Approximations of the Cramér–Rao bound for multiple-target motion analysis

J.-P. Le Cadre, H. Gauvrit and F. Trarieux

Abstract: The study is concerned with multiple target motion analysis (MTMA), when the system state is not directly observed. The Cramér–Rao lower bound (CRLB) is widely used reference for assessing estimation performance. The lack of explicit bounds on the performance of MTMA remains an important issue in the tracking community. The problem is an aspect of the estimation of normal mixture parameters. A general formulation of the CRLB is given. The authors contribute the calculation of convenient explicit approximations of the bounds relative to source kinematic parameters, especially for close tracks.

1 Introduction

This study is concerned with multiple target motion analysis (MTMA for the sequel), when the system state is not directly observed. a classical example is that of passive MTMA where measurements are only made of estimated bearings [1]. Such systems are used in passive sonar [1], infrared tracking or electronic warfare. The Cramér–Rao lower bound (CRLB) is widely used reference for assessing estimation performance. The lack of explicit bounds on the performance of MTMA remains an important issue in the tracking community [2–4]. As a result, a great deal of attention has been paid to measures of performance, such as track purity, correct assignment ratio [5;6], etc. These methods are based on the discrete assignments of measurements to tracks and are thus not universally applicable. Their interest is, for a large part, due to the fact that numerous MTMA algorithms rely on "hard" association. This type of analysis is quite pertinent, and sophisticated tools have thus been developed. However, there is a need for simple and (relatively) explicit formulations of the CRLB in the MTMA context. These bounds are developed here in a general framework which employs a probabilistic structure on the measurement to target association.

The difficulty of obtaining CRLB for MTMA is due to a need for an association between measurements and tracks, and to incorporate this basic step in the CRLB calculation. In fact, when properly cast, a CRLB for the MTMA does exist, even if its evaluation may be difficult [7]. This problem will be overcome by means of a "mixture modelling" of the likelihood [8, 9]. It is then possible to immerse the problem in the general framework of the estimation of normal mixture parameters, for which important statistical literature exists. Furthermore, this modelling has been widely used in the derivation of the probabilistic multiple hypothesis tracking (PMHT) developed by Streit and Luginbuhl [10, 11]. Practically, the main difficulty is to obtain an explicit expression of the Fisher information matrix by using approximations of the interaction terms (associated with the mixture components) on the one hand, and by means of the special structure induced by modified polar co-ordinates on the other.

This study emerges from the general framework developed by Graham and Streit [2], which will be of constant use subsequently. It is also motivated by the development of MTMA methods that do not explicitely estimate measurements to target associations [10, 12]. Our contribution is in the calculation of accurate approximations of the bounds relative to source kinematic parameters. It is worth stressing that approximations of the interaction terms reduce the validity of our approach to close source tracks (Section 3.4).

2 General calculations

For this Section and the rest of the paper we consider the following scenario: two sources move with a constant velocity vector. They are (partially) observed through a (passive) receiver (sonar, IR, ESM). Measurements are bearings. For the sake of simplicity, we restrict our attention to planar problems. For deterministic motions, the source trajectories are defined by initial conditions i.e. a four-dimensional vector whose components are (x, y)-position and (x, y)-velocity. The corresponding bearing sequence (i.e. $\beta_1(X_1, k)$, $\beta_2(X_2, k)$) are completely determined by the source state vectors (i.e. X_1 and X_2).

Associated with these deterministic models and a Bayesian framework are their *a priori* probabilities π_1 and $\pi_2 (\pi_1 + \pi_2 = 1)$. Denote $q \stackrel{\Delta}{=} \pi_1$, then the scenario parameters are represented by the following Φ-vector, $\Phi = (X_1, X_2, q)$. The batch data are denoted by Z. At each scan, *two* measurements (possibly collapsed) are observed,

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each of which comes from one of the *two* models with probability q and $1-q$, respectively, i.e.

$$z_j(k) = \begin{cases} \beta_1(\boldsymbol{X}_1, k) + w_1(k), \\ \quad \text{if } z_j(k) \text{ originates from source 1} \\ \beta_2(\boldsymbol{X}_2, k) + w_2(k), \\ \quad \text{if } z_j(k) \text{ originates from source 2} \end{cases} \quad (1)$$

w_1 and w_2 are the measurement noises. We assume them to be independent (from scan to scan), gaussian, with known and constant (throughout the measurement batch) variances (σ_1^2 and σ_2^2).

The likelihood function then takes the following form (For the sake of brevity, clutter and detection processes are not included in this model, we refer to [13] for a complete modelling.):

$$p(Z|\Phi) = \prod_{k=1}^{T}\prod_{j=1}^{2} p(z_j(k)|\Phi)$$
$$= \prod_{k=1}^{T}\prod_{j=1}^{2}\{qp_1(z_j(k)|\boldsymbol{X}_1) + (1-q)p_2(z_j(k)|\boldsymbol{X}_2)\} \quad (2)$$

We are now dealing with the calculation of the Fisher information matrix (FIM), when measurements are bearings. The validity of the related bounds in our context (measurements are not identically distributed) must be considered with care. For a detailed analysis refer to [14], chapt. 4. First, recall the classical expression of the FIM [1] for the unique source case (no assignment problem then exists):

$$\mathrm{FIM} = \mathbb{E}\left\{\nabla_{x_1}(\log p(Z|\boldsymbol{X}_1)\nabla_{x_1}^*(\log p(Z|\boldsymbol{X}_1)\right\}$$
$$= \sum_{k=1}^{T}\frac{1}{\sigma^2}\boldsymbol{G}_1(k)\boldsymbol{G}_1^*(k)$$

where

$$\boldsymbol{G}_1(k) = \nabla_{x_1}\beta_1(\boldsymbol{X}_1, k)$$
$$= \left(\frac{\cos\beta_1(k)}{r(k)}, -\frac{\sin\beta_1(k)}{r(k)}, k\frac{\cos\beta_1(k)}{r(k)}, -k\frac{\sin\beta_1(k)}{r(k)}\right)^* \quad (3)$$

This calculation may be easily extended [15, 16] to the mixture model (eqns. 1 and 2), thus yielding:

Proposition 1: Let FIM be the Fisher information matrix associated with the mixture model (eqns. 1 and 2), then

$$\mathrm{FIM} = 2\sum_{k=1}^{T} I(k), \text{ where } I(k) = \begin{pmatrix} I_{11}(k) & I_{12}(k) & I_{13}(k) \\ I_{12}^*(k) & I_{22}(k) & I_{23}(k) \\ I_{13}^*(k) & I_{23}^*(k) & I_{33}(k) \end{pmatrix}$$

and

$$I_{11}(k) = \frac{q^2 M_{2,0}(p_1, p_1, k)}{\sigma_1^2}\boldsymbol{G}_1(k)\boldsymbol{G}_1^*(k),$$
$$I_{12}(k) = \frac{q(1-q)M_{1,1}(p_1, p_2, k)}{\sigma_1\sigma_2}\boldsymbol{G}_1(k)\boldsymbol{G}_2^*(k)$$
$$I_{22}(k) = \frac{(1-q)^2 M_{0,2}(p_2, p_2, k)}{\sigma_2^2}\boldsymbol{G}_2(k)\boldsymbol{G}_2^*(k),$$
$$I_{13}(k) = \frac{-M_{1,0}(p_1, p_2, k)}{\sigma_1}\boldsymbol{G}_1(k)$$
$$I_{23}(k) = \frac{M_{0,1}(p_1, p_2, k)}{\sigma_2}\boldsymbol{G}_2(k),$$
$$I_{33}(k) = \frac{1}{q(1-q)}(1 - M_{0,0}(p_1 p_2, k))$$

$$M_{m,n}(p_i, p_j, k) = \int_{-\infty}^{\infty}\frac{p_i(z|\boldsymbol{X}_i)p_j(z|\boldsymbol{X}_j)}{p(z|\phi)}\left(\frac{z-\beta_i(\boldsymbol{X}_i, k)}{\sigma_i}\right)^m \left(\frac{z-\beta_j(\boldsymbol{X}_j, k)}{\sigma_j}\right)^n dz$$
$$i, j \in \{1, 2\};\ m, n \in \{0, 1, 2\}.$$

Proof:

Consider, for instance, the calculation of I_{11}

$$I_{11} = \mathbb{E}\{\nabla_{x_1}(\log p(Z|\Phi)\nabla_{x_1}^*(\log p(Z|\Phi)\}$$

where

$$\nabla_{x_1}\log p(Z|\phi)$$
$$= \sum_{k=1}^{T}\sum_{j=1}^{2}\nabla_{x_1}\log\{qp_1(z_j(k)|\boldsymbol{X}_1) + (1-q)p_2(z_j(k)|\boldsymbol{X}_2)\}$$
$$= \sum_{k=1}^{T}\sum_{j=1}^{2}\frac{qp_1(z_j(k)|\boldsymbol{X}_1)}{\sigma_1^2 p(z_j(k)|\Phi)}(z_j(k) - \beta_1(\boldsymbol{X}_1, k))\nabla_{x_1}\beta_1(\boldsymbol{X}_1, k) \quad (4)$$

I_{11} is then obtained by calculating the expectation of the dyadic product of the term (eqn. 4). The calculation is greatly simplified by the following remark : all the cross products yield null contributions. We then obtain

$$I_{11} = \sum_{k=1}^{T}\sum_{j=1}^{2}\frac{q^2}{\sigma_1^2}\nabla_{x_1}\beta_1(\boldsymbol{X}_1, k)\nabla_{x_1}^*\beta_1(\boldsymbol{X}_1, k)$$
$$\times\,\mathbb{E}\left\{\frac{p_1^2(z_j(k)|\boldsymbol{X}_1)}{p^2(z_j(k)|\Phi)}\left(\frac{z_j(k)-\beta_1(\boldsymbol{X}_1, k)}{\sigma_1}\right)^2\right\}$$
$$= \sum_{k=1}^{T}\frac{q^2}{\sigma_1^2}\nabla_{x_1}\beta_1(\boldsymbol{X}_1, k)\nabla_{x_1}^*\beta_1(\boldsymbol{X}_1, k)$$
$$\times\, 2\mathbb{E}\left\{\frac{p_1^2(z|\boldsymbol{X}_1)}{p^2(z|\Phi)}\left(\frac{z-\beta_1(\boldsymbol{X}_1, k)}{\sigma_1}\right)^2\right\} \quad (5)$$

Denoting $\boldsymbol{G}_1(k)$ the gradient vector $\nabla_{x_1}\beta(\boldsymbol{X}_1, k)$ and $M_{2,0}$ defined as in eqn. 4, exp. 4 of I_{11} then follows. Calculation of I_{12} and I_{22} is quite similar. □

It remains to calculate and approximate the scalar interaction terms $M_{m,n}$. Using an elementary transformation [17] $(y = \varepsilon(z-\bar{\mu})/\bar{\sigma},\ \bar{\mu} \triangleq (\mu_1+\mu_2)/2,\ \bar{\sigma} \triangleq (\sigma_1\sigma_2)^{1/2},\ \varepsilon = \pm 1)$, the interaction terms $M_{m,n}(p_i, p_j, k)$ are considerably simplified. For the sequel, we adopt the very concise notations of Behboodian [17] (i.e. $d = |\mu_2 - \mu_1|/2\bar{\sigma}$, $\rho = \sigma_1/\sigma_2$, $d_1 = -d$, $d_2 = d$ and $\rho_1 = \rho$, $\rho_2 = 1/\rho$), yielding ($\mu_1 \triangleq \beta_1(\boldsymbol{X}_1, k)$, $\mu_2 \triangleq \beta_2(\boldsymbol{X}_2, k)$) [17].

Lemma 1: Let $M_{m,n}(p_i, p_j, k)$ be the scalar interaction terms of proposition 1, the following simplifications hold:

$$M_{m,n}(p_i, p_j, k) = \varepsilon^{m+n}\rho_i^{-m/2}\rho_j^{-n/2}G_{m,n}(g_i, g_j, k)$$

where

$$G_{m,n}(g_i, g_j, k) = \int_{-\infty}^{\infty}(y - d_{i,k})^m(y - d_{j,k})^n(g_i(y)g_j(y))/g(y)dy$$

and

$$d_k = \frac{|\beta_1(\boldsymbol{X}_1, k) - \beta_2(\boldsymbol{X}_2, k)|}{2\sqrt{\sigma_1\sigma_2}}$$
$$g_i(y) = \frac{1}{\sqrt{2\pi\rho_i}}\exp\left(-\frac{1}{2\rho_i}(y - d_{i,k})^2\right);\ i = 1, 2$$
$$d_{2,k} = -d_{1,k} = d_k,\ \varepsilon = 1 \ for\ \mu_1 \le \mu_2,\ \varepsilon = -1\ for\ \mu_1 > \mu_2$$
$$g(y) = qg_1(y) + (1-q)g_2(y) \quad (6)$$

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Now, our analysis is divided into two parts. First, we examine approximations of the scalar interaction terms $M_{m,n}$. The second part consists in using these results for approximating the CRLB bounds relative to the kinematic parameters of the sources. Since this analysis is multi-dimensional, this part is essentially based on (linear) algebra.

3 Approximation of interaction terms

We now restrict to tracts in close proximity (i.e. $d_k \leq 1$). The parameter d_k (eqn. 6 for its definition) represents the normalised angular separation between the two tracks at the instant k. Since expansions of the interaction terms $M_{m,n}$ will play a fundamental role, we restrict to the close track hypothesis (i.e. $d_k < 1$).

First, for reasons we present later, the case $\rho = 1$ is a special one for which approximations of terms $M_{m,n}(p_i, p_j, k)$ are particularly simple and easy to obtain. More precisely, considering a fourth-order expansion of the functions (eqn. 6) $G_{m,n}(g_i, g_j, k)$, around 0 and relatively to d_k, the following approximations are obtained [16], for $\rho = 1$:

3.1 Result 1

$$\begin{aligned}
M_{0,0}(p_1, p_2, k) &\approx 1 - 4q(1-q)d_k^2, \\
M_{1,1}(p_1, p_2, k) &\approx 1 - 12q(1-q)d_k^2 \\
M_{2,0}(p_1, p_1, k) &\approx 1 - 4(3q-2)(1-q)d_k^2, \\
M_{0,2}(p_2, p_2, k) &\approx 1 - 4q(1-3q)d_k^2 \\
M_{1,0}(p_1, p_2, k) &\approx -2qd_k + 8(3q-1)q(1-q)d_k^3 \\
M_{0,1}(p_1, p_2, k) &\approx 2(1-q)d_k + 8q(3q-2)(1-q)d_k^3
\end{aligned} \quad (7)$$

An illustration of the accuracy of their second-order approximations is provided with Fig. 1. The value of q is 0.5, the parameter d_k is varying from 0 to 3 (horizontal axis), and we compare ($\rho = 1$), the exact values of $M_{m,n}$ (eqn. 6) with its approximations given by (eqn. 7). The approximations are satisfactory for values of the normalised separation d_k as great as 0.7; which is, here, a convenient hypothesis (close tracks). For greater values, these approximations become quite inaccurate.

Rather surprisingly, the results obtained for the general case (i.e. $\rho \neq 1$) are fundamentally different. Considering ρ as a free parameter, the previous approach does not provide explicit results since there is no explicit expression of the integrals of the expansion of the terms $(g_i(y)g_j(y))/g(y)$. Then, analogously to [17, 18] a natural and rigorous approach consists in using a series expansion of the function $(g_i(y)g_j(y))/g(y)$. More precisely, we observe that

$$(g_i(y)g_j(y))/g(y) = \left(\frac{1}{\sqrt{2\pi\rho_i\rho_j/\rho}}\right)(h_i(y)h_j(y))/h(y) \quad (8)$$

where $h(y) = qh_1(y) + (1-q)\rho h_2(y)$ and $h_i(y) = \exp[-(y - d_{i,k})^2/2\rho_i]$, $i = 1,2$. Now, it is easy to show that $qh_1(y)/(1-q)\rho h_2(y) < 1$ if y is in the interval $(-\infty, \alpha_1)$ or (α_2, ∞), with $\alpha_1 < \alpha_2$, and the converse (i.e. $(1-q)\rho h_2(y)/qh_1(y) < 1$) if y is in the interval (α_1, α_2), where α_1 and α_2 are the real roots of the following second-order equation:

$$(1-\rho^2)y^2 + 2d_k(1+\rho^2)y + (d_k^2)(1-\rho^2) + 2\rho\log\left[\left(\frac{1-q}{q}\right)\rho\right] = 0 \quad (9)$$

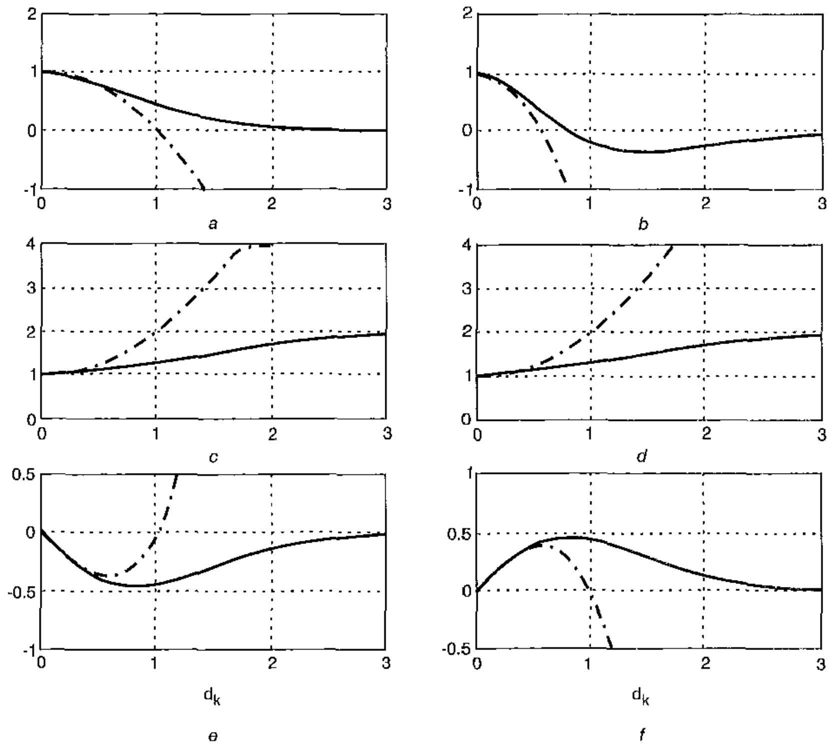

Fig. 1 *Order 2 approximation of $M_{m,n}$; $q = 0.5$*
—— exact value of $M_{m,n}(d_k)$, eqn. 6
–·–· approximations of $M_{m,n}(d_k)$, eqn. 7
a $M_{0,0}$; *b* $M_{1,1}$; *c* $M_{2,0}$; *d* $M_{0,2}$; *e* $M_{1,0}$; *f* $M_{0,1}$

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If real roots exist. Using the method presented in [18, 17], the following expression of $G_{m,n}(g_i,g_j)$ is obtained:

$$G_{m,n}(g_i, g_j) = \left(\frac{1}{\sqrt{2\pi\rho_i\rho_j/\rho}}\right) \sum_{l=0}^{\infty}\left[\int_{-\infty}^{a_1} H_1(y)\,dy + \int_{a_1}^{a_2} \bar{H}_l(y)\,dy + \int_{a_2}^{\infty} H_l(y)\,dy\right] \quad (10)$$

where the functions $H_l(y)$ and $\bar{H}_l(y)$ are straightforwardly deduced from above calculations and detailed in [17]. The computation of the integrals leads to deal with truncated moments of a normal distribution, which is already known. The advantage of this method lies in the fact that we approximate $G_{m,n}(g_i, g_j)$ by an alternating series.

The calculation is simplified if we assume that eqn. 9 has no real root, and obtain (these approximations differ slightly from that of Behboodian [17]) (see Appendix, Section 9.1)

3.2 Result 2

$$\begin{aligned}
G_{0,0}(g_1, g_2, k) &= (1/(1-q)) \textstyle\sum_{l=0}^{\infty}\left(-\frac{q}{(1-q)\rho}\right)^l \\
&\quad a_l^{-1/2} \exp\left[2d_k^2 \frac{\rho}{a_l} l(l+1)\right] \\
G_{2,0}(g_1, g_1, k) &= (1/(1-q)) \textstyle\sum_{l=0}^{\infty}\left(-\frac{q}{(1-q)\rho}\right)^l \\
&\quad a_{l+1}^{-3/2}\left[4d_k^2(1+l)^2 \frac{\rho^3}{a_{l+1}} + 1\right] \\
&\quad \exp[2d_k^2(l+1)(l+2)\rho/a_{l+1}] \\
G_{1,1}(g_1, g_2, k) &= (\rho/(1-q)) \textstyle\sum_{l=0}^{\infty}\left(-\frac{q}{(1-q)\rho}\right)^l \\
&\quad a_l^{-3/2}\left[4d_k^2 l(l+1)\frac{\rho}{a_l} + 1\right] \\
&\quad \exp[2d_k^2 l(l+1)\rho/a_l] \\
G_{0,2}(g_2, g_2, k) &= (\rho/(1-q)) \textstyle\sum_{l=0}^{\infty}\left(-\frac{q}{(1-q)\rho}\right)^l \\
&\quad a_{l-1}^{-3/2}\left[\rho + 4d_k^2 \frac{l^2}{a_{l-1}}\right] \\
&\quad \exp[2d_k^2 l(l-1)\rho/a_{l-1}]
\end{aligned} \quad (11)$$

where $a_l = l(1-\rho^2)+1$. A less rigorous but simpler approach consists in using a second-order expansion of g_i and g_j, both with respect to d (around 0) and ρ (e.g. around 1). Calculations are performed by means of symbolic computation and yield

$$\begin{aligned}
G_{0,2}(g_2, g_2) &\approx P_{0,2}(q,\rho) + d^2 Q_{0,2}(q,\rho) \\
G_{1,1}(g_1, g_2) &\approx P_{1,1}(q,\rho) + d^2 Q_{1,2}(q,\rho) \\
G_{2,0}(g_1, g_1) &\approx P_{2,0}(q,\rho) + d^2 Q_{2,0}(q,\rho)
\end{aligned} \quad (12)$$

The polynomials $P_{i,j}$ and $O_{i,j}$ are detailed in the Appendix (Section 9.2). Their complexity is inherent to the case $\rho \neq 1$.

4 Approximations of CRLB

4.1 Performance analysis for MTMA (reduced state vector)

We show now that it is possible to obtain explicit approximations of the bounds for the variance of estimated kinematic parameters. The two following ingredients are fundamental:

- kinematic parameters are modified polar co-ordinates (MPC)
- approximations of interaction terms (i.e. :$M_{m,n}$) given in Section 4

The fundamental role of MPC (β_0, $\dot{\beta}$, $\dot{r}/r$, $1/r$) in TMA was proposed by Aidala and Hammel [19] and is now well recognised. Further, recognising that the TMA problem is nonlinear leads us to consider the Lie derivatives of the observation (i.e. the bearing), themselves spanned by the MPC [20]. We stress that the co-ordinate ($1/r$) plays a particular role, since it is a 'control' co-ordinate; so estimation of the related component will be treated separately.

To facilitate the calculations, the following (partial) source state vectors are considered throughout this Section:

$$\boldsymbol{X}_1 = (\beta_1^0, \dot{\beta}_1, \ddot{\beta}_1)^*, \quad \boldsymbol{X}_2 = (\beta_2^0, \dot{\beta}_2, \ddot{\beta}_2)^* \quad (13)$$

β_i^0, $\dot{\beta}_i$, $\ddot{\beta}_i$, 'respectively' denote the initial (i.e. at time 0) bearing, the bearing-rate and the time derivative of the bearing-rate of the ith source. Also, we assume that the probability q is known. Further, note that the 'usual' MPC have been slightly modified since we use $\ddot{\beta}$ in place of $\dot{r}/r$. This is quite justified since, in the absence of observer maneuver, we have $\ddot{\beta} = -2\dot{\beta}\dot{r}/r$ (see [20] for the general case) and higher order derivatives of β can be expressed as polynomials in $\{\dot{\beta}, \ddot{\beta}\}$ of increasingly homogenous degree [20]. Thus, the following quadratic bearing model is considered in this section:

$$\beta_i(k) = \beta_i(0) + k\dot{\beta}_i + \frac{k^2}{2}\ddot{\beta}_i$$

Then, from eqn. 4 the FIM (relative to $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$) takes the following form:

$$\mathrm{FIM} = \sum_{k=1}^{T} \mathcal{M}_k \otimes \boldsymbol{G}_k \boldsymbol{G}_k^*$$

where

$$\mathcal{M}_k = \begin{pmatrix} \frac{q^2}{\sigma_1^2} M_{2,0}(k) & \frac{q(1-q)}{\sigma_1\sigma_2} M_{1,1}(k) \\ \frac{q(1-q)}{\sigma_1\sigma_2} M_{1,1}(k) & \frac{(1-q)^2}{\sigma_2^2} M_{0,2}(k) \end{pmatrix} \quad (14)$$

$$\boldsymbol{G}_k = (1, k, k^2/2)^*$$

Note that now the gradient vector $\boldsymbol{G}_k$ is identical for the two sources. This is due to the co-ordinate choice (i.e. MPC).

It is quite reasonable to assume that the parameter d_k is sufficiently small (i.e. $d_k \leq 1$). A 3rd-order expansion (w.r.t. d_k) of the components of the matrix $\mathcal{M}_k$ yields

$$\mathcal{M}_k = \mathcal{M}_0(k) + d_k^2 \mathcal{M}_1(k) \quad (15)$$

Calculation of the CRLB will require convenient approximations of the interaction matrices $\mathcal{M}_0$ and $\mathcal{M}_1$. These approximations have been derived in Section 4. In this Section it has been shown that the cases $\rho = 1$ and $\rho \neq 1$ must be considered separately since approximations are quite different. Indeed, algebraically, a major difference exists: the approximated matrix $\mathcal{M}_0$ is rank-one when $\rho = 1$, while it is full rank otherwise. The corresponding CRLB calculations will thus be considerably different.

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4.1.1 The case $\rho = 1$: For the case $\rho = 1$, the matrices $\mathcal{M}_0$ and $\mathcal{M}_1$ are straightforwardly deduced from eqn. 7, yielding

$$\mathcal{M}_0 = \begin{pmatrix} q^2 & q(1-q) \\ q(1-q) & (1-q)^2 \end{pmatrix},$$
$$\mathcal{M}_1 = \begin{pmatrix} q^2(1-q)(3q-2) & 3q^2(1-q)^2 \\ 3q^2(1-q)^2 & q(1-q)^2(1-3q) \end{pmatrix}$$

so that

$$\sigma^2\text{FIM} = \underbrace{\mathcal{M}_0 \otimes \left(\sum_k \boldsymbol{G}_k \boldsymbol{G}_k^*\right)}_{A} - \underbrace{\mathcal{M}_1 \otimes \left(\sum_k 4d_k^2 \boldsymbol{G}_k \boldsymbol{G}_k^*\right)}_{B} \tag{16}$$

It is now convenient to define the following matrices which will play a major role for the analysis:

$$\begin{aligned} \mathcal{C}_0 &\triangleq \textstyle\sum_k \boldsymbol{G}_k \boldsymbol{G}_k^*, \mathcal{C}_1 \triangleq \sum_k (4d_k^2 \boldsymbol{G}_k \boldsymbol{G}_k^*) \\ A &= \mathcal{M}_0 \otimes \mathcal{C}_0, B = \mathcal{M}_1 \otimes \mathcal{C}_1 \end{aligned} \tag{17}$$

Here, we note that since rank $(\mathcal{M}_0) = 1 (\mathcal{M}_0 = \boldsymbol{V}_0 \boldsymbol{V}_0^*)$ and $\mathcal{C}_0$ is invertible, the rank of A is 3. On the another hand, the $\mathcal{M}_1$ matrix is invertible, as well as $\mathcal{C}_1$, hence B is invertible. However, inversion of FIM must be considered with a certain care, since for values of d_k as small as 1/2, the norm of B is (generally) quite smaller than the A one. In fact, since A is rank deficient, we *cannot* use the general formula for inverting the sum of invertible matrices. This difficulty requires us to consider the eigensystem of A and rather technical calculations. Despite rather indirect intermediate steps, the following result yields an explicit bound, remarkable by its simplicity; which is the main result of this paper. Denote FIM^{-1} [1] and FIM^{-1} [2] as the 3×3 diagonal block-matrices of FIM^{-1} corresponding to variance bounds for source 1 and source 2 parameters, respectively, then we have

Proposition 2: For $\rho = 1$, the following approximations of the variance bounds hold ($\mathcal{P} \triangleq (\mathcal{C}_0^{-1} + \alpha\ \mathcal{C}_1^{-1})^{-1}$, $\alpha = -1/q(1-q)$; Note that if q is interchanged with $1-q$, FIM^{-1} [1] becomes FIM^{-1} [2]):

$$\begin{aligned} \sigma^{-2}\text{FIM}^{-1}[1] &= \frac{-\alpha(1-3q)}{2q}\mathcal{C}_1^{-1} - \frac{1}{q^2(1-q)^2}\mathcal{C}_1^{-1}\mathcal{P}\mathcal{C}_1^{-1} \\ \sigma^{-2}\text{FIM}^{-1}[2] &= \frac{-\alpha(2-3q)}{2(q-1)}\mathcal{C}_1^{-1} - \frac{1}{q^2(1-q)^2}\mathcal{C}_1^{-1}\mathcal{P}\mathcal{C}_1^{-1} \end{aligned} \tag{18}$$

Proof: We are now dealing with the calculation of an explicit form of FIM^{-1}, where FIM is given by eqn. 16. Denote $\{\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3\}$ as the eigenvectors of $\mathcal{C}_0$, and $\Delta = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ the (diagonal) matrix of the eigenvalues. Further, note that under the assumption $\rho = 1$, the rank of the matrix $\mathcal{M}_0$ is one ($\mathcal{M}_0 = \boldsymbol{V}_0\boldsymbol{V}_0^*$ with $\boldsymbol{V}_0^* = (q, (1-q))$). Then it is easily shown that the vectors $\{\boldsymbol{W}_1 = \boldsymbol{V}_0 \otimes \boldsymbol{V}_1, \boldsymbol{W}_2 = \boldsymbol{V}_0 \otimes \boldsymbol{V}_2, \boldsymbol{W}_3 = \boldsymbol{V}_0 \otimes \boldsymbol{V}_3\}$ are eigenvectors of A, $\{\lambda_i\}_{i=1}^3$ being the associated eigenvalues. We then have

$$A = \mathcal{U}\Delta\mathcal{U}^*, \quad \mathcal{U} = \{\boldsymbol{W}_1, \boldsymbol{W}_2, \boldsymbol{W}_3\} = \boldsymbol{V}_0 \otimes \mathcal{V}$$

where

$$\mathcal{V} \triangleq (\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3)$$

The following inversion formula, valid for B invertible, is then instrumental [21]:

$$(B + \mathcal{U}\Delta\mathcal{U}^*)^{-1} = B^{-1} - B^{-1}\mathcal{U}(\Delta^{-1} + \mathcal{U}^*B^{-1}\mathcal{U})^{-1}\mathcal{U}^*B^{-1} \tag{19}$$

So we have now to deal with the calculation of the various terms of the right member of eqn. 19.

Step 1. Calculation of $B^{-1}\mathcal{U}$: Since $\mathcal{M}_1$ and $\mathcal{C}_1$ are invertible, and invoking the classical results [22, 23], i.e.

$$\begin{aligned} (A \otimes B)^{-1} &= A^{-1} \otimes B^{-1} \\ (A \otimes B)(C \otimes D) &= AC \otimes BD \end{aligned} \tag{20}$$

we obtain ($\mathcal{V} \triangleq \{\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3\}$)

$$\begin{aligned} B^{-1}\mathcal{U} &= -(\mathcal{M}_1^{-1} \otimes \mathcal{C}_1^{-1})(\boldsymbol{V}_0 \otimes \mathcal{V}) \\ &= -(\mathcal{M}_1^{-1}\boldsymbol{V}_0) \otimes (\mathcal{C}_1^{-1}\mathcal{V}) \end{aligned}$$

Of course, a similar result holds for the conjugate term (i.e. $\mathcal{U}^*B^{-1} = -(\boldsymbol{V}_0^*\mathcal{M}_1^{-1}) \otimes (\mathcal{V}^*\mathcal{C}_1^{-1})$).

Step 2. Calculation of $\mathcal{U}^*B^{-1}\mathcal{U}$: Using the previous result we obtain

$$\begin{aligned} \mathcal{U}^*B^{-1}\mathcal{U} &= -(\boldsymbol{V}_0^* \otimes \mathcal{V}^*)(\mathcal{M}_1^{-1}\boldsymbol{V}_0 \otimes \mathcal{C}_1^{-1}\mathcal{V}) \\ &= -(\boldsymbol{V}_0^*\mathcal{M}_1^{-1}\boldsymbol{V}_0) \otimes \mathcal{V}^*\mathcal{C}_1^{-1}\mathcal{V} \\ &= \alpha\mathcal{V}^*\mathcal{C}_1^{-1}\mathcal{V} \end{aligned} \tag{21}$$

α is simply a scalar ($\alpha = -\boldsymbol{V}_0^*\mathcal{M}_1^{-1}\boldsymbol{V}_0$) factor of the 3×3-matrix $\mathcal{V}^*\mathcal{C}_1^{-1}\mathcal{V}$. The value of α is given later.

Step 3. Calculation of $(\Delta^{-1} + \mathcal{U}^*B^{-1}\mathcal{U})^{-1}$: From previous calculations we deduce ($\Delta'^{-1} = \mathcal{V}\Delta^{-1}\mathcal{V}^*$):

$$\begin{aligned} \Delta^{-1} + \mathcal{U}^*B^{-1}\mathcal{U} &= \Delta^{-1} + \alpha\mathcal{V}^*\mathcal{C}_1^{-1}\mathcal{V} \\ &= \mathcal{V}^*(\Delta'^{-1} + \alpha\mathcal{C}_1^{-1})\mathcal{V} \end{aligned} \tag{22}$$

Now, the following implication holds true ($\mathcal{V}$ unitary matrix):

$$\mathcal{C}_0 = \mathcal{V}\Delta\mathcal{V}^* \Rightarrow \Delta'^{-1} = \mathcal{V}\Delta^{-1}\mathcal{V}^* = \mathcal{C}_0^{-1}$$

so that

$$(\Delta^{-1} + \mathcal{U}^*B^{-1}\mathcal{U})^{-1} = \mathcal{V}^*(\mathcal{C}_0^{-1} + \alpha\mathcal{C}_1^{-1})^{-1}\mathcal{V} \tag{23}$$

Considering the preceding formula as well as the basic inversion formula (eqn. 19), a last step is required, namely the calculation of the term $B^{-1}\mathcal{U}\mathcal{V}^*$ and of associated simplifications.

Step 4. Calculation of $B^{-1}\mathcal{U}(\Delta^{-1} + \mathcal{U}^*B^{-1}\mathcal{U})^{-1}\mathcal{U}^*B^{-1}$: Collecting previous results we obtain

$$\begin{aligned} &B^{-1}\mathcal{U}(\Delta^{-1} + \mathcal{U}^*B^{-1}\mathcal{U})^{-1}\mathcal{U}^*B^{-1} \\ &\quad = [\mathcal{M}_1^{-1}\boldsymbol{V}_0 \otimes \mathcal{C}_1^{-1}\mathcal{V}]\mathcal{V}^*(\mathcal{C}_0^{-1} + \alpha\mathcal{C}_1^{-1})^{-1} \\ &\quad\quad \mathcal{V}[\boldsymbol{V}_0^*\mathcal{M}_1^{-1} \otimes \mathcal{V}^*\mathcal{C}_1^{-1}]. \end{aligned} \tag{24}$$

A last simplification step is then

$$\begin{aligned} \mathcal{V}[\boldsymbol{V}_0^*\mathcal{M}_1^{-1} \otimes \mathcal{V}^*\mathcal{C}_1^{-1}] &= (1 \otimes \mathcal{V})(\boldsymbol{V}_0^*\mathcal{M}_1^{-1} \otimes \mathcal{V}^*\mathcal{C}_1^{-1}) \\ &= (\boldsymbol{V}_0^*\mathcal{M}_1^{-1}) \otimes (\mathcal{V}\mathcal{V}^*\mathcal{C}_1^{-1}) \\ &= (\boldsymbol{V}_0^*\mathcal{M}_1^{-1}) \otimes \mathcal{C}_1^{-1} \end{aligned} \tag{25}$$

The following result summarises all the preceding calculations:

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Lemma 2: Under the Section 5 hypotheses ($\rho = 1$), the FIM inverse takes the following form:

$$\begin{aligned}
&\sigma^{-2}(\mathrm{FIM})^{-1} \\
&\quad = -\mathcal{M}_1^{-1} \otimes \mathcal{C}_1^{-1} \\
&\quad\quad - (\mathcal{M}_1^{-1} \boldsymbol{V}_0 \otimes \mathcal{C}_1^{-1})(\mathcal{C}_0^{-1} + \alpha \mathcal{C}_1^{-1})^{-1}(\boldsymbol{V}_0^* \mathcal{M}_1^{-1} \otimes \mathcal{C}_1^{-1})
\end{aligned}$$

It simply remains to calculate explicit expressions of elementary terms (i.e. α, $\mathcal{M}_1^{-1} \boldsymbol{V}_0 \otimes \mathcal{C}_1^{-1}$). For $\rho = 1$, the following results are obtained:

$$\alpha = \frac{-1}{q(1-q)} \tag{26}$$

$$\mathcal{M}_1^{-1} \boldsymbol{V}_0 \otimes \mathcal{C}_1^{-1} = -\alpha \begin{pmatrix} \mathcal{C}_1^{-1} \\ \mathcal{C}_1^{-1} \end{pmatrix} \tag{27}$$

So ending the proof. □

A further step of approximation may be considered for very close sequence of bearings. More precisely, if we assume that $d_k \ll 1$ then we can reasonably assume that (element-wise) $\mathcal{C}_0^{-1} \ll \mathcal{C}_1^{-1}$, so that ($\mathcal{P} \approx \alpha^{-1}\, \mathcal{C}_1$), yielding

Result 3

$$\begin{aligned}
\mathrm{FIM}^{-1}[1] &\approx \frac{\sigma^2}{2q^2} \mathcal{C}_1^{-1} \\
\mathrm{FIM}^{-1}[2] &\approx \frac{\sigma^2}{2(q-1)^2} \mathcal{C}_1^{-1} \\
\mathcal{C}_1 &= \sum_k 4 d_k^2 \boldsymbol{G}_k \boldsymbol{G}_k^*.
\end{aligned} \tag{28}$$

The simplicity is rather striking, and the result appears quite natural. In particular, the sequence of 'normalised separation' (i.e. $\{d_k\}_k$) plays the fundamental role.

4.1.2 The case $\rho \neq 1$: This time, the matrices $\mathcal{M}_0$ and $\mathcal{M}_1$ are both invertible. Of course, it is possible to extend the previous calculations to this case, but no explicit result can be obtained by this way. However, it is possible to apply the general inversion formula [21], valid for A and B invertible ($A = \mathcal{M}_0 \otimes \mathcal{C}_0$, $B = \mathcal{M}_1 \otimes \mathcal{C}_1$)

$$(A + B)^{-1} = A^{-1} - A^{-1}(A^{-1} + B^{-1})^{-1} A^{-1} \tag{29}$$

Under the hypothesis $d_k \ll 1$, it may be reasonably assumed that (element-wise) $\mathcal{C}_0^{-1} \ll \mathcal{C}_1^{-1}$, so

$$(A^{-1} + B^{-1})^{-1} \approx B$$

therefore, using classical properties of Kronecker products (eqn. 20), we have

$$\begin{aligned}
\sigma^2 \mathrm{FIM}^{-1} &\approx \mathcal{M}_0^{-1} \otimes \mathcal{C}_0^{-1} \\
&\quad - (\mathcal{M}_0^{-1} \otimes \mathcal{C}_0^{-1})(\mathcal{M}_1 \otimes \mathcal{C}_1)(\mathcal{M}_0^{-1} \otimes \mathcal{C}_0^{-1}) \\
&= \mathcal{M}_0^{-1} \otimes \mathcal{C}_0^{-1} - (\mathcal{M}_0^{-1} \mathcal{M}_1 \mathcal{M}_0^{-1}) \otimes (\mathcal{C}_0^{-1} \mathcal{C}_1 \mathcal{C}_0^{-1})
\end{aligned} \tag{30}$$

In general (see Section 4), the matrices $\mathcal{M}_0$ and $\mathcal{M}_1$ are relatively complicated. Furthermore, the validity of such approximations is limited to "reasonable" values of ρ. For large (or low) values of ρ, the problem is more relevant to hypothesis testing [5].

4.2 Performance analysis for MTMA (complete state vector)

The previous analysis may be easily extended to the estimation of the complete source state vector. Again, the analysis becomes possible by considering MPC as the general framework. To avoid consideration of particular scenario we shall deal with a system constituted of two separated receivers [24]. The TMA problem then becomes completely observable. First, kinematic relations are considered, allowing us to utilise the framework of the previous Section.

The problem is defined as follows. Two (fixed) receivers are placed on the x-line, the first one at (0,0) and the second at (d, 0). For both receivers, measurements are bearings-only (β_1 and β_2). Direct calculations yield

$$\begin{aligned}
\dot{\beta}_1 &= \frac{\det(\boldsymbol{v}, \boldsymbol{r}_1)}{r^2} = \frac{1}{r}(v_x \cos\beta_1 - v_y \sin\beta_1) \\
\dot{\beta}_2 &= \frac{\det(\boldsymbol{v}, \boldsymbol{r}_2)}{r_2^2} = \frac{r(v_x \cos\beta_1 - v_y \sin\beta_1) - d v_y}{r^2 + 2dr \sin\beta_1 + d^2} \\
&= (\dot{\beta}_1 - d v_y / r^2)(1 + 2d/r \sin\beta_1 + d^2/r^2)^{-1}
\end{aligned} \tag{31}$$

We then consider an expansion of β_2, $\dot{\beta}_2$, $\ddot{\beta}_2$, with respect to $\varepsilon \triangleq d/r$, around 0. The source trajectory itself is determined by the four-dimensional state vector $\boldsymbol{X}$, $\boldsymbol{X} = (\beta_1^0, \dot{\beta}_1, \ddot{\beta}_1, 1/r)$ and, for a source, the gradient vectors $\boldsymbol{G}_k^1$ and $\boldsymbol{G}_k^2$ associated with the measurements of receiver 1 and receiver 2, respectively, stand as follows:

$$\mathbf{G}_k^1 = \begin{pmatrix} 1 \\ k \\ \dfrac{k^2}{2} \\ 0 \end{pmatrix};$$

$$\mathbf{G}_k^2 \approx \begin{pmatrix} 1 - 2\dfrac{kd}{r}\dot{\beta}_1 \cos\beta_1 - \dfrac{k^2 d}{2r}(3\ddot{\beta}_1 \cos\beta_1 - 2\dot{\beta}_1^2 \sin\beta_1) \\ k(1 - 2\dfrac{d}{r}\sin\beta_1 - k\dfrac{d}{r}\dot{\beta}_1 \cos\beta_1) \\ \dfrac{k^2}{2}(1 - 3\dfrac{d}{r}\sin\beta_1) \\ -\left(d + 2kd\dot{\beta}_1 \sin\beta_1 + \dfrac{k^2 d}{2}(2\dot{\beta}_1^2 \cos\beta_1 + 3\ddot{\beta}_1 \sin\beta_1)\right) \end{pmatrix} \tag{32}$$

Since we are interested with close source trajectories it is quite reasonable to assume that these gradients are independent of the source index. Now, let us denote FIM_1 and FIM_2 the Fisher information matrices associated with receivers 1 and 2, and FIM the global one. The measurements on receivers 1 and 2 being assumed uncorrelated, we have

$$\mathrm{FIM} = \mathrm{FIM}_1 + \mathrm{FIM}_2$$

where

$$\begin{aligned}
\mathrm{FIM}_1 &= \mathcal{M}_0 \otimes \left(\textstyle\sum_k \boldsymbol{G}_k^1 \boldsymbol{G}_k^{1,*}\right) - \mathcal{M}_1 \otimes \left(\textstyle\sum_k 4 d_k^2 \boldsymbol{G}_k^1 \boldsymbol{G}_k^{1,*}\right) \\
\mathrm{FIM}_2 &= \mathcal{M}_0 \otimes \left(\textstyle\sum_k \boldsymbol{G}_k^2 \boldsymbol{G}_k^{2,*}\right) - \mathcal{M}_1 \otimes \left(\textstyle\sum_k 4 d_k^2 \boldsymbol{G}_k^2 \boldsymbol{G}_k^{2,*}\right)
\end{aligned}$$

so that

$$\begin{aligned}
\mathrm{FIM} &= \mathcal{M}_0 \otimes \underbrace{\left(\sum_k \boldsymbol{G}_k^1 \boldsymbol{G}_k^{1,*} + \sum_k \boldsymbol{G}_k^2 \boldsymbol{G}_k^{2,*}\right)}_{\mathcal{C}_r} \\
&\quad - \mathcal{M}_1 \otimes \underbrace{\left(\sum_k 4 d_k^2 \boldsymbol{G}_k^1 \boldsymbol{G}_k^{1,*} + \sum_k 4 d_k^2 \boldsymbol{G}_k^2 \boldsymbol{G}_k^{2,*}\right)}_{\mathcal{C}_\infty}
\end{aligned} \tag{33}$$

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We restrict attention to the case $\rho = 1$. Then the matrix $\mathcal{M}_0$ is also rank-one, so the calculation of FIM^{-1} is identical in its principle, yielding again

$$\sigma^{-2}\mathrm{FIM}^{-1}[1] = \frac{-\alpha(1-3q)}{2q} C_1^{-1} - \frac{1}{q^2(1-q)^2} C_1^{-1} \mathcal{P} C_1^{-1} \tag{34}$$

$$\approx \frac{\sigma^2}{2q^2} C_1^{-1} \tag{35}$$

Consider now the the estimation of both the kinematic parameters and the mixing probability q. The FIM is then a hermitian 7×7 matrix, of the following form:

$$\mathrm{FIM} = \begin{pmatrix} \mathrm{FIM}_{ki} & | & \frac{-1}{\sigma_1} \sum_k m_{10}(k) \boldsymbol{G}_k \\ & | & \frac{1}{\sigma_2} \sum_k m_{01}(k) \boldsymbol{G}_k \\ \overline{(``")^*} & \overline{|} & \overline{\beta} \end{pmatrix} \tag{36}$$

Consider again the case $\rho = 1$. Then, using the partitioned matrix inversion and the Woodbury lemmas [21, 22], we obtain (Appendix, Section 9.3)

$$\mathrm{FIM}_{ki,q}^{-1} \approx \mathrm{FIM}_{ki}^{-1} \tag{37}$$

This result is important since it means that the CRLB (relative to the estimation of kinematic parameters, i.e. $\mathrm{FIM}_{ki,q}^{-1}$) is not significantly affected by the estimation of the mixing parameter q.

5 Numerical results

First we present the multiple-source scenario, common for all the results of this Section. Two (close) sources move with a constant velocity vector (rectilinear and uniform motion). Their trajectories are represented in the (x, y)-plane in Fig. 2. The kinematic parameters ($r_x(0) = 15$ km, $r_y(0) = 15$ km, $v_x = 12$ m/s, $v_y = 6$ m/s) of target 1 are fixed (solid line), while that of target 2 take 15 different values corresponding ($r_x(0) = 22$ km $\rightarrow$ 20.6 km, $r_y(0) = 18$ km, $v_x = 11.5$ m/s, $v_y = 5$ m/s) to various initial positions (dashed lines). The corresponding trajectories are thus deduced by a translation, marked 1 to 15. The receiver is fixed, at the origin. The (exact) observations (i.e. the bearings) associated with these scenarios are represented in Fig. 2. The measurements noise is identical for the two sources and constant throughout the whole scenario ($\sigma(\hat{\beta}) = 1/8$ rd ≈ 7 deg). Note that the two targets have close bearings, so that the assumption $d_k \leq 1$ be satisfied.

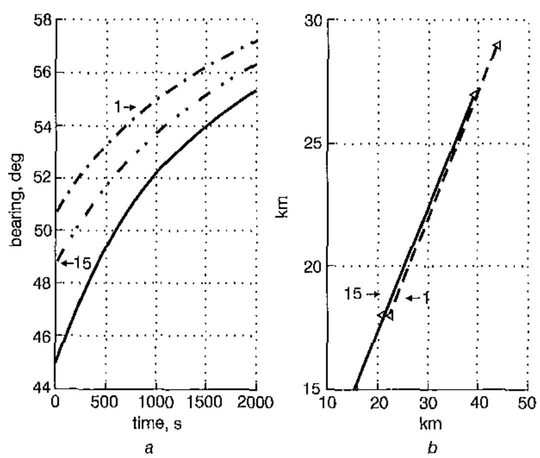

Fig. 2 *Multiple-source scenario*

a Bearings against time
b Source trajectories in (x, y) plane
-·- — target 1
-·- target 2

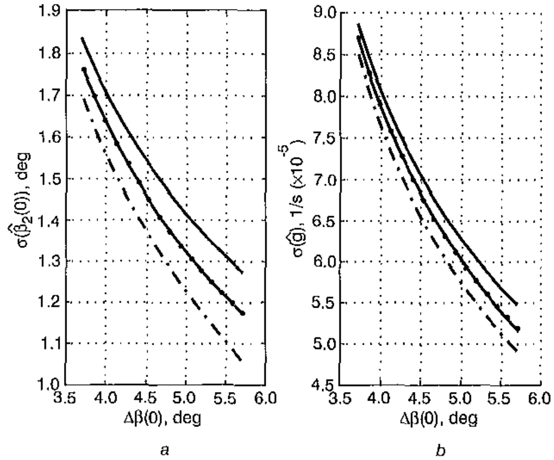

Fig. 3 *Accuracy of CRLB approximations ($\sigma(\hat{\beta}_2(0))$ and $\sigma(\hat{g})$)*

—— prop. 1
---- prop. 2
-·- res. 3

Accuracy of the approximations of the variance bounds is illustrated in Fig. 3. The values of $\Delta\beta(0)$ (ranging from 5.7 to 3.7 deg) correspond to the various initial positions of target 2 (marked from 1 to 15). The solid lines represent the exact values of the lower bound relative to the estimation of $\beta_2(0)$ (left), respectively $g_2 (g = \dot{r}/r)$, as given by prop. 1. The continuous-dotted lines illustrate approximations given by prop. 2, while the dashed lines represent the simpler and more explicit approximation obtained in res. 3. The approximation given by prop. 2 performs satisfactorily; while the simpler one (res. 3) is still quite acceptable. Note that, relative to the initial measurement variance, the variance of $\hat{\beta}_2(0)$ is considerably reduced.

We are now dealing with the estimation of the complete state vector. The source trajectories are unchanged; but this time two (fixed) receivers are considered (both on the x-axis, separated by a distance of 2 km). In Fig. 4, exact bounds (prop. 1) are compared with approximations given

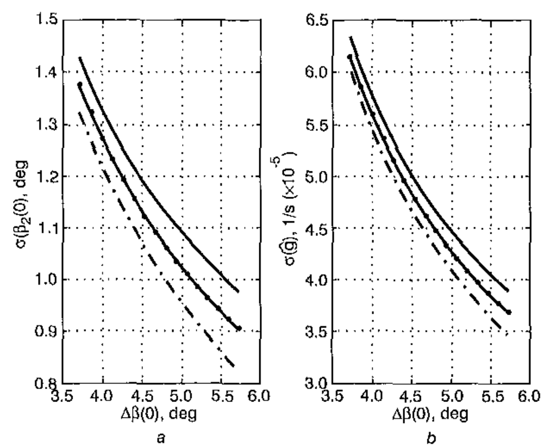

Fig. 4 *Accuracy of CRLB approximations ($\sigma(\hat{\beta}_2(0))$ and $\sigma(\hat{g})$, two receivers)*

—— prop. 1
--- prop. 2 (eqn. 34)
-·-· res. 3 (eqn. 35)

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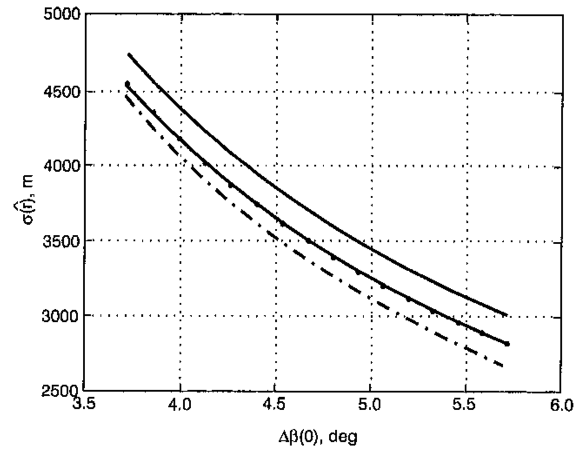

Fig. 5 *Accuracy of CRLB approximations ($\sigma(\dot{r})$, two receivers)*
—— prop. 1
--•-- prop. 2 (eqn. 34)
-·- res. 3 (eqn. 35)

by prop. 2, eqn. 34 and res. 3, eqn. 35. Then this analysis is extended to the lower bound (Fig. 5) relative to the estimation of the "missing" co-ordinate (i.e. $1/r$). Again, the quality of the approximations is satisfactory. Note however that, in comparison with the unique source case, the value of $\sigma(\dot{r})$ is very important. This is due to the track interaction.

6 Conclusions

Based on the use of modified polar co-ordinates and of convenient approximations of the interaction matrices, explicit approximations of the CRLB for MTMA have been derived. Their pertinence has been illustrated by numerical comparisons. In this way future directions, could include connections with multiple-target tracking algorithms and sensor management as well as incorporating the track coalescence phenomenon in the observation model.

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9 Appendix

9.1 Calculation of $G_{0,0}(g_1, g_2, k)$

Consider the series expansion presented in eqn. 10, but this time assume that the second-order eqn. 9 has no real root. Then $(c_l = 1/((1-q)\rho)[(-q/(1-q)\rho)^l])$

$$G_{0,0}(g_1, g_2) = \sqrt{\rho/2\pi}\sum_{l=0}^{\infty}\int_{-\infty}^{\infty} H_l(y)\,dy = \sqrt{\rho/2\pi}\sum_{l=0}^{\infty} c_l \int_{-\infty}^{\infty} h_1^{l+1}(y)h_2^{-1}(y)\,dy \quad (38)$$

Thus, the basic point is the calculation of the integral $\int_{-\infty}^{\infty} h_1^{l+1}(y)h_2^{-1}(y)dy$ (where $h_i(y) = \exp[-(y-d_i)^2/2\rho_i]$, $i = 1, 2$), i.e. $\int_{-\infty}^{\infty}\exp[-(l+1)(y+d)^2/2\rho + l(y-d)^2\rho/2]$. Considering the following quadratic-form factorisation $(\alpha = l\rho - (l+1)/\rho,\ \beta = l\rho + (l+1)/\rho)$:

$$-(l+1)(y+d)^2/2\rho + l(y-d)^2\rho/2 = \frac{1}{2}[\alpha y^2 - 2d\beta y + \alpha d^2] = \frac{1}{2}\left[\alpha\left(y - d\frac{\beta}{\alpha}\right)^2 + d^2\left(\frac{\alpha^2-\beta^2}{\alpha}\right)\right] \quad (39)$$

we obtain

$$\int_{-\infty}^{\infty} h_1^{l+1}(y)h_2^{-1}(y)dy = \exp\left[\frac{d^2}{2}\left(\frac{\alpha^2-\beta^2}{\alpha}\right)\right]\int_{-\infty}^{\infty}\exp\left[\frac{\alpha}{2}\left(y - d\frac{\beta}{\alpha}\right)^2\right]dy \quad (40)$$

Under the hypothesis $\rho < 1$, we have $\alpha < 0$. Thus, the preceding integral exists. Making necessary variable

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changes, we have (integration of a normal density)

$$\int_{-\infty}^{\infty} \exp\left[\frac{\alpha}{2}\left(y - d\frac{\beta}{\alpha}\right)^2\right] dy = \sqrt{2\pi}\left(\frac{\rho}{l(1-\rho^2)+1}\right)^{1/2} \tag{41}$$

so that finally (with $a_l \triangleq l(1-\rho^2)+1$)

$$G_{0,0}(g_1, g_2, k) = \frac{1}{(1-q)} \sum_{l=0}^{\infty} (-1)^l a_l^{-1/2} \exp\left[2d^2 \frac{\rho}{a_l} l(l+1) - l \log\left(\frac{(1-q)\rho}{p}\right)\right] \tag{42}$$

yielding eqn. 11.

9.2 Polynomials $P_{i,j}$ and $Q_{i,j}$

The polynomials, eqn. 12, are detailed as follows:

$$\begin{aligned} P_{0,2} &= 3(1-\rho) + 3q(1-\rho) + q\rho(\rho-1) \\ &\quad + \rho^2 + 10q^2(\rho-1)^2 \\ P_{1,1} &= -7q + \rho + 16q\rho - 9q\rho^2 + 10q^2(\rho-1)^2 \\ P_{2,2} &= 7 - 15\rho + 9\rho^2 + q(-17 + 36\rho - 19r^2) \\ &\quad + 10q^2(\rho-1)^2 \end{aligned} \tag{43}$$

and

$$\begin{aligned} Q_{0,2} &= 468q^4(\rho-1)^2 - 12q^3(37 - 77\rho + 40\rho^2) \\ &\quad + q^2(70 - 156\rho + 89\rho^2) - q \\ Q_{1,1} &= 468q^4(\rho-1)^2 - 18q^3(49 - 100\rho + 51\rho^2) \\ &\quad + 3q^2(158 - 332\rho + 175\rho^2) \\ &\quad - 3q(20 - 44\rho + 25\rho^2) \\ Q_{2,0} &= (q-1)(468q^3(\rho-1)^2 - 12q^2(71 - 145\rho + 74\rho^2) \\ &\quad + q(436 - 916\rho + 483\rho^2) - 51 + 112\rho - 63\rho^2] \end{aligned} \tag{44}$$

9.3 Complete state vector

Denote $\boldsymbol{H}$ the bordering vector

$$\boldsymbol{H}^* = \left(\frac{-1}{\sigma}\sum_k m_{10}(k)\boldsymbol{G}_k^*, \frac{1}{\sigma}\sum_k m_{01}(k)\boldsymbol{G}_k^*\right) = \boldsymbol{m} \otimes \boldsymbol{G})^*,$$

where (see eqn. 7, $\rho = 1$)

$$\boldsymbol{G} = \sum_k 2d_k \boldsymbol{G}_k, \quad \boldsymbol{m} = \left(\frac{q}{\sigma}, \frac{(1-q)}{\sigma}\right)^* \tag{45}$$

and $\mathrm{FIM}_{ki,q}^{-1}$, the 6×6 block of FIM^{-1} relative to kinematic parameters. Using the partitioned matrix inversion and the Woodbury lemmas [21], we obtain

$$\mathrm{FIM}_{ki,q}^{-1} = \mathrm{FIM}_{ki}^{-1} + \frac{\beta^{-1}}{1 - \beta^{-1}\boldsymbol{H}^*\mathrm{FIM}_{ki}^{-1}\boldsymbol{H}} \mathrm{FIM}_{ki}^{-1}\boldsymbol{H}\boldsymbol{H}^*\mathrm{FIM}_{ki}^{-1} \tag{46}$$

We have now to deal with the calculation of the corrective terms $\boldsymbol{H}^*\mathrm{FIM}_{ki}^{-1}\boldsymbol{H}$ and $\mathrm{FIM}_{ki}^{-1}\boldsymbol{H}\boldsymbol{H}^*\mathrm{FIM}_{ki}^{-1}$. Using prop. 2, we obtain ($\boldsymbol{m} = \boldsymbol{V}_0$)

$$\boldsymbol{H}^*\mathrm{FIM}_{ki}^{-1}\boldsymbol{H} = -\alpha(\boldsymbol{G}^*\mathcal{C}_1^{-1}\boldsymbol{G}) + \alpha^2(\mathcal{C}_1^{-1}\boldsymbol{G})^*\mathcal{P}(\mathcal{C}_1^{-1}\boldsymbol{G}) \tag{47}$$

$$\begin{aligned} &= \alpha\boldsymbol{G}^*(\mathcal{C}_1^{-1} - \alpha\mathcal{C}_1^{-1}\mathcal{P}\mathcal{C}_1^{-1})\boldsymbol{G} \\ &\approx 0 \end{aligned} \tag{48}$$

Similarly, we have

$$\mathrm{FIM}_{ki}^{-1}\boldsymbol{H}\boldsymbol{H}^*\mathrm{FIM}_{ki}^{-1} \approx 0$$

so that, finally

$$\mathrm{FIM}_{ki,q}^{-1} \approx \mathrm{FIM}_{ki}^{-1} \tag{49}$$

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