Scheduling Active and Passive Measurements

J.-P. Le Cadre
IRISA/CNRS
Campus de Beaulieu
35 042, Rennes (cedex), France.
lecadre@irisa.fr

Abstract - ***Many tracking systems involve basically active and passive subsystems. If it can be reasonably assumed that passive measurements have no "cost", this is not true for active measurements. So, a general problem is to scheduling active measurements, so as to combine them optimally with the passive ones. More generally, we are interested by optimizing controls in the estimation procedure.***

Keywords: Target Motion Analysis, Tracking, estimation, resource allocation, multilinear algebra, sensor management.

1 Introduction

A recent trend in systems for detection/tracking is the availability of multiple sensing modalities that differ in such crucial measures as detection, estimation, geographical coverage, cost/risk of operation.

In these systems, the determination of optimal sensor design and measurement scheduling are important issues in estimating system status. The problem of measurement scheduling has a long history. Mehra [1] considered different norms of the Fisher information Matrix as criteria for the optimization of measurements scheduling. Van Keuk et al. [2] examined the problem of efficient allocation of radar resources for maintaining existing tracks. Avitzour and Rogers [3] considered the problem of optimizing the time-distribution of measurement variances for estimating a scalar random variable y, given that: 1) the total measurement budget is fixed, 2) the cost of an individual measurement varies inversely with the (controllable) measurement variance, and 3) the autocorrelation matrix of the quantity to be measured $\{x(i) \ : \ i = 1, \cdots, N\}$, as well as the cross-correlation between $\{x(i) \ : i = 1, \cdots, N\}$ and y are known.

This work has been extended to discrete-time, vector stochastic-processes by Shakeri et al. [4]. However, we stress that previous systems are mainly devoted to linear systems. Opposite, our contribution will be to analyze performance for optimal scheduling of the multiple estimation modes for a non-linear system. In particular, we shall focuse on non-linear effects in target motion analysis and global optimization. After a general presentation of the problem and its principal difficulties (section 2), general tools are presented in section 3 (essentially multilinear algebra). It is then possible to consider general formulations of measurement scheduling; first in the deterministic case (section 4), then for the stochastic case (section 5).

2 A basic formulation

We shall restrict, in a first time, to a target in rectilinear and uniform motion. Then the equations of motion takes the following form [6] :

$$\mathbf{r}(t) = \mathbf{r}(0) + t\mathbf{v}(0) - \int_0^t (t-\tau)\mathbf{a}_o(\tau)d\tau \qquad (2.1)$$

where :

$$\left| \begin{array}{l} \text{the reference time is } 0\,, \\ \mathbf{r} \text{ and } \mathbf{v} \text{ are resp. the relative target range} \\ \quad \text{and velocity vectors}\,, \\ \mathbf{a}_o \text{ describes own-ship accelerations}\,. \end{array} \right.$$

For bearings-only (planar problem) target motion anlysis (TMA for the sequel) these measurements consist of line of sight angles which satisfy the following relation :

$$\beta(t) = \tan^{-1}\left[r_x(t)/r_y(t)\right] \quad . \qquad (2.2)$$

For active measurements (e.g. radar, active sonar), target range $r(t)$ is also available and related to the target state by :

$$r(t) = \left[r_x^2(t) + r_y^2(t)\right]^{1/2} \quad . \qquad (2.3)$$

The *available* measurements are the estimated angles $\hat{\beta}_k$ from the observer to the source and *when available*, estimated ranges. The measurement noises $w_{\beta,k}, w_{r,k}$ are usually modelled by i.i.d. zero-mean, gaussian process. Associated variance σ^2 may depend upon the relative source-receiver positions.

Elementary calculations yield $\mathbf{M}_k$ and $\mathbf{N}_k$, the gradient vectors of measurements β_k and r_k w.r.t. the state vector $\mathbf{X}$ (the symbol $*$ denotes transposition throughout this text) :

$$\begin{array}{l} \mathbf{M}_k = \frac{1}{r_k}\left(\cos\beta_k, -\sin\beta_k, k\cos\beta_k, -k\sin\beta_k\right)^*\,, \\ \mathbf{N}_k = \left(\sin\beta_k, \cos\beta_k, k\sin\beta_k, k\cos\beta_k\right)^*\,. \end{array} \qquad (2.4)$$

Assuming independence of bearing and range measurements, calculating the Fisher Information Matrix (FIM) is a routine exercise, yielding :

$$\text{FIM} = \sum_k \left(\frac{1}{\sigma^2_{\beta,k}} \mathbf{M}_k \mathbf{M}^*_k + \frac{\delta_{r,k}}{\sigma^2_{r,k}} \mathbf{N}_k \mathbf{N}^*_k \right) . \quad (2.5)$$

In 2.5, $\delta_{r,k}$ is equal to 1 when an active measurement is available at time k, and to 0 else.

In this context, a "measure" of the system estimability is a functional of the FIM. The determinant is a convenient "measure" for the estimability of the problem. The difficulty and the originality of the problem stem from the two following facts. First, the source motion is unknown which means that the sequence of bearings is unpredictable. Second, we deal with a global optimization problem which means that we seek for an optimal *sequence* of controls [1] maximizing a global cost function like the matrix trace (i.e. $tr(\text{FIM})$).

Of course, the problem is drastically eased by considering an additive (matrix) like the trace. However, this functional is not relevant [2] for information measuring even if its calculation is quite direct.

However, note that the function det does not satisfy the MDP (Matrix Dynamic Programming Property) defined below.

Definition 1 *The function f ($\mathcal{H}_n \to \mathbb{R}$, differentiable, C^2) has the* MDP *if the following implication holds, whatever $C \in \mathcal{H}_n$* [3] *:*
$f(B) > f(A) \Rightarrow f(B+C) > f(A+C)$.

An interpretation of this definition in terms of dynamic programming is the following type of inequality [4] :

$$\max_{\bar{\mathcal{U}}^*_k} f\left(\sum_{i=n}^{k} \mathbf{G}_{\mathbf{X}_i} \mathbf{G}^*_{\mathbf{X}_i}\right) \leq \max_{\mathbf{u}_k} [f\{\mathbf{G}_{\mathbf{X}_k} \mathbf{G}^*_{\mathbf{X}_k} + F(\mathbf{X}_0, \bar{\mathcal{U}}^*_{k+1})\}] .$$

which must be valid for the strategy $\bar{\mathcal{U}}^*_k$, optimal up to time k, and for $k = n-1, \cdots, 0$. Roughly, the MDP appears as a matricial form of a "comparison" principle. A fundamental question consists in determining the functionals having it. An answer is provided with the following result.

Proposition 1 *Let f satisfying the* MDP *property then :*

$$f(A) = g\,(tr(AR))$$

where g is any monotonic increasing function and R is a fixed matrix.

[1] In this context, the instants of active measurements.

[2] On the contrary, $tr(\text{FIM}^{-1})$ is meaningful. However, calculations are, at best, as complicated as with the determinant

[3] $\mathcal{H}_n$: vector space of Hermitian matrices.

[4] F denotes a FIM matrix, $\bar{\mathcal{U}}_k$: optimal sequence of controls from n to k

We refer to [5] for the proof of Prop. 1. It is simply based on the fact that if $\nabla f(A)$ and $\nabla f(B)$ are not colinear, then their respective orthogonal subspaces are distinct which implies that there exists a matrix C for which the MDP is not satisfied. Therefore, $\nabla f(A)$ and $\nabla f(B)$ must be colinear, whatever A and B. This is a very strong property. In turn, this yields the general form of f. Consider for instance $f(A) = \log\det A$; then, for a non-singular matrix A, we have :

$$Df_A(C) = tr\left(A^{-1}C\right) = \left(\nabla^* f(A),\, C\right) ,$$

and we see immediately that f does not have the MDP property. The same remark is valid for functionals as simple as $f(A) = tr\left(A^{-1}\right)$.

3 General results

A *local approximation* of this determinant may be calculated by considering an expansion of the vectors $\mathbf{M}_{t+i}$. For instance, let us consider the following third order expansion of $\mathbf{M}_{t+i}$.

$$\mathbf{M}_{t+i} \stackrel{3}{=} \mathbf{M}_t + i\mathbf{M}^{(1)}_t + \frac{i^2}{2}\mathbf{M}^{(2)}_t + \frac{i^3}{6}\mathbf{M}^{(3)}_t . \quad (3.1)$$

Note that the above expansion must be considered as an expansion of $\mathbf{M}_{t+i}$, relatively to the arbitrary small sampling time τ_e (the time separating two consecutive measurements). For the sake of brevity, τ_e is omitted ($\tau_e \equiv 1$).
Using exterior algebra and the above expansion, the following basic result is obtained :

Proposition 2 *Consider a third order expansion of the vectors $\mathbf{M}_{t+i}$, then the determinant of the 4×4 matrix $\mathcal{M}(t, t+3) \triangleq (\mathbf{M}_t, \cdots, \mathbf{M}_{t+3})$:*
$\det(\mathcal{M}(t,t+3)) \stackrel{3}{=} \det(\mathbf{M}_t, \mathbf{M}^{(1)}_t, \mathbf{M}^{(2)}_t, \mathbf{M}^{(3)}_t)$.

Proof : First, let us briefly recall the definition of the exterior powers of a vector space [5]. Let V be an n-dimensional vector space over $\mathbb{R}$, then $\Lambda^2 V$ consists of all formal sums $\sum_i \alpha_i\, (\mathbf{U}_i \wedge \mathbf{V}_j)$, where the "wedge product" $\mathbf{U} \wedge \mathbf{V}$ is bilinear and alternate. This definition is straightforwardly extended to higher exterior powers [9]. For any basis $\{\mathbf{V}_1, \cdots, \mathbf{V}_n\}$ of V, the set of p-vectors $\{\mathbf{V}_{i_1} \wedge \cdots \wedge \mathbf{V}_{i_p},\, 0 \leq i_1 < \cdots < i_p \leq n\}$ forms a basis of the $n!/(n-p)!p!$-dimensional vector space $\Lambda^p V$. In particular, $\Lambda^4 \mathbb{R}^4$ is *one-dimensional*, and throughout the paper we make intensive use of the isomorphism $\Lambda^4\mathbb{R}^4 \equiv \Lambda^2\mathbb{R}^4 \wedge \Lambda^2\mathbb{R}^4$. The exterior algebra formalism thus appears as an economical way to conduct determinant calculations.

Denoting $\mathbf{M}\wedge\mathbf{N}$ the elements of the exterior product $\Lambda^2\mathbb{R}^4$, we have :

$$\det(\mathcal{M}(t,t+3)) = (\mathbf{M}_t \wedge \mathbf{M}_{t+1}) \wedge (\mathbf{M}_{t+2} \wedge \mathbf{M}_{t+3}) . \quad (3.2)$$

[5] For a complete presentation, we refer e.g. to [9][10]

It remains to calculate the two vectors $\mathbf{M}_t \wedge \mathbf{M}_{t+1}$ and $\mathbf{M}_{t+2} \wedge \mathbf{M}_{t+3}$ of the exterior power [9] $\Lambda^2\mathbb{R}^4$. Invoking the basic properties of exterior algebra, we obtain straightforwardly [9][10] :

$$
\begin{aligned}
\mathbf{M}_t \wedge \mathbf{M}_{t+1} &= \mathbf{M}_t \wedge \mathbf{M}_t^{(1)} + \tfrac{1}{2}\,\mathbf{M}_t \wedge \mathbf{M}_t^{(2)} + \tfrac{1}{6}\,\mathbf{M}_t \wedge \mathbf{M}_t^{(3)} \\
\mathbf{M}_{t+2} \wedge \mathbf{M}_{t+3} &= 3\,\mathbf{M}_t^{(1)} \wedge \mathbf{M}_t^{(2)} + 3\,\mathbf{M}_t^{(2)} \wedge \mathbf{M}_t^{(3)}\ , \\
&\quad +\ 5\,\mathbf{M}_t^{(1)} \wedge \mathbf{M}_t^{(3)}.
\end{aligned}
\tag{3.3}
$$

Note that the terms involving $\mathbf{M}_t$ in $\mathbf{M}_{t+2} \wedge \mathbf{M}_{t+3}$ are not considered since their contribution in $\det(\mathcal{M}(t,t+3))$ is null. Then from 3.3, we deduce $\det(\mathcal{M}(t,t+3))$:

$$
\begin{aligned}
\det(\mathcal{M}(t,t+3)) \;=\;& 3\,\mathbf{M}_t \wedge \mathbf{M}_t^{(1)} \wedge \mathbf{M}_t^{(2)} \wedge \mathbf{M}_t^{(3)}\ , \\
&+\frac{5}{2}\,\mathbf{M}_t \wedge \mathbf{M}_t^{(2)} \wedge \mathbf{M}_t^{(1)} \wedge \mathbf{M}_t^{(3)}\ , \\
&+\frac{1}{2}\,\mathbf{M}_t \wedge \mathbf{M}_t^{(3)} \wedge \mathbf{M}_t^{(1)} \wedge \mathbf{M}_t^{(2)}\ ,
\end{aligned}
$$

so that, finally :

$$\det(\mathcal{M}(t,t+3)) \overset{3}{=} \det(\mathbf{M}_t, \mathbf{M}_t^{(1)}, \mathbf{M}_t^{(2)}, \mathbf{M}_t^{(3)})\ .$$

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Furthermore, considering a third order expansion of the relative range r_t, we shall prove that the calculation of $\det(\mathrm{FIM}_{t,t+k})$ is unchanged . The hypothesis of (approximately) constant relative range can thus be removed, more precisely :

Proposition 3 *Consider a third order expansion of the vectors $\mathbf{M}_{t+i}$ and of the relative range r_{t+i}, then :* $\det(\mathrm{FIM}^{\beta}_{t,t+3}) \overset{3}{=} (\sigma\, r_t)^{-8}\left[\det\left(\mathbf{M}_t, \mathbf{M}_t^{(1)}, \mathbf{M}_t^{(2)}, \mathbf{M}_t^{(3)}\right)\right]^2$.

Proof : The FIM takes the following general form [6] :

$\mathrm{FIM} = (\mathbf{G}_t, \cdots, \mathbf{G}_{t+3})(\mathbf{G}_t, \cdots, \mathbf{G}_{t+3})^*$ where :
$\mathbf{G}_{t+i} = \frac{1}{r_{t+i}}\mathbf{M}_{t+i}$.

We can then invoke Prop. 1, thus obtaining :

$$\det(\mathrm{FIM}) = \left[\det\left(\mathbf{G}_t, \cdots, \mathbf{G}_t^{(3)}\right)\right]^2 .$$

The calculation of the derivative vector $\mathbf{G}_t^{(i)}$ ($i = 1,2,3$)) is straightforward, yielding [7] :

$$
\left|
\begin{aligned}
&\mathbf{G}^{(1)} = \tfrac{1}{r}\mathbf{M}^{(1)} - \tfrac{g}{r}\mathbf{M}\ , \\
&\mathbf{G}^{(2)} = \tfrac{1}{r}\mathbf{M}^{(2)} - 2\tfrac{g}{r}\mathbf{M}^{(1)} + (\tfrac{g^2}{r^2} - \tfrac{g^{(1)}}{r})\mathbf{M}\ , \\
&\mathbf{G}^{(3)} = \tfrac{1}{r}\mathbf{M}^{(3)} - 3\tfrac{g}{r}\mathbf{M}^{(2)} + 3(\tfrac{g^2}{r^2} - \tfrac{g^{(1)}}{r})\mathbf{M}^{(1)}\ , \\
&\quad +(3\tfrac{g^{(1)}g}{r^2} - 2\tfrac{g^3}{r^2} - \tfrac{g^{(2)}}{r})\mathbf{M}\ , \quad g = \dot{r}/r\ .
\end{aligned}
\right.
\tag{3.4}
$$

Now, the following equalities are direct consequences of exterior algebra properties:

$$
\begin{aligned}
&\mathbf{G} \wedge \mathbf{G}^{(1)} = \tfrac{1}{r^2}\,\mathbf{M} \wedge \mathbf{M}^{(1)}\ , \\
&\mathbf{G}^{(2)} \wedge \mathbf{G}^{(3)} = \tfrac{1}{r^2}\,\mathbf{M}^{(2)} \wedge \mathbf{M}^{(3)} + \textit{other terms}\,.
\end{aligned}
\tag{3.5}
$$

Other terms is only formed of exterior products of either the vector $\mathbf{M}$ or $\mathbf{M}^{(1)}$ with another vector ($\mathbf{M}^{(i)}$, $i = 0,1,2$). Since $\mathbf{G} \wedge \mathbf{G}^{(1)} = \frac{1}{r^2}\,\mathbf{M} \wedge \mathbf{M}^{(1)}$, the contribution of the *other terms* in the calculation of $\det(\mathrm{FIM})$ is null. Prop. 2 is thus proved. We stress that this property is essentially due to the fact that the term $\frac{1}{r}$ appears as a multiplicative factor.

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The calculation of the approximation of $\det(\mathrm{FIM}_{t,t+k})$, where: $k \geq 4$ is also surprisingly simple. Using exterior algebra and more precisely the Binet-Cauchy formula [8], we obtain :

$$
\begin{aligned}
&\det(\mathrm{FIM}_{t,t+k}) = (\sigma\, r_t)^{-8}\ , \\
&\times \sum_{0 \leq i_1 < i_2 < i_3 < i_4 \leq k} [\det(\mathbf{M}_{t+i_1}, \cdots, \mathbf{M}_{t+i_4})]^2,
\end{aligned}
$$

with :

$$
\begin{aligned}
&\det(\mathbf{M}_{t+i_1}, \cdots, \mathbf{M}_{t+i_4}) = P_{(i_1,i_2,i_3,i_4)}\ (\det A(t))\ , \\
&\det A(t) \triangleq \det\,(\mathbf{M}_t, \mathbf{M}_{t+1}, \mathbf{M}_{t+2}, \mathbf{M}_{t+3})\ ,
\end{aligned}
\tag{3.6}
$$

so that :

$$
\begin{aligned}
\det(\mathrm{FIM}_{t,t+k}) \;=\;& (\sigma\, r_t)^{-8}\left[\sum_{0 \leq i_1 < i_2 < i_3 < i_4 \leq k} P^2_{(i_1,i_2,i_3,i_4)}\right], \\
&\times\ \ (\det(A(t)))^2\ , \\
&= c_k (\det(A(t)))^2\ .
\end{aligned}
\tag{3.7}
$$

In 3.7, $P_{(i_1,i_2,i_3,i_4)}$ is a polynomial in i_1, i_2, i_3, i_4 of degree homogeneously equal to 12.

The problem we deal with is to obtain an explicit formula of $\det(\mathrm{FIM}_{t,t+k})$. For that purpose the following result is instrumental.

Proposition 4 $\det A(t)$ *is independent of the value of β_t, furthermore we can consider that the reference time is zero .*

Proof : Define[9] the matrix $\mathcal{R}_t$ as follows :

$$\mathcal{R}_t = \begin{pmatrix} R_t & 0 \\ tR_t & R_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \otimes R_t$$

where R_t is the rotation matrix associated with the angle $-\beta_t$. Then, the following equality is straightforwardly verified :

$$\mathbf{M}_t = \mathcal{R}_t \mathbf{e}_1\ ,\ \ \mathbf{e}_1 = (1,0,0,0)^*$$

hence :

$$\mathbf{M}_t^{(1)} = \mathcal{R}_t^{(1)}\mathbf{e}_1, \cdots, \mathbf{M}_t^{(3)} = \mathcal{R}_t^{(3)}\mathbf{e}_1\ . \tag{3.8}$$

Furthermore the following equalities are straightforwardly deduced :

$$\mathcal{R}_t \;=\; C_t \otimes R_t\ ,$$

[6] In fact, $\mathbf{G}_{t+i} = \frac{1}{\sigma\, r_{t+i}}\mathbf{M}_{t+i}$, but the constant coefficient σ is omitted for the sake of brevity

[7] For the sake of simplicity the time index t is ommitted in the following formula

[8] we assume that σ and r are constant for the duration of the analysis i.e. $\{t, \cdots, t+k\}$

[9] {$\otimes$: Kronecker product }

$$\mathcal{R}_t^{(1)} = \beta_t^{(1)}\left(C_t \otimes R_t^{(1)}\right) + D \otimes R_t ,$$
$$\mathcal{R}_t^{(2)} = \beta_t^{(2)}\left(C_t \otimes R_t^{(1)}\right) - \beta_t^{(1)}\mathcal{R}_t , + \left(\beta_t^{(1)}+1\right)\left(D \otimes R_t^{(1)}\right) ,$$
$$\mathcal{R}_t^{(3)} = \beta_t^{(3)}\left(C_t \otimes R_t^{(1)}\right) - 2\,\beta_t^{(2)}\mathcal{R}_t + \cdots ,$$

where :

$$C_t \triangleq \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, D \triangleq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \quad (3.9)$$

The above expression may be somewhat simplified by means of the following remark :

$$R_t^{(1)} = R_t\, J \text{ where } J \triangleq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \quad (3.10)$$

Using the multilinearity property of the determinant, we deduce from 3.8, 3.9 that $\det A(t)$ is a sum of elementary expressions of the following type :

$$\det\left[\mathcal{R}_t \mathbf{e}_1, (C_t \otimes R_t J)\,\mathbf{e}_1, (D \otimes R_t J)\,\mathbf{e}_1, (C_t \otimes R_t J)\,\mathbf{e}_1\right] . \quad (3.11)$$

The following classical property of the tensor product is then instrumental [10] :

$$(H_1\, F_1) \otimes (H_2\, F_2) = (H_1 \otimes H_2)\,(F_1 \otimes F_2) , \quad (3.12)$$

where H and F are endomorphisms of the state space. Applying this general property yields [10] :

$$\begin{aligned} \mathcal{R}_t &= C_t \otimes R_t , \quad (3.13) \\ &= (Id\, C_t) \otimes (R_t\, Id) , \\ &= (Id \otimes R_t)\,(C_t \otimes Id) , \end{aligned}$$

and similarly :

$$\begin{aligned} C_t \otimes R_t J &= (Id\, C_t) \otimes (R_t\, J) \quad (3.14) \\ &= (Id \otimes R_t)\,(C_t \otimes J) \ldots \end{aligned}$$

Thus, each of the terms 3.11 admits the following factorization ($\det(A \otimes B) = (\det A)^2 (\det B)^2$, A and B 2×2 matrices) :

$$\begin{aligned} (3.11) = & (\det R_t)^2 \det\left[(C_t \otimes Id_2)\,\mathbf{e}_1, \right. \quad (3.15) \\ & \left., (C_t \otimes J)\,\mathbf{e}_1, (D \otimes Id_2)\,\mathbf{e}_1, (C_t \otimes J)\,\mathbf{e}_1\right] . \end{aligned}$$

Since $\det R_t$ is equal to 1 we deduce from 3.11 and 3.15 that $\det A(t)$ itself is independent of β_t.

The last step is proved by invoking the same property of tensor products. More precisely the following factorizations are obtained :

$$\left| \begin{array}{l} C_t \otimes J = (C_t \otimes Id)\,(Id \otimes J) , \\ D \otimes Id = (C_t D) \otimes Id = (C_t \otimes Id)\,(D \otimes Id) . \end{array} \right. \quad (3.16)$$

From what, the following equality holds :

$$\begin{aligned} (16) = & \det(C_t \otimes Id) \times , \quad (3.17) \\ & \det\left[\mathbf{e}_1, (Id \otimes J)\,\mathbf{e}_1, (D \otimes Id_2)\,\mathbf{e}_1, (Id \otimes J)\,\mathbf{e}_1\right] . \end{aligned}$$

Since the above reasoning holds for any expression of the type of 3.11, Prop. 3 is thus proved.
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[10] Id : identity matrix

4 Applications

4.1 Active and passive measurement, the deterministic case

4.1.1 Non-Maneuvering Target

In the absence of observer maneuver, the TMA problem is not observable. But, if we consider *multiple* measurement modes (e.g. passive *and* active measurements), the TMA problem becomes observable [12]. Consider, for instance, the case of two modes (passive and active): the scalar observation $y(t)$ is replaced by a vectorial one $\mathbf{y}(t) = (y_1(t), y_2(t))^*$ but the statistical nature of the problem is unchanged. We shall assume that passive measurements are available at each time, while active ones are scarce. The problem is to optimally scheduling these active measurements. Again, let us denote $\mathcal{M}_1$ and $\mathcal{M}_2$ the $\mathcal{M}$ matrices associated with passive and active measurements, we have :

$$\begin{aligned} \det(\text{FIM}_{t,t+k}) &= c(k)\, \det(\mathcal{M}_1\mathcal{M}_1^* + \mathcal{M}_2\mathcal{M}_2^*) , \\ &= c(k)\, \det\left[(\mathcal{M}_1, \mathcal{M}_2)\begin{pmatrix} \mathcal{M}_1^* \\ \mathcal{M}_2^* \end{pmatrix}\right] \quad (4.1) \end{aligned}$$

The matrix $\mathcal{M}_1$ and $\mathcal{M}_2$ are not detailed but have the standard form. Opposite to the previous case, a direct calculation of $\det(\text{FIM}_{t,t+k})$ is not an easy task. However, the Binet-Cauchy formula allows us to perform them. More precisely, denoting $\text{col}(\mathcal{M}_i)$ the columns (vectors) of $\mathcal{M}_i$, we have :

$$\begin{aligned} \det(\text{FIM}) = & \sum_k [\, (\mathbf{M}_1 \wedge \mathbf{M}_2) \wedge (\mathbf{N}_1 \wedge \mathbf{N}_2)\,]^2 , \quad (4.2) \\ & + \sum_k [\, (\mathbf{M}_1) \wedge (\mathbf{N}_1 \wedge \mathbf{N}_2 \wedge \mathbf{N}_3)\,]^2 , \\ & + \sum_k [\, (\mathbf{M}_1 \wedge \mathbf{M}_2 \wedge \mathbf{M}_3) \wedge \mathbf{N}_1)\,]^2 , \end{aligned}$$

where : $\mathbf{M}_i \in \text{col}(\mathcal{M}_1)$, $\mathbf{N}_i \in \text{col}(\mathcal{M}_2)$.

The calculation of $\det(\mathbf{M}_1, \mathbf{M}_2, \mathbf{N}_1, \mathbf{N}_2)$ is easily achieved by means of exterior algebra. We assume that $\mathbf{N}_1$ corresponds to an active measurement occuring at the instant 0 (associated target bearing β_0); while $\mathbf{N}_2$ is also associated with an active measurement occuring at time τ (associated bearing $\beta_0+\delta$. Similarly, $\mathbf{M}_1$ and $\mathbf{M}_2$ corresponds to passive measurements occuring at time t and t' (associated bearings β_t and $\beta_{t'}$). We thus have :

$$\begin{aligned} \sigma_r\,\mathbf{N}_1 &= (\sin\beta_0, \cos\beta_0, 0, 0)^* , \\ \sigma_r\,\mathbf{N}_2 &= (\sin(\beta_0+\delta), \cos(\beta_0+\delta) , \\ & \quad \tau\sin(\beta_0+\delta), \tau\cos(\beta_0+\delta))^* , \\ \sigma_\beta\, r\,\mathbf{M}_1 &= (\cos\beta_t, -\sin\beta_t, t\cos\beta_t, -t\sin\beta_t)^* , \\ \sigma_\beta\, r\,\mathbf{M}_2 &= (\cos\beta_{t'}, -\sin\beta_{t'}, t'\cos\beta_{t'}, -t'\sin\beta_{t'})^* . \end{aligned}$$

We are now calculating the components (denoted $\alpha_1, \cdots, \alpha_6$) of $\mathbf{N}_1 \wedge \mathbf{N}_2$ as well as those of $\mathbf{M}_1 \wedge \mathbf{M}_2$ (denoted $\gamma_1, \cdots, \gamma_6$) in the canonical basis of $\Lambda^2\mathbb{R}^4$. More precisely, if $\mathbf{e}_1, \cdots, \mathbf{e}_4$ denotes the canonical basis of $\mathbb{R}^4$, then a basis of $\Lambda^2\mathbb{R}^4$ is :

$$\{\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_3, \mathbf{e}_1 \wedge \mathbf{e}_4, \mathbf{e}_2 \wedge \mathbf{e}_4, \mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_3 \wedge \mathbf{e}_4\} ,$$

and associated components of $\mathbf{M}_1 \wedge \mathbf{M}_2$ and $\mathbf{N}_1 \wedge \mathbf{N}_2$ are :

$$\left\{ \begin{array}{l} \alpha_1 = -\sin\delta \, ; \, \gamma_1 = -\cos\beta_t \sin\beta_{t'} + \sin\beta_t \cos\beta_{t'} \; , \\ \alpha_2 = \tau \sin\beta_0 \sin(\beta_0 + \delta) \; , \\ \gamma_2 = (t' - t) \cos\beta_t \cos\beta_{t'} \; , \\ \alpha_3 = \tau \sin\beta_0 \cos(\beta_0 + \delta) \; , \\ \gamma_3 = t \cos\beta_{t'} \sin\beta_t - t' \sin\beta_{t'} \cos\beta_t \; , \\ \alpha_4 = \tau \cos\beta_0 \cos(\beta_0 + \delta) \; , \; \gamma_4 = (t' - t) \sin\beta_t \sin\beta_{t'} , \\ \alpha_5 = \tau \cos\beta_0 \sin(\beta_0 + \delta) \; , \\ \gamma_5 = t \cos\beta_t \sin\beta_{t'} - t' \sin\beta_t \cos\beta_{t'} \; , \\ \alpha_6 = 0 \; , \; \gamma_6 = t\,t' \, (\cos\beta_{t'} \sin\beta_t - \sin\beta_{t'} \cos\beta_t) \; . \end{array} \right. \tag{4.3}$$

We are now in position to examine the calculation of $(\mathbf{M}_1) \wedge (\mathbf{N}_1 \wedge \mathbf{N}_2 \wedge \mathbf{N}_3)$, $(\mathbf{M}_1) \wedge (\mathbf{N}_1 \wedge \mathbf{N}_2 \wedge \mathbf{N}_3)$ and $(\mathbf{M}_1 \wedge \mathbf{M}_2 \wedge \mathbf{M}_3) \wedge \mathbf{N}_1)$.

Consider, for instance, that we have *two* active measurements (at time periods 0 and τ), and T passive measurements, then :

$$\begin{array}{l} (r\sigma_r\sigma_\beta)^2 \;\; (\mathbf{N}_1 \wedge \mathbf{N}_2 \wedge \mathbf{M}_1 \wedge \mathbf{M}_2) \; , \\ = \; \alpha_1\gamma_6 - \alpha_2\gamma_4 + \alpha_3\gamma_5 - \alpha_4\gamma_2 + \alpha_5\gamma_3 \; . \end{array} \tag{4.4}$$

Up to now this calculation is exact. Using a linear approximation (i.e. $\delta = \tau\dot\beta$, $\beta_t = \beta_0 + t\,\dot\beta$ and $\beta_{t'} = \beta_0 + t'\,\dot\beta$), we deduce the following approximation of $\det(\text{FIM}_{r,2\beta,2r})$:

Proposition 5 *Let* $\det(\text{FIM}_{\tau,T,\beta,r})$ *the determinant of the* FIM *associated with* **two** *active measurements (separated by* τ*) and* T *passive measurements :*

$$\begin{array}{l} \det(\text{FIM}_{\tau,T}) \simeq \frac{\tau^2}{(r\sigma_r\sigma_\beta)^4} \; , \\ \times \displaystyle\sum_{0 \le t < t' \le T} (t - t')^2 \left[1 + \dot\beta^2 \left(-t\,t' + \tau\,(t + t') \, \right) \right]^2 . \end{array} \tag{4.5}$$

In order to deal with the general case, it is thus sufficient to consider the following measurement scheduling :

- 1 active measurement, T passive measurements,
- 2 active measurements, T passive measurements,
- 3 active measurements, T passive measurements.

In the case of *three* active measurements, calculations are almost similar to the previous case. except that $\Lambda^2\mathbb{R}^4$ is replaced by $\Lambda^3\mathbb{R}^4$ (canonical basis : $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$, $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4$, $\mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$, $\mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$). The following result is then obtained :

Proposition 6 *Consider that we have* **three** *active measurements (at time periods* 0, τ_2 *and* τ_3*), and* T *passive measurements, then :*

$$\det(\text{FIM}_{\tau_2,\tau_3,T}) \simeq \frac{T}{r^2\sigma_\beta^2\sigma_r^6} \sum_{0 < \tau_2 < \tau_3 \le T} \left[\tau_2\,\tau_3(\tau_2 - \tau_3)\dot\beta \right]^2 . \tag{4.6}$$

Finally consider the case of a unique active measurement, then whatever the active measurement instant we obtain.

Proposition 7 *Consider that we have a* **unique** *active measurement and* T *passive measurements, then :*

$$\begin{array}{l} \det(\text{FIM})_T \simeq \frac{1}{r^6\,\sigma_\beta^2\sigma_r^2\sigma_\beta^6} \; , \\ \qquad \times \displaystyle\sum_{0 \le t_1 < t_2 < t_3 \le T} \left[(t_3 - t_1)(t_2 - t_1)(t_3 - t_2)\dot\beta \right]^2 . \end{array} \tag{4.7}$$

Assume now that a given number of active measurements is available. These active measurements lies in a set E, of fixed cardinality ($\text{Card}(E) = N$) and are indexed by their periods. Then, thanks to the Binet-Cauchy formula and the previous results we are now in position to derive the general form of the determinant of the FIM, yielding.

Proposition 8 *Assume that active measurements are in a set* E *(*$Card(E) = N$*) and that* T *passive measurements are available and let* FIM_E *the associated* FIM*, then :*

$$\begin{array}{rcl} \det(\text{FIM}_E) & = & N.\det(\text{FIM})_T + \displaystyle\sum_{\tau \in E} \det(\text{FIM}_{\tau,T}) \\ & & + \displaystyle\sum_{\tau_1,\tau_2 \in E} \det(\text{FIM}_{\tau_1,\tau_2,T}) \quad , \end{array} \tag{4.8}$$

where $\det(\text{FIM})_T$*,* $\det(\text{FIM}_{\tau,T})$ *and* $\det(\text{FIM}_{\tau_1,\tau_2,T})$ *are given by 4.7, 4.5 and 4.6.*

Finally, let us consider the case where we have an equal number (say T) of simultaneous *active* and *passive* measurements. A direct calculation of det(FIM) is impossible, due to the very large number of elementary terms. However, using Prop. 4, the following result can be derived.

Proposition 9 *Assume that we have* T **simultaneous** *active and passive measurements, then :*

$$\det(\text{FIM}_{T,T}) \propto T^{16} \; \left(\frac{1}{r^4\sigma_\beta^4\sigma_r^4} + \frac{4\dot\beta^2}{r^2\sigma_\beta^2\sigma_r^6} + \frac{4\dot\beta^2}{r^6\sigma_\beta^6\sigma_r^2} \right) . \tag{4.9}$$

As $\dot\beta = \frac{1}{\|\mathbf{r}\|^2}(\mathbf{r} \wedge \mathbf{v})$, the two terms in $\dot\beta$ cannot be neglected. In the absence of observer maneuver and *active* measurements, we know that $\det(\text{FIM}_\beta)$ is strictly zero. However, this calculation is not realistic and Finally, practical considerations plead for including small variations of the source or observer trajectories in the motion model. Actually, this is the general case. Then, an interesting modelling consists in considering independent increments of $\dot\beta$, i.e. :

$$\dot\beta_{t+i} = \dot\beta_t + w_i \; , w_i \;\; \text{w.g.n} \; \mathcal{N}(0, \tau^2) \; ,$$

Using exterior algebra, we easily obtain :

$$\begin{array}{rcl} \det(A_t) & = & 4\,\sin(\dot\beta + w_1)\sin(\dot\beta \; , \\ & & +w_3 - w_2) - \sin(2\dot\beta + w_3 - w_1)\sin(2\dot\beta + w_2) \; . \end{array}$$

We are now in position to calculate the mean value of det(FIM) (denoted $\mathbb{E}\,[\det(\text{FIM}_{t,t+k})]$). More precisely, the expectations of the $\det(A_t)$ are calculated by means of the characteristic functions of $\cos(w_i)$, yielding :

$$\mathbb{E}\,[\det(\text{FIM}_{t,t+3})] \approx \frac{16\,c_3}{r^8} \, \exp\left(-\frac{3}{2}\tau^2 \right) \, \left(\sin(\dot\beta) \right)^8 . \tag{4.10}$$

This result may be extended to the general case, the only change being the exponential term which becomes $\exp(-m_k\,\tau^2)$. We thus see that $\left(\frac{\dot{\beta}}{r}\right)^8$ is a convenient upper bound of det(FIM). The effect of small variations is also evident ($(\exp\left(-\frac{3}{2}\tau^2\right))$ multiplicative term).

From the previous results, we thus have a convenient measure of the information gain induced by active measurements, which is dramatic. This is easily understood if we recall the respective expressions of the vectors $\mathbf{M}_i$ (passive measurements) and $\mathbf{N}_i$ (active measurements) . Indeed it is easily seen that for a given period these two vectors are orthogonal. Now, det(FIM) is also the volume of the Grammian matrix asociated with the gradient vectors or, equivalently, with the volume of the parallelotope overbounding this vector set. This volume is dramatically increased by adding orthogonal vectors, which are spanned by the gradient vectors associated with active measurements.

Optimal measurement scheduling, in this context, need also some comments. In fact, the interest of the above results (see Props. 5 – 9) is to provide us explicit expressions of the performance criterion, in terms of available parameters. Furthermore, it is easily seen that the term $\det(\text{FIM}_{\tau,T})$ is generally predominant in the calculation of det(FIM) by means of Prop. 8. Actually, this is due to the following fact: the gradient spaces, induced by ($\mathbf{M}_1, \mathbf{M}_2$) on the one hand and ($\mathbf{N}_1, \mathbf{N}_2$) on the other are (approximately) orthogonal subspaces of the same dimension.

Assuming r, σ_β, σ_r constant the optimal measurement scheduling reverts in considering the following elementary optimization problem :

Find τ maximizing :

$$\tau^2 \sum_{0\le t<t'\le T} (t-t')^2 \left[1+\dot{\beta}^2\left(-t\,t'+\tau\,(t+t')\right)\right]^2 .$$

Since $\dot{\beta}$ is generally available (from measurements), it is possible to replace it by its estimate. However, we see that except for very long-time scenarios the term $\dot{\beta}^2\left(-t\,t'+\tau\,(t+t')\right)$ is small w.r.t. 1. So, our optimization problem can be further simplified, reducing to the optimiszation of $\tau^2 \sum_{0\le t<t'\le T} (t-t')^2$. The general conclusion is that $\det(\text{FIM}_{\tau,T})$ (and therefore $\det(\text{FIM})$) is maximum when τ is maximum. The optimal measurement scheduling consists then in concentrating all the active measurements at the two extremities (starting and end) of the total measurement batch. Of course, this conclusion is valid only for a deterministic target model. Practically, simplifying hypotheses have been made (σ_β, σ_r, r constant). Previous calculations may be straighforwardly extended to handle the case of time-varying σ_β, σ_r, r.

4.1.2 Maneuvering Target

Consider now a maneuvering source, whose trajectory is made of two legs. Then the target state vector $\mathbf{X}$ becomes 6-dimensional ($\mathbf{X} = (x_0, y_0, v_{x,1}, v_{y,1}, v_{x,2}, v_{y,2})^*$). Assuming that the target maneuver instant be known and denoting it t_m, the gradient vectors take the following form :

$$\left\{\begin{array}{l} r_t\,\sigma_b\,\nabla_{\mathbf{X}}\,\beta_t = (\cos\beta_t, -\sin\beta_t, t\,\cos\beta_t, -t\,\sin\beta_t, 0, 0)^* , \\ \text{for } t \le t_m , \\ r_t\,\sigma_b\,\nabla_{\mathbf{X}}\,\beta_t = (\cos\beta_t, -\sin\beta_t, \\ \quad , t_m\,\cos\beta_t, -t_m\,\sin\beta_t, (t-t_m)\cos\beta_t, -(t-t_m)\sin\beta_t)^* , \\ \text{for } t > t_m . \end{array}\right. \tag{4.11}$$

Similar formulas hold for active measurements. Considering the maximization of the parallelotope overbounding the uncertainty ellipsoid, we can prove that the predominant term of det(FIM) is associated with the exterior products $(\mathbf{M}_1\wedge\mathbf{M}_2\wedge\mathbf{M}_3)\wedge(\mathbf{N}_1\wedge\mathbf{N}_2\wedge\mathbf{N}_3)$ ($\mathbf{M}_i \rightarrow$ passive measurements, $\mathbf{N}_i \rightarrow$ active measurements). Let (t_1, t_2) the instants corresponding to $(\mathbf{M}_1, \mathbf{N}_1)$, (t_3, t_4) the instants corresponding to $(\mathbf{M}_2, \mathbf{N}_2)$, idem for (t_5, t_6); then :

$$\begin{array}{l} \det(\text{FIM}) \approx c \displaystyle\sum_{t_1,\cdots,t_6} \left[(t_1-t_2)(t_3-t_4)(t_5-t_6)\right]^2 \\ \left[\,2\,t_2\cos(2\beta_0 - 2(t_2 - 2t_m + t_2\sin 2\beta_0)\right. , \\ -2\,t_2\dot{\beta}\,(t_2 - t_1 + (t_1+t_2)(\cos 2\beta_0 + \sin 2\beta_0) , \\ +\dot{\beta}^2\left((t_1-t_2)^2(t_2 - 2\,t_m) - t_2(t_1+t_2)^2(\cos 2\beta_0 - \sin 2\beta_0)\right) , \\ -(t_3+t_4-t_5-t_6)^2\,\left(-t_2 + 2\,t_m + t_2(\cos 2\beta_0 - \sin 2\beta_0))\,\right)]^2 . \end{array} \tag{4.12}$$

At a first glance, this formula appears rather formidable and even useless. However, it worth stressing that the predominant term inside the brackets [ ] is simply $(2\,t_2\cos(2\beta_0 - 2(t_2 - 2t_m + t_2\sin 2\beta_0))$. So again, we conclude that optimal measurement scheduling consists in concentrating active measurements on the leg extremities.

Considering a multileg target trajectory, with legs of same length (say j), the FIM takes the following form :

$$\begin{array}{l} \text{FIM}_{1,n\,j} = \text{FIM}_{1,j} + \cdots + \text{FIM}_{(n-1)j,nj} , \\ \quad = \displaystyle\sum_{m=1}^{n}\sum_{(m-1)j+1}^{mj} \left[\mathbf{d}_{m-1,n+1}(k)\,\mathbf{d}^*_{m-1,n+1}(k)\right]\otimes\mathbf{\Omega}_k , \end{array} \tag{4.13}$$

where :

$$\begin{array}{l} \mathbf{d}^*_{p,q}(k) = (1, j, \cdots, j, (k-p\,j), 0, \cdots, 0) , \\ \mathbf{\Omega}_k = \begin{pmatrix} \cos^2\beta_k & \cos\beta_k\sin\beta_k \\ -\cos\beta_k\sin\beta_k & \cos^2\beta_k \end{pmatrix} . \end{array} \tag{4.14}$$

Thus, the previous analysis can be extended to the general case.

The uncertainty about the maneuver instant τ can be modelled by a randomization (of τ). This leads to replace the classical expression of $p(\tilde{\mathbf{B}}|\mathbf{X})$ by the following :

$$p(\tilde{\mathbf{B}}|\mathbf{X}, \bar{\tau}) = \sum_i p\left[\tilde{\mathbf{B}}|\mathbf{X}, \tau_i(\bar{\tau})\right]\, p(\tau_i(\bar{\tau}))\, .$$

For this modelling, the distribution of $\{\tau_i\}$ is centered around $\bar{\tau}$. The problem (MAP) then consists in determining the parameters $\{\mathbf{X}, \bar{\tau}\}$ which maximizes $p(\tilde{\mathbf{B}}|\mathbf{X}, \bar{\tau})$ and estimability analysis can be achieved by means of previous results. Another approach is to consider maneuver detection. For instance, for a two-leg target trajectory, an approximate localization of the source maneuvers can be obtained by sequential detection based, for instance, on the following type of test which may be viewed as an approximation [11] of a likelihood ratio around the parameter $\theta_0 \in \Theta_0$:

$$\begin{array}{l} \Lambda_n(\tilde{\mathbf{B}}_2) = (1/n)\ \mathbf{e}_2^*\, U_2^{-1}\, \mathbf{e}_2\ , \\ \text{where :} \\ \mathbf{e}_2 \triangleq \tilde{\mathbf{B}}_2 - \mathbf{B}_2(\hat{\mathbf{X}}_1)\ , \\ U_2 = \Sigma_2 + H_2(\hat{\mathbf{X}}_1)\, F_1^{-1}\, H_2^*(\hat{\mathbf{X}}_1)\ , \\ n = \dim(\mathbf{e}_2)\ . \end{array} \tag{4.15}$$

In 4.15, $\hat{\mathbf{X}}_1$ represents the source state vector estimated using the first batch of measurements ($\tilde{\mathbf{B}}_1$) of the first source leg which assumes that no source change-point occurs during this first set. The vector $\mathbf{B}_2(\hat{\mathbf{X}}_1)$ is the vector of extended (from $\mathbf{X}_1$) bearings on the second set of measurements while $\tilde{\mathbf{B}}_2$ is the second set of measurements. The matrix U_2 is the covariance matrix of $\mathbf{e}_2$ and $H_2(\hat{\mathbf{X}}_1)$ is a gradient matrix. Let us denote $\mathcal{H}_0$ and $\mathcal{H}_1$ the hypotheses associated respectively with no source maneuver and a maneuver. Under $\mathcal{H}_0$, $n\Lambda$ is (asymptotically) distributed as a central chi-squared random variable χ_n^2, while under $\mathcal{H}_1$ it is distributed as a non-central chi-squared $\chi_n^2(\delta)$ whith the non-centrality parameter δ ($\delta = (\mathbf{B}_2 - \mathbf{B}_2(\mathbf{X}_1))^* U_2{}^{-1}(\mathbf{B}_2 - \mathbf{B}_2(\mathbf{X}_1))$).

The power of the test may then be easily derived and classical calculations yield (η :value of the threshold) :

$$\left\{ \begin{array}{c} P_{\mathrm{fa}} = \exp(-\eta/2)\sum_{i=0}^{n/2-1}\frac{(n/2)^i}{i!} \\ P_{\mathrm{d}} = 1 - \exp(-\frac{\eta+\delta}{2})\sum_{j=0}^{\infty}\frac{\delta^j}{j!}[\sum_{i=0}^{(n/2)+j-1}\frac{(n/2)^i}{i!}]. \end{array} \right. \tag{4.16}$$

Of course, this analysis may be extended to the case of active and passive measurements. Considering the optimization of $\det(U_2)$, we can use the previous calculations to derive an approximation of $\det(U_2)$ and thus perform measurement scheduling.

5 . Stochastic observability and estimability:

We shall now consider a markovian sequence of state vectors:

$$\mathbf{X}_{k+1} = F\mathbf{X}_k + \mathbf{U}_k + \mathbf{W}_k$$

$$\text{with : } \mathbf{W}_k \text{ i.i.d. sequence } (\mathrm{cov}(\mathbf{W}) = Q). \tag{5.1}$$

According to the definition of Boguslavskij [13], we shall say that the system is stochastically observable if, in estimating its states from its outputs, the posterior error variances of all the state components are strictly smaller than the priors.

Let $\hat{\mathbf{X}}_k$ be the linear least-mean-square estimate of $\mathbf{X}_k$ given the measurements $\{y_k, \cdots, y_0\}$ and define the matrices Π_k and P_k ($\Pi_k = \mathrm{cov}(\mathbf{X}_k)$, $P_k = \mathrm{cov}(\mathbf{X}_k - \hat{\mathbf{X}}_k)$) then we shall consider the following definition of observability:

Definition : *The system is said to be observable* [12] *iff :*

$$\mathbf{e}_i^* P_k \mathbf{e}_i < \mathbf{e}_i^* \Pi_k \mathbf{e}_i \quad 1 \leq i \leq n\ . \tag{5.2}$$

It can easily shown that the general form of P_k is:

$$P_k = \Pi_k - L_k \Sigma_k^{-1} L_k^*\ . \tag{5.3}$$

The rectangular ($n \times k$) L_k will be defined later. Since the matrix Σ_k (the covariance matrix of the noise measurements) is positive definite, the matrix $\Pi_k - P_k$ is positive semi-definite. So, the inequality $\mathbf{e}_i^* P_k \mathbf{e}_i \leq \mathbf{e}_i^* \Pi_k \mathbf{e}_i$ always holds, whatever i. The values of i ensuring a strict inequality are the observable state components. In fact, the above definition is rather arbitrary and do not take into account the possible coupling between the estimates of the state components. A convenient definition may then be the estimability condition of Baram and Kailath [14].

Definition : *The system is said to be estimable iff :* $\Pi_k - P_k$ *is positive definite.*

Denote $\theta(L_k)$ the number of rows of the matrix L_k with non-zero elements, then a direct consequence of 5.2 is that the system is stochastically observable iff $\theta(L_k)$ is equal to n. Another direct consequence of 5.2 is that the system is estimable iff the rank of L_k is n. Direct calculations [13],[14] yield the following expression of L_k :

$$L_k = \{\Phi_{k,0}N_0, \Phi_{k,1}N_1, \cdots, \Phi_{k,k}N_k\}\ ,$$

where:

$$\Phi_{k,j} = F^{k-j}\ \ (j \leq k)\ ,\ N_j = \Pi_j H_j^*,$$

and Π_j satisfies the following Lyapunov state equation:

$$\Pi_{j+1} = F\Pi_j F^* + Q\ . \tag{5.4}$$

As pointed in [13] stochastic observability is thus less demanding than deterministic observability which requires a maneuver of the observer.

We shall now consider estimability for BOT. In view of the following factorization of the matrix L_k [43,44]:

$$L_k = F^k(\Pi_0 H_0^*, \Pi_1' H_1^*, \cdots, \Pi_k' H_k^*)\ \ (\Pi_i' = F^{-i}\Pi_i), \tag{5.5}$$

it is easily shown ($F^k = kF - (k-1)Id$) that the matrices Π_i' are spanned by the three matrices Π_0 , Π_1' , Π_2' i.e.:

$$\begin{array}{l} \Pi_3' = 4\,\Pi_0 - 6\,\Pi_1' + 4\,\Pi_2'\ , \\ \Pi_4' = 13\,\Pi_0 - 20\,\Pi_1' + 10\,\Pi_2'\ , \cdots\ . \end{array} \tag{5.6}$$

[11] More precisely, denoting $\hat{\theta}$ the MLE over Θ and θ^* the MLE over Θ_0 we consider the following expansion of the likelihood : $l(\theta^*) = l(\hat{\theta}) + \dot{l}_{\hat{\theta}}\,(\theta^* - \hat{\theta}) - (\theta^* - \hat{\theta})^*\, I(\hat{\theta})\,(\theta^* - \hat{\theta})$, $I(\hat{\theta})$ Fisher matrix .

[12] $\{\mathbf{e}_i\}_{i=1}^n$ usual orthogonal basis of $\mathbb{R}^n$ ($n = \dim\mathbf{X}$)

From 5.6 we deduce that the rank of L_k is generally equal to 3 for a zero-bearing-rate scenario. The system is then not estimable. Excepting this case, the BOT system is generally estimable since the rank of L_k is equal to 4.

A convenient "measure" of the information contained in the measurements may be $\det(L_k \Sigma_k^{-1} L_k^*)$. Direct calculations yield the following results (bearings-only) :

$$\begin{aligned} (\sigma r)^8 \det(L_4 \Sigma_4^{-1} L_4^*) &= 16[43 - 47\cos 2\dot{\beta}]^2 (\sin\dot{\beta})^4 \ , \\ (\sigma r)^8 \det(L_k \Sigma_k^{-1} L_k^*) &= Q_l[\cos 2\dot{\beta}, \cdots, \cos(4(k-4)\dot{\beta})] \ , \\ & \quad \times (\sin\dot{\beta})^4 \ . \end{aligned}$$

In order to optimize the controls (sequence of emissions), a reasonable criterion may consists in maximizing $\det(L_k \Sigma_k^{-1} L_k^*)$ for whom an explicit form is given by (9.4) (9.7). Note, however, that this formula is valid only if $\dot{\beta}$ is approximatively constant and that the true problem is a difficult stochastic control problem. There are many possibilities for the cost criterion. One of them is the trace of the state prediction matrix. Other are based upon the determinant (e.g. Hellinger distance). The trace functional has the great advantage to be linear, thus allowing us to use the methods of optimal control (see [4]). Determinant-based criteria are more demanding (see [15]). In fact, the following result due to Potter and Fraser [13] is particularly useful.

Result 1 *Let P_+, the updated state prediction matrix, given by the following recursive Ricatti equation :*

$$P_+ = P - PH\,(R + H\,R\,H)^{-1}\,H^*\,P \ ,$$

then the following determinant updating formula holds :

$$\det(P_+)/detP = detR/[det(R + H^*\,P\,H)] \ .$$

6 Conclusions

This paper provides an original approach to measurement scheduling. Using a general formalism explicit expressions of the information criteria have been obtained. In this framework, the related optimization problems are (relatively) simple. However, we stress that the target trajectory (even a Markovian one) modelling is particularly simple. Practical modelling should involve reactive sources, thus giving a strong "game" flavor to this problem.

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