde{\varphi}$ satisfy constraint $\int_E \tilde{\varphi}(x)\,dx = \phi_o$. The probability increases so that $P_{nd}(\alpha_o, \varphi_o) \le P_{nd}(\alpha_o, \tilde{\varphi})$. Equation $\alpha_o(a) p'_a(\varphi_o(a)) \ge \alpha_o(b) p'_b(\varphi_o(b))$ is obtained after simplifications. Thus, we have just proven:

$$\left.\begin{matrix} \varphi_o(x) = \varphi_1(x) \\ \varphi_o(y) = \varphi_2(y) \end{matrix}\right\} \Rightarrow \alpha_o(x) p'_x(\varphi_o(x)) \ge \alpha_o(y) p'_y(\varphi_o(y))\ .$$

Since $\varphi_1 \le \varphi_o \le \varphi_2$, properties $\varphi_o(x) = \varphi_1(x)$ and $\varphi_o(y) = \varphi_2(y)$ are equivalent to $\varphi_o(x) \le \varphi_1(x)$ and $\varphi_o(y) \ge \varphi_2(y)$ respectively. The previous equation then enable the existence of η such that:

$$\begin{cases} \varphi_o(x) \le \varphi_1(x) \Longrightarrow \alpha_o(x) p'_x(\varphi_o(x)) \ge \eta\ , \\ \varphi_o(y) \ge \varphi_2(y) \Longrightarrow \alpha_o(y) p'_y(\varphi_o(y)) \le \eta\ . \end{cases}$$

In other words, proposition 1 is also verified in this case.

Constantness equation: This part is almost similar to the preceding. First, it is noticed that:

$$\alpha_o \in \arg\max_\alpha P_{nd}(\alpha, \varphi_o)\ .$$

Constraint $\alpha_1 \le \alpha_o \le \alpha_2$ applies to this minimization. Let $a \in E$ and $b \in E$ so that $\alpha_o(a) > \alpha_1(a)$ and $\alpha_o(b) < \alpha_2(b)$. Let $dt > 0$ be a positive infinitesimal variation, and define a new mixed strategy $\tilde{\alpha}$ by:

$$\begin{cases} \tilde{\alpha}(a) = \alpha_o(a) - dt \text{ and } \tilde{\alpha}(b) = \alpha_o(b) + dt\ , \\ \tilde{\alpha}(x) = \alpha_o(x) \text{ for } x \neq a, b\ . \end{cases}$$

Constraint $\int_E \tilde{\alpha}(x)\,dx = A_o$ still holds true. Thus, since α_o is a maximizer, the probability decreases, i.e. $P_{nd}(\alpha_o, \varphi_o) \ge P_{nd}(\tilde{\alpha}, \varphi_o)$. Since $dt > 0$ we obtain, after simplification:

$$p_a(\varphi_o(a)) \ge p_b(\varphi_o(b))\ .$$

There is also a dual (positive) variable λ such that:

$$\begin{cases} \alpha_1(x) < \alpha_o(x) < \alpha_2(x) \Rightarrow p_x(\varphi_o(x)) = \lambda\ , \\ \alpha_o(x) = \alpha_1(x) \text{ or } \alpha_2(x) \text{ else}\ . \end{cases} \tag{8}$$

A more precise optimality equation will be proven now. Again, assumptions $\alpha_1 < \alpha_2$, $\exists x,\ \alpha_o(x) > \alpha_1(x)$ and $\exists x,\ \alpha_o(x) < \alpha_2(x)$ are made without loss of generality.

Proposition 2 *There exists a positive scalar λ for which, the following alternatives hold true for all $x \in E$:*

$$\begin{cases} \alpha_1(x) < \alpha_o(x) < \alpha_2(x) \Rightarrow p_x(\varphi_o(x)) = \lambda \;, \\ \alpha_o(x) = \alpha_1(x) \text{ or } \alpha_2(x) \text{ else } , \end{cases}$$

in accordance with the following discriminating equations:

$$\begin{cases} \varphi_o(x) < p_x^{-1}(\lambda) \Longrightarrow \alpha_o(x) > \alpha_1(x) \;, \\ \varphi_o(x) > p_x^{-1}(\lambda) \Longrightarrow \alpha_o(x) < \alpha_2(x) \;. \end{cases} \tag{9}$$

case a : For this case, the existence of a cell $b \in E$ such that $\alpha_1(b) < \alpha_o(b) < \alpha_2(b)$ is assumed. Let $a \in E$ be a cell such that $\alpha_o(a) = \alpha_1(a)$ and $\varphi_o(a) < p_a^{-1}(\lambda)$. Let $dt > 0$ be a positive variation. Construct a perturbation $\tilde{\alpha}$ of α_o defined by:

$$\begin{cases} \tilde{\alpha}(a) = \alpha_o(a) + dt \text{ and } \tilde{\alpha}(b) = \alpha_o(b) - dt \;, \\ \tilde{\alpha}(x) = \alpha_o(x) \text{ for } x \neq a, b \;. \end{cases}$$

Constraint $\int_E \tilde{\alpha}(x)\,dx = A_o$ still holds true. The probability decreases so that $P_{nd}(\alpha_o, \varphi_o) \geq P_{nd}(\tilde{\alpha}, \varphi_o)$. Equation $p_a(\varphi_o(a)) \leq p_b(\varphi_o(b))$ is obtained after simplifications. Now, from hypothesis on b and equation (7) we have $p_b(\varphi_o(b)) = \lambda$. A combination of the two previous results yields $p_a(\varphi_o(a)) \leq \lambda$. This contradicts hypothesis on a. The existence of $x \in E$ such that $\alpha_o(x) = \alpha_1(x)$ and $\varphi_o(x) < p_x^{-1}(\lambda)$ has been refuted. Similarly, there is no $x \in E$ such that $\alpha_o(x) = \alpha_2(x)$ and $\varphi_o(x) > p_x^{-1}(\lambda)$. Thus, equations (9) are proven, whenever b is located.

case b : Assume now, there is no cell $b \in E$, such that $\alpha_1(b) < \alpha_o(b) < \alpha_2(b)$. Since $\exists x,\ \alpha_o(x) < \alpha_2(x)$ and $\exists x,\ \alpha_o(x) > \alpha_1(x)$, there is both a cell a and a cell b so that $\alpha_o(a) = \alpha_1(a)$ and $\alpha_o(b) = \alpha_2(b)$. Consider variation $dt > 0$ and perturbation $\tilde{\alpha}$ of α_o defined by:

$$\begin{cases} \tilde{\alpha}(a) = \alpha_o(a) + dt \text{ and } \tilde{\alpha}(b) = \alpha_o(b) - dt \;, \\ \tilde{\alpha}(x) = \alpha_o(x) \text{ for } x \neq a, b \;. \end{cases}$$

Function $\tilde{\alpha}$ obeys to constraint $\int_E \tilde{\alpha}(x)\,dx = A_o$. The probability decreases so that $P_{nd}(\alpha_o, \varphi_o) \geq P_{nd}(\tilde{\alpha}, \varphi_o)$. Equation $p_a(\varphi_o(a)) \leq p_b(\varphi_o(b))$ is obtained after simplifications. Thus, we have just proven:

$$\left.\begin{matrix} \alpha_o(x) = \alpha_1(x) \\ \alpha_o(y) = \alpha_2(y) \end{matrix}\right\} \Rightarrow p_x(\varphi_o(x)) \leq p_y(\varphi_o(y)) \;.$$

Again, this property proves the existence of a dual variable λ satisfying proposition 2.

Clarification: The previous propositions 1 and 2 have a geometric interpretation. For a given cell x, the optimal strategies (α_o, φ_o) are locally defined by the intersection of two curves H_η^x and Λ_λ^x. In other words, $(\alpha_o(x), \varphi_o(x)) \in H_\eta^x \cap \Lambda_\lambda^x$. These two curves are defined respectively from propositions 1 and 2:

$$\boxed{(\mathfrak{a}, \mathfrak{f}) \in H_\eta^x \Leftrightarrow \begin{cases} \mathfrak{a} \leq \frac{\eta}{p_x'(\varphi_1(x))} \Rightarrow \mathfrak{f} = \varphi_1(x) \\ \frac{\eta}{p_x'(\varphi_1(x))} < \mathfrak{a} < \frac{\eta}{p_x'(\varphi_2(x))} \Rightarrow \mathfrak{a}\, p_x'(\mathfrak{f}) = \eta \\ \mathfrak{a} \geq \frac{\eta}{p_x'(\varphi_2(x))} \Rightarrow \mathfrak{f} = \varphi_2(x) \end{cases}} \tag{10}$$

and

$$\boxed{(\mathfrak{a}, \mathfrak{f}) \in \Lambda_\lambda^x \Leftrightarrow \begin{cases} \mathfrak{f} < p_x^{-1}(\lambda) \Rightarrow \mathfrak{a} = \alpha_2(x) \\ \mathfrak{f} = p_x^{-1}(\lambda) \Rightarrow \mathfrak{a} \in [\alpha_1(x), \alpha_2(x)] \\ \mathfrak{f} > p_x^{-1}(\lambda) \Rightarrow \mathfrak{a} = \alpha_1(x) \end{cases}} \tag{11}$$

It is not very difficult to derive these curves from the propositions. But proofs are left to the reader. Since p_x is convex, p_x' is increasing and $\mathfrak{a} \mapsto {p_x'}^{-1}\left(\frac{\eta}{\mathfrak{a}}\right)$ is increasing ($\eta < 0$). Thus, H_η^x is flat ($\mathfrak{f} = \varphi_1(x)$) for $\mathfrak{a} \leq \frac{\eta}{p_x'(\varphi_1(x))}$, then becomes an increasing curve and is flat again ($\mathfrak{f} = \varphi_2(x)$) for $\mathfrak{a} \geq \frac{\eta}{p_x'(\varphi_2(x))}$. On the other hand, Λ_λ^x is vertically decreasing down to $p_x^{-1}(\lambda)$ for $\mathfrak{a} = \alpha_1(x)$. Then the curve becomes flat ($\mathfrak{f} = p_x^{-1}(\lambda)$) for $\alpha_1(x) \leq \mathfrak{a} \leq \alpha_2(x)$ and, at last, the curve is vertically decreasing down from $p_x^{-1}(\lambda)$ for $\mathfrak{a} = \alpha_2(x)$. These two curves are schematized in figure 1. However, propositions 1 and 2 have a more precise meaning. There is a common choice of dual variables, which defines the whole optimal strategies as local intersection of the associated curves.

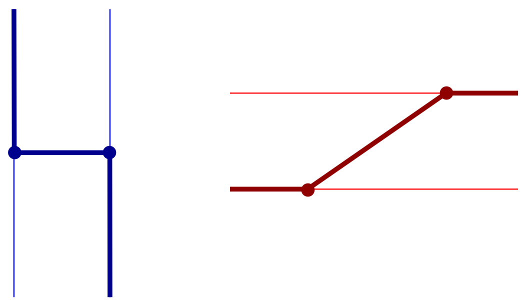

Figure 1: Curves Λ_λ^x and H_η^x.

$$\exists \eta_o,\ \exists \lambda_o,\ \forall x \in E,\ (\alpha_o(x), \varphi_o(x)) \in H_{\eta_o}^x \cap \Lambda_{\lambda_o}^x \;. \tag{12}$$

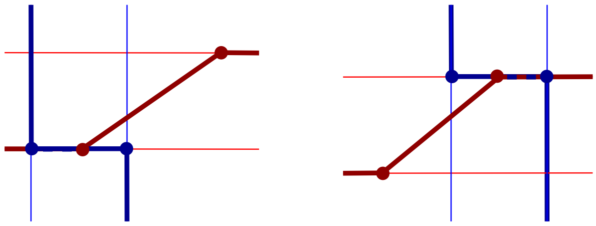

Figure 2: Undefined intersections.

We will use this viewpoint to develop an algorithmic resolution. However, these intersections may be (even locally) non unique, as it is shown in figure 2. The confusing cases are precised by means of the constraint equations $\int_E \alpha = A_o$ and $\int_E \varphi = \phi_o$. But it may happen that several solutions are optimal. Our algorithm is defined in next section and takes into accounts the previous remarks.

Mapping $(\eta,\lambda)\mapsto(\alpha^{\eta\lambda},\varphi^{\eta\lambda})$**:** The previous remarks permit us to build a mapping from the dual variable (η,λ) to the associated strategies $(\alpha^{\eta\lambda},\varphi^{\eta\lambda})$, which inverts the optimality equations. As seen previously, this mapping may point to more than one strategy. What we have to define is a multivalued function. Now, the curves shape induces that $H^x_{\eta_o}\cap\Lambda^x_{\lambda_o}$ is always an horizontal closed interval. In other word, the mapping is $1:1$ for $\varphi^{\eta\lambda}$; while, for each $\alpha^{\eta\lambda}(x)$, it is given by a continuum from a minimum value $\alpha^{\eta\lambda}_{min}(x)$ to a maximum value $\alpha^{\eta\lambda}_{max}(x)$. Generally, $\alpha^{\eta\lambda}_{min}(x)=\alpha^{\eta\lambda}_{max}(x)$. In fact, because of the middle flatness of Λ^x_λ, **there is at most one** λ **such that** $\alpha^{\eta\lambda}_{min}(x)<\alpha^{\eta\lambda}_{max}(x)$. Now, the following mapping may be defined, for the solutions associated to the optimality constraints on (η,λ) :

$$(\eta,\lambda)\longmapsto\left[\alpha^{\eta\lambda}_{min},\alpha^{\eta\lambda}_{max}\right]\times\{\varphi^{\eta\lambda}\}\;.$$

The crucial point, is that $\alpha^{\eta\lambda}_{min}$, $\alpha^{\eta\lambda}_{max}$ and $\varphi^{\eta\lambda}$ are simply and entirely defined and computable by means of the problem data. However, we shall not give an explicit definition of these functions, since a lot of case checking is required.

Knowing $\alpha^{\eta\lambda}_{min}$, $\alpha^{\eta\lambda}_{max}$ and $\varphi^{\eta\lambda}$ it is useful to define the following global values:

$$\begin{cases}\phi^{\eta\lambda}=\displaystyle\int_E\varphi^{\eta\lambda}(x)\,dx\\ A^{\eta\lambda}_{min}=\displaystyle\int_E\alpha^{\eta\lambda}_{min}(x)\,dx\\ A^{\eta\lambda}_{max}=\displaystyle\int_E\alpha^{\eta\lambda}_{max}(x)\,dx\end{cases}$$

Values $\phi^{\eta\lambda}$, $A^{\eta\lambda}_{min}$ and $A^{\eta\lambda}_{max}$ will be of constant use in the development of our algorithm.

Variation of $\phi^{\eta\lambda}$**,** $A^{\eta\lambda}_{min}$ **and** $A^{\eta\lambda}_{max}$**:** Our interest now focuses on the variation of $\phi^{\eta\lambda}$, $A^{\eta\lambda}_{min}$ and $A^{\eta\lambda}_{max}$ according to the variables η and λ. First, it appears that an increase of η produces an up swelling (associated to a left shifting) of the curve H^x_η (recall η and p'_x are negative and ${p'_x}^{-1}$ is increasing), more precisely:

$$\eta_1<\eta_2\Rightarrow\left[\forall x,\;\forall\mathfrak{a},\;\left.\begin{array}{l}(\mathfrak{a},\mathfrak{f}_1)\in H^x_{\eta_1}\\(\mathfrak{a},\mathfrak{f}_2)\in H^x_{\eta_2}\end{array}\right\}\Rightarrow\mathfrak{f}_2\geq\mathfrak{f}_1\right]\tag{13}$$

This property is a direct consequence of definition (10). Now Λ^x_λ is a decreasing curve and H^x_η is an increasing curve. Thus, the increase of η (i.e. the up increase of H^x_η) then yields an up-left move of the intersection $H^x_\eta\cap\Lambda^x_\lambda$. Thus, the incoming result :

$$\eta_1<\eta_2\Longrightarrow\begin{cases}\alpha^{\eta_1\lambda}_{min}(x)\geq\alpha^{\eta_2\lambda}_{min}(x)\\ \alpha^{\eta_1\lambda}_{max}(x)\geq\alpha^{\eta_2\lambda}_{max}(x)\\ \varphi^{\eta_1\lambda}(x)\leq\varphi^{\eta_2\lambda}(x)\end{cases}\tag{14}$$

Thanks to definition (11), an increase of λ produces similarly a left swelling (associated to a down shifting) of curve Λ^x_λ (p_x^{-1} is decreasing). In other words:

$$\lambda_1<\lambda_2\Rightarrow\left[\forall x,\;\forall\mathfrak{f},\;\left.\begin{array}{l}(\mathfrak{a}_1,\mathfrak{f})\in\Lambda^x_{\lambda_1}\\(\mathfrak{a}_2,\mathfrak{f})\in\Lambda^x_{\lambda_2}\end{array}\right\}\Rightarrow\mathfrak{a}_2\leq\mathfrak{a}_1\right]\tag{15}$$

Similarly to the previous case, the intersection $H^x_\eta\cap\Lambda^x_\lambda$ moves down-left. Again, the variations of α and φ are deduced:

$$\lambda_1<\lambda_2\Longrightarrow\begin{cases}\alpha^{\eta\lambda_1}_{min}(x)\geq\alpha^{\eta\lambda_2}_{min}(x)\\ \alpha^{\eta\lambda_1}_{max}(x)\geq\alpha^{\eta\lambda_2}_{max}(x)\\ \varphi^{\eta\lambda_1}(x)\geq\varphi^{\eta\lambda_2}(x)\end{cases}\tag{16}$$

But there is, in fact, a stronger property. Since there is at most one λ such that $\alpha^{\eta\lambda}_{min}(x)<\alpha^{\eta\lambda}_{max}(x)$, the above property yields :

$$\lambda_1<\lambda_2\Longrightarrow\alpha^{\eta\lambda_1}_{min}(x)\geq\alpha^{\eta\lambda_2}_{max}(x)\;.$$

Global results are then derived:

$$\forall\lambda,\;\eta_1<\eta_2\Rightarrow\begin{cases}A^{\eta_1\lambda}_{min}\geq A^{\eta_2\lambda}_{min}\\ A^{\eta_1\lambda}_{max}\geq A^{\eta_2\lambda}_{max}\\ \phi^{\eta_1\lambda}\leq\phi^{\eta_2\lambda}\end{cases}\tag{17}$$

and

$$\forall\eta,\;\lambda_1<\lambda_2\Rightarrow\begin{cases}A^{\eta\lambda_1}_{min}\geq A^{\eta\lambda_2}_{max}\\ \phi^{\eta\lambda_1}\geq\phi^{\eta\lambda_2}\end{cases}\tag{18}$$

Implicit definition of $\eta(\lambda)$: Let λ be fixed. In this situation, the curve Λ_λ^x is also fixed. Then, what happen, when η is varying? Define:

$$\eta_{min} = \min_x \Big(\alpha_2(x) p'_x(\varphi_1(x)) \Big) \ ,$$

and

$$\eta_{max} = \max_x \Big(\alpha_1(x) p'_x(\varphi_2(x)) \Big) \ .$$

Inverting the above equations gives us $\varphi^{\eta_{min}\lambda} = \varphi_1$ and $\varphi^{\eta_{max}\lambda} = \varphi_2$, whence :

$$\phi^{\eta_{min}\lambda} = \int_E \varphi_1(x)\, dx \text{ and } \phi^{\eta_{max}\lambda} = \int_E \varphi_2(x)\, dx \ .$$

Now, curve H_η^x is continuously increasing and the shifting of H_η^x due to the η-variation is also continuous. Thus, when η varies from η_{min} to η_{max}, the value $\phi^{\eta\lambda}$ also increases continuously from $\int_E \varphi_1(x)\, dx$ to $\int_E \varphi_2(x)\, dx$. It follows that every $\phi \in \left[\int_E \varphi_1, \int_E \varphi_2\right]$ admits a non empty set of antecedents. It is in particular true for ϕ_o. The set of antecedents is often reduced to one element, otherwise it is an interval:

$$\phi^{\eta\lambda} = \phi_o \Leftrightarrow \eta \in [\eta_{min}(\lambda), \eta_{max}(\lambda)] \ .$$

Now, property $\{\ \lambda_1 < \lambda_2 \Rightarrow \varphi^{\eta\lambda_1} \geq \varphi^{\eta\lambda_2}\ \}$ and property $\{\ \eta_1 < \eta_2 \Rightarrow \varphi^{\eta_1\lambda} \leq \varphi^{\eta_1\lambda}\ \}$ hold from equations (14) and (16). Consequently, if λ increases and $\phi^{\eta\lambda}$ is maintained equal to ϕ_o, η has to "increase" also. Variations of $\eta_{min}(\lambda)$ and $\eta_{max}(\lambda)$ are deduced:

$$\lambda_1 < \lambda_2 \Longrightarrow \begin{cases} \eta_{min}(\lambda_1) \leq \eta_{min}(\lambda_2) \\ \eta_{max}(\lambda_1) \leq \eta_{max}(\lambda_2) \end{cases}$$

Now, $A_{min}^{\eta\lambda}$ and $A_{max}^{\eta\lambda}$ are decreasing for both η and λ. Thus previous results yield:

$$\lambda_1 < \lambda_2 \Longrightarrow \begin{cases} A_{min}^{\eta_{min}(\lambda_1)\lambda_1} \geq A_{min}^{\eta_{min}(\lambda_2)\lambda_2} \\ A_{max}^{\eta_{min}(\lambda_1)\lambda_1} \geq A_{max}^{\eta_{min}(\lambda_2)\lambda_2} \\ A_{min}^{\eta_{max}(\lambda_1)\lambda_1} \geq A_{min}^{\eta_{max}(\lambda_2)\lambda_2} \\ A_{max}^{\eta_{max}(\lambda_1)\lambda_1} \geq A_{max}^{\eta_{max}(\lambda_2)\lambda_2} \end{cases} \qquad (19)$$

4 Algorithm

The previous properties are a guideline for developing our algorithm. Since optimality equations are almost invertible and signs of variation are fixed for $A_{min}^{\eta\lambda}$, $A_{max}^{\eta\lambda}$ and $\phi^{\eta\lambda}$, bi-sectional methods were chosen. Our algorithm is made of two parts. First part find the optimal dual parameter λ_o. At this point, convergence is almost achieved. The second part sharpens the convergence and renders more precise some subdefinitions, by calibrating the optimal dual parameter η_o.

Computing λ_o and η_o: The first ingredient is to build up the procedure, which defines $\eta(\lambda)$, that is, which computes $\eta_{min}(\lambda)$ and $\eta_{max}(\lambda)$. Thanks to the increaseness property associated with the definition of $\eta_{min}(\lambda)$ and $\eta_{max}(\lambda)$, two bi-sectional processes around ϕ_o are in use to compute $\eta_{min}(\lambda)$ and $\eta_{max}(\lambda)$. Then, the main part of the process will consist in finding λ, such that $A_o \in \left[A_{min}^{\eta_{max}(\lambda)\lambda}, A_{max}^{\eta_{min}(\lambda)\lambda}\right]$. Thanks to the increaseness evoked in property (19), this is done again by a bi-sectional process. However, this process will call the procedure for $\eta_{min}(\lambda)$ and $\eta_{max}(\lambda)$ computation, constituting in fact a double bi-sectional procedure. This procedure yields as result the optimal dual variable λ_o. It is noteworthy that for $\eta \in [\eta_{min}(\lambda_o), \eta_{max}(\lambda_o)]$, $\phi^{\eta\lambda_o} = \phi_o$ and we will not have to care about the constraint on ϕ_o, now. Otherwise, since $A_o \in \left[A_{min}^{\eta_{max}(\lambda_o)\lambda_o}, A_{max}^{\eta_{min}(\lambda_o)\lambda_o}\right]$, exists $\eta \in [\eta_{min}(\lambda_o), \eta_{max}(\lambda_o)]$ so that $A_o \in \left[A_{min}^{\eta\lambda_o}, A_{max}^{\eta\lambda_o}\right]$. This η will be our optimal dual variable η_o. To compute it, a bi-sectional process is again instrumental, because of the constant sign variations of $A_{min}^{\eta\lambda_o}$ and $A_{max}^{\eta\lambda_o}$ (refer to property (17)). The whole process is summed up below:

i. Find λ_o such that $A_o \in \left[A_{min}^{\eta_{max}(\lambda_o)\lambda_o}, A_{max}^{\eta_{min}(\lambda_o)\lambda_o}\right]$; *do it by means of a bi-sectional process; a sub-procedure is used to compute $\eta_{min}(\lambda)$ and $\eta_{max}(\lambda)$,*

ii. Find η_o, element of $[\eta_{min}(\lambda_o), \eta_{max}(\lambda_o)]$, such that $A_o \in \left[A_{min}^{\eta_o\lambda_o}, A_{max}^{\eta_o\lambda_o}\right]$; *do it by means of a bi-sectional process.*

sub-procedure: Compute $\eta_{min}(\lambda)$ and $\eta_{max}(\lambda)$ by means of a bi-sectional process.

Finalization: Now, η_o and λ_o are found. Function φ_o is entirely defined by $\varphi^{\eta_o\lambda_o}$. However, there could be some indetermination for α_o, in particular when $A_{min}^{\eta_o\lambda_o} < A_{max}^{\eta_o\lambda_o}$. Now, definitions of $A_{min}^{\eta\lambda}$ and $A_{max}^{\eta\lambda}$ say $A_{min}^{\eta\lambda} = \int_E \alpha_{min}^{\eta\lambda}$ and $A_{max}^{\eta\lambda} = \int_E \alpha_{max}^{\eta\lambda}$. Thus, a candidate α_o, such that $\int_E \alpha_o = A^o$, may be defined as the barycenter of $\alpha_{min}^{\eta_o\lambda_o}$ and $\alpha_{max}^{\eta_o\lambda_o}$, where weights are given by the relative positions of $A_{min}^{\eta_o\lambda_o}$, $A_{max}^{\eta_o\lambda_o}$ and A_o:

$$\boxed{\begin{cases} \varphi_o = \varphi^{\eta_o\lambda_o} \ , \\ \alpha_o = \alpha_{min}^{\eta_o\lambda_o} + \dfrac{A_o - A_{min}^{\eta_o\lambda_o}}{A_{max}^{\eta_o\lambda_o} - A_{min}^{\eta_o\lambda_o}} \left(\alpha_{max}^{\eta_o\lambda_o} - \alpha_{min}^{\eta_o\lambda_o} \right) \end{cases}}$$

5 Results

In this section, we present an exemple computed by the algorithm. The search space E is a set of 30×20 cells. Values $A_o = 1$ and $\phi_o = 30$ are used. The local bounds α_1 and α_2 are represented in figure 3. In the figures, dark cells are representing low values, while bright cells represent high values. The local bounds φ_1 and φ_2 are represented in the two first frames of figure 4. The conditional probability, p, is of exponential form $p_x(\varphi) = \exp(-\omega_x \varphi)$. The visibility parameter ω_x is low for bad detection and high for good detection. The parameter ω is represented by last frame of figure 4. The functions α_o and φ_o obtained after convergence are represented in figures 5. Again, low values correspond to dark cells whereas bright cells represent high values. Moreover, the color of the cell contours indicate if bounds are reached or not. Blue contour on cell x means $\varphi_o(x) = \varphi_1(x)$ or $\alpha_o(x) = \alpha_1(x)$. Green contour on cell x signifies $\varphi_1(x) < \varphi_o(x) < \varphi_2(x)$ or $\alpha_1(x) < \alpha_o(x) < \alpha_2(x)$. Red contour on cell x corresponds to $\varphi_o(x) = \varphi_2(x)$ or $\alpha_o(x) = \alpha_2(x)$.

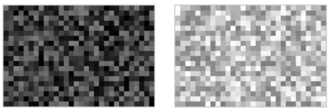

Figure 3: Bounds α_1 and α_2.

Figure 4: Bound φ_1, φ_2 and visibility parameter ω.

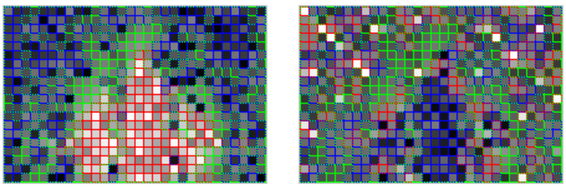

Figure 5: Target strategy α_o and searcher strategy φ_o.

6 Conclusion

Our aim was to solve a spatial resource allocation problem, in a game context between the target and the searcher. A great enhancement and a generalization of both Nakai's game and de Guenin's optimization problem were obtained. The viewpoint considered is versatile, allowing subtle modeling of the target and resource behavior. It is not limited to simple priors on available target position. The algorithm, we developed, is quite original and fast. It is reliable and may be involved in more intricate processes. It particular, extension to multi-type resource game has been solved, by means of an iterative method based on this algorithm. For the sake of brevity, it is not presented here. Details may be found in [13].

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