Searching Tracks

J.-P. Le CADRE, Member, IEEE

G. SOURIS
IRISA/CNRS
France

Search theory is the discipline that studies the problem of how best to search for an object when the amount of searching efforts is limited and only probabilities of the possible position of the object are given. Then, the problem is to find the *optimal* **distribution of this total effort that maximizes the probability of detection. Although the general formalism of search theory will be used subsequently, we consider now a radically different problem. The problem is to detect target tracks. In the "classical" search theory, the target is said detected if a detection occurs during any time of the time frame. Here, on the contrary, the target track will be said to be detected if elementary detections occur at various times. That means that there is a test for acceptance (or detection) of a target track and that the problem is to optimize the allocation of the search effort for track detection. So, specific optimization problems are solved by means of the primal-dual formalism, in an original setup. Other aspects concern Markovian targets and two-sided search for which simple and efficient algorithms are derived.**

Manuscript received April 17, 1999; revised March 29, 2000; released for production May 15, 2000.

IEEE Log No. T-AES/36/4/11363.

Refereeing of this contribution was handled by J. Tague.

This work was supported by DCN/Ingénierie/Sud, (Dir. Const. Navales), France.

Authors' current addresses: J.-P. Le Cadre, IRISNA/CNRS, Campus de Beaulieu, 35 042, Rennes Cedex, France, E-mail: (lecadre@irisa.fr); G. Souris, Ligeron S. A.: Les Algorithmes, Bat. Euclide, F91 194, Saint-Aubib Cedex, France.

0018-9251/00/$10.00 © 2000 IEEE

I. INTRODUCTION

Search theory is the discipline that treats the problem of how best to search for an object when the amount of searching efforts is limited and only probabilities of the possible position of the object are given. Search theory came into being during World War II with the work of B. O. Koopman and his colleagues [1] in the Antisubmarine Warfare Operations Research Group (ASWORG). Since that time, search theory has grown to be a major discipline within the field of operations research. Important literature has been devoted to this subject; the interested reader may consult various extensive surveys [2], introductive texts [3–5] and specialized books [6–10].

A search theory problem is characterized by three pieces of data: 1) the probabilities of the searched object (the "target") being in various possible locations; 2) the local *detection probability* that a particular amount of local search effort could detect the target; 3) the total amount of searching effort available. The problem is to find the *optimal* distribution of this total effort that maximizes the probability of detection.

Major steps in the development of search theory have been summarized by Stone [4], describing stationary target problems, moving target problems, optimal searcher path algorithms, and dynamic search games.

The rapid growth of the search theory literature is chronicled in [2]. For instance, the last item (search games) is the primary focus of recent researches, including numerous subdomains such as mobile evaders, avoiding target, ambush games, inspection games, and tactical games. For moving target problems, decisive progress have been made in developing search strategies that maximize the probability of detecting the (moving) target within a fixed amount of time. In particular, Brown [11] and Washburn [12] have proposed an iterative algorithm in which the motion space and the time frame have been discretized, and the search effort available in each period is infinitely divisible between the grid cells of the target motion space. In this approach, the search effort available in each period is bounded above by a constant that does not depend on the allocations made during any other periods.

However, although the general formalism of search theory will be used subsequently, we study a radically different problem. The problem is to detect target tracks. In "classical" search theory, the target is said to be detected if a detection occurs any time during the measurement epoch. Here, the target track will be said detected if (*multiple*) elementary detections occur at various times, and this is the *fundamental* difference. Thus there is a test for acceptance (or detection) of a target track [13, 14]. Track detection is also associated with a spatio-temporal modeling of the target track. Moreover, we do not consider (in general) search effort bounds at each period. The bound is relative to the global search effort (i.e., for the *entire* measurement epoch), since our objective is to detect target tracks.

This has multiple consequences. A quite common hypothesis in classical search theory is that the objective function, which is generally the nondetection probability, is a convex functional relatively to the search efforts. This is a quite reasonable assumption in this context, which ensures convergence of iterative algorithms. However, it is not at all valid in our context which leads us to consider alternative optimization approaches. In particular, the dual formalism will play a crucial role. Actually, the dual function being concave whatever the primal problem, a key point is precisely to derive efficient methods for calculating this function. From the maximum of the dual function, solutions of the primal problem are easily recovered.

The paper is organized as follows. In Section II, the optimization framework is presented, followed by the general formulation of the search problem (Section III). In Section IV, we deal with the 2-period search problem, for the AND detection rule. Then, the optimization problems are posed and solved (see Appendix A), and are extended to the n-period search (also for the AND rule) in Section V. Another detection rule is considered in Section VI, the MAJORITY detection rule. The related optimization problems are rather intricate, but we show that they can be solved by means of the dual formalism. Section VII is of a different nature since we consider here the general problem of search for Markovian tracks. The previous problems are extended to two-sided search in Section VIII. Finally, the theoretical results of the previous sections are illustrated by simulation results in Section IX.

The following notations are used throughout this work.

- $x_{i,\theta}$ is the search effort associated with the cell (i,θ),
- i temporal index, θ: track index,
- $X_{i,\theta}$ $= \exp(-w_i x_{i,\theta})$,
- $\underline{x}_{i,\theta}$ value of $x_{i,\theta}$ at the optimum, idem for $X_{i,\theta}$, $y_{i,\theta}$ and λ,

KKT conditions means Karush–Kuhn–Tucker optimality conditions.

II. OPTIMIZATION FRAMEWORK

Assuming that the search associated with the cell indexed by i is denoted x_i, the elementary search problem consists in determining the x_i which minimizes the nondetection probability [1] (denoted P_{nd}):

$$\begin{cases} \text{Find the } \{x_i\}_{i\in\mathcal{C}} \text{ minimizing:} \\ P_{\text{nd}} = \sum_{i\in\mathcal{C}} \alpha_i \exp(-w_i x_i) \\ \text{subject to:} \\ \sum_{i\in\mathcal{C}} x_i = \Phi, \qquad x_i \geq 0 \quad \forall\, i. \end{cases} \tag{1}$$

This optimization problem[1] is easily solved by means of duality (see Appendix A), even if the the elementary probability of nondetection (i.e., $\exp(-w_i x_i)$) is replaced by another one. Numerous variations and extensions, of increasing complexity, of this generic problem exist in the literature, including extensions to multiperiod search for a Markovian moving target [11, 12]. However, a different problem is considered here.

More precisely, the major part of this paper is centered around the following (primal) optimization problem:

$$\mathcal{P}\begin{cases} \min -P \quad \text{with:} \quad P = \sum_{\theta\in\Theta} F(x_{1,\theta}, x_{2,\theta}, \ldots, x_{n,\theta}) \\ \text{where:} \\ F(x_{1,\theta}, x_{2,\theta}, \ldots, x_{n,\theta}) \stackrel{\Delta}{=} f(p(x_{1,\theta}), p(x_{2,\theta}), \ldots, p(x_{n,\theta})) \\ \text{under the resource constraints:} \\ \sum_{\theta\in\Theta} x_{1,\theta} + x_{2,\theta} \cdots + x_{n,\theta} = \Phi; \\ x_{1,\theta} \geq 0, \quad x_{2,\theta} \geq 0, \ldots, x_{n,\theta} \geq 0 \qquad \forall \quad \theta \in \Theta. \end{cases} \tag{2}$$

In (2), $x_{k,\theta}$ represents a search effort, affected to the cell indexed by the parameter θ, at the search period k. The index k takes its values in the subset $\{1,\ldots,n\}$. The parameter θ takes its values in a multidimensional space or set (denoted Θ), characterizing the target trajectory (e.g., initial position and velocity) and the n-dimensional vector $\mathbf{X}_\theta \stackrel{\Delta}{=} (x_{1,\theta}, x_{2,\theta}, \ldots, x_{n,\theta})^*$ represents the effort vector associated with the target trajectory (or track) indexed by θ. Furthermore, $p(x_{k,\theta})$ is the elementary probability of detection in the cell (k,θ), for a search effort $x_{k,\theta}$; while f is a given differentiable function. The following simple remarks are then fundamental.

1) The functional $F(x_{1,\theta},\ldots,x_{n,\theta})$ is a differentiable[2] functional of the variables $x_{k,\theta}$.

2) The constraints are qualified [16] since they are linear.

3) The "hard constraint" is the equality constraint (i.e., $\sum_\theta x_{1,\theta} + x_{2,\theta} \cdots + x_{n,\theta} = \Phi$), the inequality

[1] It seems that this type of problem has been considered for the first time by J. W. Gibbs.

[2] Note that, in our context, $F(x_{1,\theta},\ldots,x_{n,\theta})$ is not generally assumed separable.

constraints ($x_{1,\theta} \geq 0, \ldots, x_{1,\theta} \geq 0$) being *implicitly* taken into account (see Appendix A).

A fundamental assumption made in all the search theory literature is that the detection functionals $F(x_{1,\theta}, x_{2,\theta}, \ldots, x_{n,\theta})$ are concave. In turn, the objective functional P is also concave. This assumption is essential for proving the necessity of the classical optimality conditions for the search plan. The role of the concavity hypothesis for a general multiperiod search plan is recognized in its full generality for instance in [15]. Unfortunately, this assumption is not at all valid in our context,[3] compelling us to develop a fundamentally different formalism.

In this work, the following dual function $\psi(\lambda)$ is examined:

$$\begin{cases} \psi(\lambda) = \inf_{x_{1,\theta},\ldots,x_{n,\theta}} \mathcal{L}(\lambda) \\ \text{where:} \\ \mathcal{L}(\lambda) = -P + \lambda\left(\sum_{\theta} x_{1,\theta} + x_{2,\theta} \cdots + x_{n,\theta} - \Phi\right). \end{cases} \tag{3}$$

We stress that, in our framework, the function $\psi(\lambda)$ may be explicitly determined on the subset defined by the inequality constraints. The dual problem ($\mathcal{D}$) then takes the following form:

$$\mathcal{D} : \max_{\lambda} \psi(\lambda). \tag{4}$$

The benefits of this approach are 1) the maximization of $\psi(\lambda)$ is an (unconstrained) *monodimensional*[4] problem, 2) the function $\psi(\lambda)$ is differentiable, 3) from the solution $\underline{\lambda}$ of the dual problem, the solution $\underline{\mathbf{X}}$ of the primal problem $\mathcal{P}$ is deduced (say $\underline{\mathbf{X}}, \underline{\lambda}$)). The pair $(\underline{\lambda}, \underline{\mathbf{X}})$ is a saddle point of the primal-dual problem.

Throughout this work, we make constant use of the following (classical) result [16].

PROPOSITION 1 *Let $\underline{\lambda}$ be a solution to the dual problem. Then:*

a) *If the primal problem has a saddle point, there exists $\mathbf{X}$ solution to $\mathcal{P}$ such that $(\underline{\mathbf{X}}, \underline{\lambda})$ is a saddle point.*

b) *If the primal problem $\mathcal{P}$ has a saddle point and if $\mathcal{L}(\mathbf{X}, \underline{\lambda})$ has a unique minimum in $\mathbf{X}$ (say $\underline{\mathbf{X}}$), then $(\underline{\mathbf{X}}, \underline{\lambda})$ is a saddle point and $\underline{\mathbf{X}}$ solves $\mathcal{P}$.*

c) *If ψ is differentiable at $\underline{\lambda}$, and if $\underline{\mathbf{X}}$ is the unique minimum (in $\mathbf{X}$) of $\mathcal{L}(\mathbf{X}, \underline{\lambda})$, then $\mathcal{P}$ has a saddle point $(\underline{\mathbf{X}}, \underline{\lambda})$.*

The last property is especially important since we prove that $\psi(\lambda)$ is everywhere differentiable on its definition domain and that the minimum of $\mathcal{L}(\mathbf{X}, \lambda)$ is attained for a unique vector $\mathbf{X}$. Practically, this means that no duality gap does exist when the condition c of Proposition 1 is satisfied.

[3] Consider for example the following detection function: $f(x_1, x_2) = (1 - e^{-wx_1})(1 - e^{-wx_2})$.

[4] In the case of a unique "hard" resource constraint.

III. MODELING AND FORMULATION OF THE PROBLEM

In most of this article, except for Section 7, we make the assumption that the target motion is rectilinear and uniform. So, in this case, the target trajectory is completely defined by its initial position vector (i) and velocity vector $(v) : \theta \equiv (i, v)$. Also, we restrict ourselves to discrete problems, both in time and in space. Assumptions of our search problem are as follows.

1) A target moves in a search space consisting of a finite number of search cells $\mathcal{C}_t = \{c_{\theta,t}\}_\theta$ in discrete time $\mathbf{T} = \{1, 2, \ldots, n\}$. We further assume that the sequence of (searched) cells $\{c_{\theta,t}\}_t$ is completely defined by the parameter (θ) [17, 18] (conditionally deterministic motion). Thus, the mapping $c_{\theta,1} \to c_{\theta,2} \cdots \to c_{\theta,n}$ is a bijection. In the simpler case of rectilinear target motion, this mapping is simply a translation of vector v.

2) The search effort applied to cell $c_{\theta,t}$ is denoted $x_{t,\theta}$ ($x_{t,\theta} \geq 0$).

3) The conditional probability of detecting the target given that the target is in the cell $c_{\theta,t}$ and that the search effort applied to this cell is $x_{t,\theta}$ is $p(x_{t,\theta})$. This probability is a classical exponential law, i.e., $p(x_{t,\theta}) = 1 - \exp(-w_{t,\theta} x_{t,\theta})$. The term $w_{t,\theta}$ stands for the particular conditions of detection (visibility) for the cell $c_{\theta,t}$.

The exponential assumption is very general and is obtained by the following elementary reasoning [1]. Consider now that the search effort is represented by time (t) and let us denote $q(t)$ ($q(t) = 1 - p(t)$) the probability of nondetection. Denoting w as the "instantaneous" probability of detection, the increment in probability of detection associated with the time increment dt will be $w\,dt$, so that:

$$\begin{aligned} &q(t + dt) = q(t)(1 - w\,dt) \\ &\text{or:} \\ &\frac{d}{dt} q(t) = -wq(t) \qquad \text{and:} \qquad p(t) = 1 - e^{-wt}. \end{aligned} \tag{5}$$

A more general presentation of the exponential density (for the probability of detection) can be found in [5]. For instance, elementary calculations yield (with the notations of [5]):

$$P(\text{det}) = 1 - e^{-wL/A} \tag{6}$$

where A denotes the area of the region containing the target, L the length of the search segment and w the visibility parameter (here the sweep width). The term wL/A then represents the elementary area coverage.

IV. THE 2-PERIOD SEARCH FOR THE AND TRACK DETECTION RULE

First, we deal with the two period search problem (i.e., $n = 2$). Actually, this optimization problem is quite representative of the general case (n periods). We say that the target track has been detected if the target has been detected at *each* (temporal) period of the search. Therefore, we must solve the following search problem:

$$\mathcal{P}\begin{cases} \min -P \qquad \text{where:} \quad P = \sum_\theta g_1(\theta)p(x_{1,\theta})p(x_{2,\theta}) \\ \text{under the constraints:} \\ \sum_\theta (x_{1,\theta}+x_{2,\theta}) = \Phi, \qquad x_{1,\theta} \geq 0, \quad x_{2,\theta} \geq 0, \quad \forall\ \theta. \end{cases} \tag{7}$$

In the above equation $x_{1,\theta}$ ($x_{2,\theta}$) denotes the search effort applied to the cell $c_{\theta,1}$ ($c_{\theta,2}$). Then, we form the Lagrangian of the primal problem (7), i.e.,

$$\mathcal{L}(\lambda) = -\sum_\theta g_1(\theta)(1-e^{-wx_{1,\theta}})(1-e^{-wx_{2,\theta}}) \tag{8}$$

$$+\lambda\left(\sum_\theta x_{1,\theta} + \sum_\theta x_{2,\theta} - \Phi\right) - \sum_\theta \mu_{1,\theta}x_{1,\theta}$$

$$-\sum_\theta \mu_{2,\theta}x_{2,\theta}; \qquad \mu_{1,\theta} \geq 0, \quad \mu_{2,\theta} \geq 0. \tag{9}$$

In order to apply the Karush–Kuhn–Tucker (KKT) conditions [16] of optimality, we must consider two cases.

A. KKT Optimality Conditions and Their Consequences

Case 1 ($x_{1,\theta} > 0$).

In this case, the KKT condition $\{\mu_{1,\theta}x_{1,\theta} = 0\}$ implies $\{\mu_{1,\theta} = 0\}$. Then, the KKT stationarity condition (for the Lagrangian) simply results in

$$\frac{\partial}{\partial x_{1,\theta}}\mathcal{L}(\lambda) = -wg_1(\theta)e^{-wx_{1,\theta}}(1-e^{-wx_{2,\theta}}) + \lambda = 0. \tag{10}$$

From (10), we note that the assumption $x_{1,\theta} > 0$ implies $x_{2,\theta} > 0$, otherwise the multiplier λ should be zero. Indeed, if $\lambda = 0$ then it is easily seen (see (8)) that the value of the dual function $\psi(\lambda) = \inf_{(x_{1,\theta},x_{2,\theta})}\mathcal{L}(\lambda)$ is $-\infty$. Since, we have to maximize $\psi(\lambda)$, we see that λ is necessarily *strictly* positive (see (10) for the sign). Thus, (10) implies the validity of the following equation:

$$\frac{\partial}{\partial x_{2,\theta}}\mathcal{L}(\lambda) = -wg_1(\theta)e^{-wx_{2,\theta}}(1-e^{-wx_{1,\theta}}) + \lambda = 0. \tag{11}$$

By collecting (10) and (11), and denoting $X_{1,\theta} = e^{-wx_{1,\theta}}$, $X_{2\theta} = e^{-wx_{2,\theta}}$, we obtain

$$\begin{aligned} &X_{1,\theta}(1-X_{2,\theta}) = X_{2,\theta}(1-X_{1,\theta}) \\ &\text{so that:} \\ &X_{1,\theta} = X_{2,\theta}, \qquad \text{i.e.,} \quad x_{1,\theta} = x_{2,\theta}. \end{aligned} \tag{12}$$

The above equality is fundamental for solving the problem.

Case 2 ($x_{1,\theta} = 0$). Assume now that $x_{2,\theta} > 0$, then the KKT condition (relative to $x_{2,\theta}$) should imply (see (11), with $x_{1,\theta} = 0$):

$$\frac{\partial}{\partial x_{2,\theta}}\mathcal{L}(\lambda) = \lambda = 0. \tag{13}$$

In this case, the value of $\psi(\lambda)$ is $-\infty$. Hence, we must restrict to the *strictly* positive values of λ, which means that the assumption $x_{1,\theta} = 0$ implies $x_{2,\theta} = 0$.

B. Solving the Dual Problem

We conclude that $x_{1,\theta} = x_{2,\theta}$, meaning that we need to solve the following (simplified) optimization problem:

$$\mathcal{P}\begin{cases} \min -P \qquad \text{where:} \quad P = \sum_\theta g_1(\theta)(p(x_{1,\theta}))^2 \\ \text{under the constraints:} \\ \sum_\theta x_{1,\theta} = \Phi/2, \qquad x_{1,\theta} \geq 0 \quad \forall\ (\theta). \end{cases} \tag{14}$$

Again, we examine the necessary conditions induced by the KKT Theorem. Consider the reduced Lagrangian functional $\mathcal{L}(\lambda)$ given by

$$\mathcal{L}(\lambda) = -\sum_\theta g_1(\theta)(1-e^{-wx_{1,\theta}})^2 + \lambda\left(2\sum_\theta x_{1,\theta} - \Phi\right). \tag{15}$$

This form of the Lagrangian corresponds to the relaxation of the positivity constraints relative to the search variables $\{x_{1,\theta}\}$, which are *implicitly* taken into account by restricting our search to positive values of the variables $x_{1,\theta}$. We refer to Appendix A for a complete justification of the positivity constraints relaxation [19]. Since this optimization problem is separated into the variables $\{x_{1,\theta}\}$, it is easily solved. Under the assumption that $x_{1,\theta}$ is *strictly* positive and differentiating $\mathcal{L}(\lambda)$ relatively to $x_{1,\theta}$, we then obtain

$$\begin{aligned} &\frac{\partial\mathcal{L}(\lambda)}{\partial x_{1,\theta}} = -2wg_1(\theta)e^{-wx_{1,\theta}}(1-e^{-wx_{1,\theta}}) + 2\lambda = 0 \\ &\text{or, equivalently:} \\ &X_{1,\theta}(1-X_{1,\theta}) = \frac{\lambda}{wg_1(\theta)}. \end{aligned} \tag{16}$$

Equation (16) is a second-order equation (in $X_{1,\theta}$), allowing us to determine $\underline{x}_{1,\theta}$, for a given value of λ. Note that we restrict to the roots (0 or 2) of (16) lying inside the interval $[0,1]$, and select the root (denoted $\underline{X}_{1,\theta}(\lambda)$) which minimizes the reduced Lagrangian functional $\mathcal{L}(\lambda)$.[5]

[5]Note that we must test and compare the value of $\mathcal{L}(\lambda)$ not only for the roots of (16), but also for its lower bound (i.e., $X_{1,\theta} = 1 \Leftrightarrow x_{1,\theta} = 0$).

We have now to deal with the maximization of the dual functional defined by

$$\psi(\lambda) = -\sum_{(\theta)_+} g_1(\theta)(1-\underline{X}_{1,\theta}(\lambda))^2 + \lambda\left(2\sum_{(\theta)_+}\underline{x}_{1,\theta}(\lambda) - \Phi\right) \tag{17}$$

$$\underline{x}_{1,\theta}(\lambda) = -\frac{1}{w}\ln(\underline{X}_{1,\theta}(\lambda)) \qquad \text{if:} \quad \underline{x}_{1,\theta}(\lambda) > 0$$

where the symbol $(\theta)_+$ denotes the values of the index for which (16) has a root inside $[0,1]$. The maximization of $\psi(\lambda)$ is rather easy since it corresponds to an unidimensional search for a concave and differentiable function. The general framework [20] is detailed in Appendix A, while the unicity of the search vector $\mathbf{X}$ is proved in the Appendix B. In turn, $\psi(\lambda)$ is differentiable.

Notation 1. The (spatio-temporal) index (θ,t) for which the research efforts are strictly positive are denoted $(\theta,t)_+$ (t: index of the search period); $(\theta)_+$ for the first search period.

C. Additional Constraints on Search Efforts

Practically speaking, it may be worth including constraints relative to the search effort, available at a given period. The (primal) problem then takes the following form:

$$\mathcal{P}\begin{cases} \min -P \qquad \text{where:} \quad P = \displaystyle\sum_\theta g_1(\theta)p(x_{1,\theta})p(x_{2,\theta}) \\ \text{under the constraints:} \\ \displaystyle\sum_\theta x_{1,\theta} \le c_1 \\ \displaystyle\sum_\theta x_{1,\theta} + \sum_\theta x_{2,\theta} = \Phi, \quad x_{1,\theta}\ge 0, \quad x_{2,\theta} \ge 0 \quad \forall\,(\theta) \end{cases} \tag{18}$$

for which the reduced Lagrangian (see (15)) is

$$\mathcal{L}(\lambda,\mu) = -\sum_\theta g_1(\theta)(1-e^{-wx_{1,\theta}})(1-e^{-wx_{2,\theta}}) + \lambda\left(\sum_\theta (x_{1,\theta}+x_{2,\theta}) - \Phi\right) + \mu\left(\sum_\theta x_{1,\theta} - c_1\right).$$

Assuming that $x_{1,\theta}$ is strictly positive, KKT conditions imply the nullity of $\mu(\sum_\theta x_{1,\theta} - c_1)$ ($\mu \ge 0$). If the inequality constraint is inactive, then μ is null, so that the KKT conditions imply that

$$\frac{\partial}{\partial x_{1,\theta}}\mathcal{L}(\lambda,\mu) = -g_1(\theta)X_{1,\theta}(1-X_{2,\theta}) + \lambda = 0. \tag{19}$$

Since we restrict to (strictly) positive values of λ, $x_{2,\theta}$ cannot be zero. We then have the same condition imposed on $X_{2,\theta}$ (i.e., $\partial/\partial x_{2,\theta}\mathcal{L}(\lambda,\mu) = -g_1(\theta)X_{2,\theta} \cdot (1-X_{1,\theta}) + \lambda = 0$), and consequently $X_{1,\theta} = X_{2,\theta}$.

Now, if the inequality constraint is active, μ is strictly positive, and $x_{1,\theta}$ and $x_{2,\theta}$ must satisfy the following system of equations:

$$\begin{cases} \dfrac{\partial}{\partial x_{1,\theta}}\mathcal{L}(\lambda,\mu) = -g_1(\theta)X_{1,\theta}(1-X_{2,\theta}) + \lambda + \mu = 0 \\ \dfrac{\partial}{\partial x_{2,\theta}}\mathcal{L}(\lambda,\mu) = -g_1(\theta)X_{2,\theta}(1-X_{1,\theta}) + \lambda = 0 \end{cases} \tag{20}$$

and we must determine the search efforts $x_{1,\theta}$ and $x_{2,\theta}$, solutions of (20). It is then necessary to consider a two-dimensional dual functional $\psi(\lambda,\mu)$. The complexity of the problem is considerably greater but the general framework is quite similar.

V. THE n-PERIOD SEARCH FOR THE AND TRACK DETECTION RULE

Quite similar to the 2-period search, we assume that the probability of detection of the track is the product of elementary detection probabilities of detection at each period and is given by:[6]

$$\begin{cases} P = \displaystyle\sum_\theta g_1(\theta)p(x_{1,\theta})p(x_{2,\theta})\cdots p(x_{n,\theta}) \\ p(x_{k,\theta}) = (1-\gamma_k e^{-w_{k,\theta}x_{k,\theta}}) \qquad k = 1,\ldots,n \end{cases} \tag{21}$$

and the optimization problem is again

$$\mathcal{P}\begin{cases} \min -P \\ \text{under the constraints:} \\ \displaystyle\sum_\theta [x_{1,\theta} + \cdots + x_{n,\theta}] = \Phi, \\ x_{1,\theta}\ge 0,\ldots,x_{n,\theta}\ge 0, \qquad \forall\,(\theta). \end{cases} \tag{22}$$

Assume $x_{1,\theta} \neq 0$, then by reasoning identical to the 2-period case, we deduce that $x_{2,\theta}\neq 0,\ldots,x_{n,\theta}\neq 0$. The optimality equations deduced from the KKT conditions then yield the following (nonlinear) system of n equations:

$$\begin{cases} \gamma_1 X_{1,\theta}(1-\gamma_2X_{2,\theta})\cdots(1-\gamma_nX_{n,\theta}) = \dfrac{\lambda}{w_{1,\theta}g_1(\theta)} = \alpha_1 & (1) \\ \gamma_2 X_{2,\theta}(1-\gamma_1X_{1,\theta})\cdots(1-\gamma_nX_{n,\theta}) = \dfrac{\lambda}{w_{2,\theta}g_1(\theta)} = \alpha_2 & (2) \\ \vdots \\ \gamma_n X_{n,\theta}(1-\gamma_1X_{1,\theta})\cdots(1-\gamma_{n-1}X_{n-1,\theta}) = \dfrac{\lambda}{w_{n,\theta}g_1(\theta)} = \alpha_n & (n). \end{cases} \tag{23}$$

Consider now the above system, dividing row (1) by row (p) and denoting $Y_{1,\theta} \stackrel{\Delta}{=} \gamma_1X_{1,\theta},\ldots,Y_{p,\theta}\stackrel{\Delta}{=}\gamma_pX_{p,\theta}$, we

[6]The scalar $w_{k,\theta}$ stands for the possibly changing visibility conditions from one period to another one.

obtain

$$\frac{(1-Y_{1,\theta})}{\alpha_1 Y_{1,\theta}} = \frac{(1-Y_{p,\theta})}{\alpha_p Y_{p,\theta}} \tag{24}$$

$$\text{or:} \quad Y_{p,\theta} = \frac{Y_{1,\theta}}{Y_{1,\theta}(1-\beta_p)+\beta_p}, \qquad \text{where:} \quad \beta_p = \frac{w_{1,\theta}}{w_{p,\theta}}.$$

Consequently, $x_{p,\theta}$ is deduced from $x_{1,\theta}$, itself given by

$$x_{1,\theta} = \frac{1}{w_{1,\theta}}\left[\ln\left(\frac{\gamma_1}{Y_{1,\theta}}\right)\right]^+.$$

The problem is thus reduced to the determination of $\underline{x}_{1,\theta}$. From (24) we see that $\underline{X}_{1,\theta}$ is a root of the following nth-order polynomial equation:

$$Y_{1,\theta}(1-Y_{1,\theta})^{(n-1)} - \frac{\lambda}{w_{1,\theta} g_1(\theta)} \prod_{p=2}^{n} [Y_{1,\theta}(\beta_p^{-1}-1)+1] = 0. \tag{25}$$

The value of $\underline{X}_{1,\theta}(\lambda)$ is the root of (25) which minimizes the Lagrangian, deduced from (21); where $\underline{x}_{2,\theta},\ldots,\underline{x}_{n,\theta}$ are determined (from $\underline{x}_{1,\theta}$) by (24). Since P is separated with respect to $\{x_{1,\theta}\}_\theta$, the computation load is relatively modest. From $\underline{x}_{1,\theta}$, the dual function $\psi(\lambda)$ is deduced, i.e.,

$$\psi(\lambda) = -\sum_{(\theta)_+}\prod_{k_+}(1-\gamma_k \underline{X}_{k,\theta}) + \lambda\left(\sum_{(\theta,k)_+} \underline{x}_{k,\theta} - \Phi\right). \tag{26}$$

The problem is simply to determine the value of λ which maximizes the concave function $\psi(\lambda)$.

So far, the problem has been considered in its full generality. To illustrate the previous calculations, assume now that the visibility coefficients $\{w_{1,\theta},\ldots,w_{n,\theta}\}$ are identical:

$$p(x_{k,\theta}) = (1-e^{-wx_{k,\theta}}) \qquad k = 1,\ldots,n.$$

Then the optimality equations (23) and (24) reduce to

$$Y_{1,\theta} = \cdots = Y_{n,\theta} \tag{27}$$

so that $X_{1,\theta} = \cdots = X_{n,\theta}$ and the probability of track detection as well as the dual function $\psi(\lambda)$ become

$$\begin{cases} P = \sum_\theta g_1(\theta)[(1-e^{-wx_{k,\theta}})]^n \\ \psi(\lambda) = -\sum_{(\theta)_+} g_1(\theta)[(1-\underline{X}_{1,\theta}(\lambda))]^n \\ \qquad +\lambda\left(n\sum_{(\theta)_+}\underline{x}_{1,\theta}(\lambda) - \Phi\right). \end{cases} \tag{28}$$

Again, we have to deal now with a simple monodimensional optimization problem, involving the concave functional $\psi(\lambda)$.

Let us denote $\Phi(\lambda)$ the optimal value of the (total) search effort for a given λ; then the following result holds.

PROPOSITION 2 *$\Phi(\lambda)$ is a decreasing function of λ.*

PROOF Denoting the track parameter as θ, the Lagrangian $\mathcal{L}(\lambda)$ of the constrained problem is $\mathcal{L}(\lambda,\theta) = -P + \lambda(\sum_{i=1}^n x_{i,\theta} - \Phi)$ ($P = \sum_\theta g_1(\theta)p(x_{1,\theta})\cdots p(x_{n,\theta})$); which implies

$$\frac{\partial\mathcal{L}(\lambda)}{\partial x_{i,\theta}} = -\frac{\partial P}{\partial x_{i,\theta}} + \lambda$$

and consequently

$$\lambda_2 > \lambda_1 \Rightarrow \frac{\partial\mathcal{L}(\lambda_2)}{\partial x_{i,\theta}} \geq \frac{\partial\mathcal{L}(\lambda_2)}{\partial x_{i,\theta}} \tag{29}$$

hence $\underline{x}_{i,\theta}(\lambda_1) \geq \underline{x}_{i,\theta}(\lambda_2)$ $(\forall i,\theta)$; and in turn, $\Phi(\lambda_2) \leq \Phi(\lambda_1)$.

VI. MAJORITY RULE FOR TRACK DETECTION

Up to now, our analysis has been restricted to an AND rule for track detection. However, for numerous applications, a MAJORITY rule is also quite realistic. This means that a track is said detected if a sufficient number of elementary detections occur along the track. We now face specific problems. First, it is difficult to give a general formulation (for the general n-period search) of the detection rule. Second, the optimization problems become far more complicated.

A. The 3-Period Case and MAJORITY Track Detection Rule

The detection function is modified in order to take into account a majority rule (MAJORITY) for decision. More precisely, the track is said to be detected if the (moving) target is detected *at least* at 2 periods. With this rule, the probability of detection becomes

$$P = \sum_\theta g_1(\theta)[\beta_{0,2,3}P_{0,2,3} + \beta_{1,2,0}P_{1,2,0} + \beta_{1,0,3}P_{1,0,3} + \beta_{1,2,3}P_{1,2,3}] \tag{30}$$

and the optimization problem is

$$\mathcal{P}\begin{cases} \min -P \\ \text{under the constraints:} \\ \sum_\theta (x_{1,\theta}+x_{2,\theta}+x_{3,\theta}) = \Phi, \\ x_{1,\theta}\geq 0,\ldots,x_{3,\theta}\geq 0, \qquad \forall\ (\theta). \end{cases} \tag{31}$$

In (30), the notation $P_{0,2,3}$ corresponds to the following hypothesis: no detection at period 1, detection at periods 2 and 3, idem for $P_{1,2,0}$ and $P_{1,0,3}$. The notation $P_{1,2,3}$ corresponds to a detection at each period. Finally, the weights $\beta_{0,2,3},\ldots,\beta_{1,2,3}$ are related to the information gain associated with an elementary event.

This gain may be expressed in terms of quality of the estimated track, probability of correct association, etc. This is described in more details in Appendix C. Thus, the elementary detection terms $P_{0,2,3},\dots,P_{1,2,3}$ have the following form:

$$\begin{cases} P_{0,2,3} = e^{-wx_{1,\theta}}(1-e^{-wx_{2,\theta}})(1-e^{-wx_{3,\theta}}) \\ P_{1,2,0} = e^{-wx_{3,\theta}}(1-e^{-wx_{1,\theta}})(1-e^{-wx_{2,\theta}}) \\ P_{1,0,3} = e^{-wx_{2,\theta}}(1-e^{-wx_{1,\theta}})(1-e^{-wx_{3,\theta}}) \\ P_{1,2,3} = (1-e^{-wx_{1,\theta}})(1-e^{-wx_{2,\theta}})(1-e^{-wx_{3,\theta}}). \end{cases} \tag{32}$$

Defining the reduced Lagrangian as:

$$\mathcal{L}(\lambda) = -P + \lambda\left(\sum_{\theta}(x_{1,\theta}+x_{2,\theta}+x_{3,\theta}) - \Phi\right)$$

we adopt the following notations for the sake of simplicity:[7]

$$\begin{cases} \beta_{0,2,3} \equiv \delta_1, \qquad \beta_{1,2,0} \equiv \delta_3, \\ \beta_{1,0,3} \equiv \delta_2, \qquad \beta_{1,2,3} \equiv \delta^*, \qquad g_1 \equiv wg_1(\theta) \\ X_{1,\theta} = e^{-wx_{1,\theta}} \equiv y_1,\dots,X_{3,\theta} = e^{-wx_{3,\theta}} \equiv y_3. \end{cases} \tag{33}$$

Assuming that y_1, y_2, y_3 differ altogether from 1, the KKT conditions then become

$$\begin{cases} -\delta_1 y_1(1-y_2)(1-y_3) + \delta_2 y_1 y_2(1-y_3) + \delta_3 y_1 y_3(1-y_2) \\ \qquad +\delta^* y_1(1-y_2)(1-y_3) = \dfrac{\lambda}{g_1} & (1) \\ \delta_1 y_1 y_2(1-y_3) - \delta_2(1-y_1)y_2(1-y_3) + \delta_3 y_2 y_3(1-y_1) \\ \qquad +\delta^*(1-y_1)y_2(1-y_3) = \dfrac{\lambda}{g_1} & (2) \\ \delta_1 y_1(1-y_2)y_3 + \delta_2 y_1 y_2(1-y_3) - \delta_3(1-y_1)(1-y_2)y_3 \\ \qquad +\delta^*(1-y_1)(1-y_2)y_3 = \dfrac{\lambda}{g_1}. & (3) \end{cases} \tag{34}$$

Subtracting row 3 from row 2 in (34), we obtain

$$y_3 = \left[\frac{y_1(\delta^*-\delta_1-\delta_2)+\delta_2-\delta^*}{y_1(\delta^*-\delta_1-\delta_3)+\delta_3-\delta^*}\right]y_2. \tag{35}$$

Then, inserting $y_3 = f(y_1)y_2$ (see (35)) in (32), the following 2nd-order equation is deduced:

$$(a-by_1)y_2^2 + (c-dy_1^2)y_2 + (ey_1^2+fy_1) = 0$$

where

$$\begin{cases} a = \beta_2(\beta_3-\alpha_2), \qquad d = (\alpha_1-\beta_2)(\alpha_1-\beta_3) \\ b = (\alpha_1-\beta_2)(\alpha_2-\beta_3), \qquad e = \beta_1(\alpha_1-\beta_3) \\ c = -\beta_2\beta_3, \qquad f = \beta_1\beta_3 \\ \text{and:} \\ \beta_1 = \delta^*-\delta_1; \qquad \beta_2 = \delta^*-\delta_2; \qquad \beta_3 = \delta^*-\delta_3. \end{cases} \tag{36}$$

In this case ($x_{k,\theta} \neq 0$; $k = 1,2,3$), the distribution of the search efforts is completely determined by the optimality equations (34). For instance, from (35) and (36) we obtain $y_3 = f(y_1)y_2$ and $y_2 = f'(y_1)$. The optimal value of y_1 is that value solving the nonlinear equation in y_1 deduced from (34) by replacing y_2 and y_3 by their expressions in terms of y_1 (see (35) and (36)); $\underline{y}_1$ is its root that minimizes the Lagrangian.

Also from (34), we see that if the search effort is zero at two periods (i.e., $y_k = y_{k'} = 1$ for $k \neq k'$), then it is zero for all the periods (i.e., $y_1 = y_2 = y_3 = 1$). So, we must consider the cases where the search effort is zero for a unique period. In this case, only two optimality equations (see (34)) are valid. Consider for instance (other cases are completely similar), the case $x_{2,\theta} = 0$, then (34) reduces to

$$\delta_2 y_1(1-y_3) = \frac{\lambda}{g_1}. \tag{37}$$

EXAMPLE

Let us now consider the following simplification: $\delta_1 = \delta_2 = \delta_3 = \delta^*$. Various cases must be considered. First, assume that $x_{1,\theta}$ is non-zero, then (34) implies that $x_{2,\theta}$ and $x_{3,\theta}$ cannot be both equal to zero. Assume now $x_{3,\theta} \neq 0$, then from (34), we deduce easily

$$\begin{aligned} &(1-y_2)(y_3-y_1) + (y_1-y_3) = -y_2(y_1-y_3) = 0 \\ &\text{so that} \quad (y_2 \neq 0): \\ &y_1 = y_3. \end{aligned} \tag{38}$$

Then, we deal with two subcases.

Case 1 $\quad (y_1 = y_3 \neq 1,\ y_2 = 1)$. From (34), the equation determining $X_{1,\theta}, X_{3,\theta}$ is

$$X_{1,\theta}(1-X_{1,\theta}) = \frac{\lambda}{wg_{1,\theta}}. \tag{39}$$

The probability of detection is then $P = \sum_{1,\theta} g_{1,\theta} \cdot (1-X_{1,\theta})^2$ ($X_{1,\theta}$ being given by (39)).

Case 2 $\quad (y_1 = y_3 \neq 1;\ y_2 \neq 1)$. Then from (34), we deduce that $y_1 = y_2 = y_3$. The probability of detection becomes

$$P = \sum_{\theta} g_{1,\theta}(1-X_{1,\theta})^2(1+2X_{1,\theta})$$

so that $X_{1,\theta}$ is

$$(1-X_{1,\theta})X_{1,\theta}^2 = \frac{\lambda}{2wg_{1,\theta}}. \tag{40}$$

More generally, we see that further the "general" case (i.e., $y_1 = y_2 = y_3$), we must consider various subcases associated with the nullity of the search effort during *one* period (i.e., y_1 or y_2 or y_3 is zero). The general form of the particular possibilities is

$$\begin{cases} y_i = 1, \qquad i = 1,2 \text{ or } 3 \\ y_j = y_k \qquad \text{for} \quad j \neq i \quad \text{and} \quad k \neq i. \end{cases} \tag{41}$$

The calculations are identical to the Case 1 ones, yielding the same solution (see (40)).

[7]The index of missed detection is the index of δ.

The next step is to calculate $\psi(\lambda)$. To do so, the values of $\{y_1, y_2, y_3\}$ are given either by (39) or (40) and $\psi(\lambda)$ is given by ($\underline{X} = (\underline{x}_{1,\theta}, \underline{x}_{2,\theta}, \underline{x}_{3,\theta})$):

$$\psi(\lambda) = -P(\underline{X}) + \lambda\left(\sum_{\theta} \underline{x}_{1,\theta} + \underline{x}_{2,\theta} + \underline{x}_{3,\theta} - \Phi\right).$$

The values of the components of the vector $\underline{\mathbf{X}}$ are given either by (39) or (40) and correspond either to the solutions that minimize the reduced Lagrangian $\mathcal{L}$ or are zero altogether. Since ψ is concave, maximization of ψ with respect to λ is easy.

B. The n-Period Search and MAJORITY Track Detection Rule

We assume that elementary detections are independent and consider that a track is detected if k elementary detections occur. The track detection P_{td} then takes the following form [21, 22]:

$$P_{\text{td}} = \sum_{i=k}^{n}\left\{\left(\sum_{p=0}^{i-k}(-1)^p C(i,p)\right)\left(\sum_{C_{i,n}}\left[\prod_j Pd_j\right]\right)\right\}$$

where: (42)

$$C(i,p) = \frac{i!}{p!\,(i-p)!}.$$

In (42), the term $\sum_{C_{i,n}}[\prod_j Pd_j]$ is the sum of all the possible products of i elementary detections that can be formed from the whole elementary detections.

We now restrict ourselves to the following track detection rule. The track is said to be detected if at least $(n-1)$ elementary detections occur in an n-period search. Thus, the probabilities of the following events are considered

$$\begin{cases} P_1 \equiv P_{0,2,\ldots,n} = y_1 \prod_{i=2}^{n}(1-y_i) \\ P_2 \equiv P_{1,0,2,\ldots,n} = y_2 \prod_{i=1,\neq 2}^{n}(1-y_i) \\ \vdots \\ P_n \equiv P_{1,2,\ldots,n-1,0} = y_n \prod_{i=1}^{n-1}(1-y_i) \\ P_* \equiv P_{1,2,\ldots,n} = \prod_{i=1}^{n}(1-y_i). \end{cases} \tag{43}$$

For the sake of simplicity, we assume that the the detection coefficients $(\beta_{0,2,\ldots,n}, \beta_{1,0,2,\ldots,n}, \ldots, \beta_{1,2,\ldots,n})$, see (33)) are equal.[8] Let us first assume that the search efforts are non-zero for all the periods (i.e., $x_1 \neq 0, \ldots, x_n \neq 0$), then the KKT conditions result in ($\alpha = \lambda / w g_{1,\theta}$):

$$\begin{cases} y_1 y_2 \prod_{i\neq 1,2}^{n}(1-y_i) + y_1 y_3 \prod_{i\neq 1,3}^{n}(1-y_i) \\ \qquad + \cdots + y_1 y_n \prod_{i\neq 1,n}^{n}(1-y_i) = \alpha, \qquad (1) \\ y_2 y_1 \prod_{i\neq 2,1}^{n}(1-y_i) + y_2 y_3 \prod_{i\neq 2,3}^{n}(1-y_i) \\ \qquad + \cdots + y_2 y_n \prod_{i\neq 2,n}^{n}(1-y_i) = \alpha, \qquad (2) \\ \text{and more generally:} \\ y_j y_1 \prod_{i\neq j,1}^{n}(1-y_i) + y_2 y_j \prod_{i\neq 2,j}^{n}(1-y_i) \\ \qquad + \cdots + y_j y_{j-1} \prod_{i\neq j-1,j}(1-y_i) \\ \qquad + y_j y_{j+1} \prod_{i\neq j,j+1}(1-y_i) \\ \qquad + \cdots + y_j y_n \prod_{i\neq j,n}^{n}(1-y_i) = \alpha, \qquad (j) \end{cases} \tag{44}$$

Subtracting (for example) row 3 from row 2, we obtain

$$(y_2 - y_3)\prod_{i\neq 2,3}^{n}\left[\frac{y_1}{(1-y_1)} + \frac{y_4}{(1-y_4)} + \cdots + \frac{y_n}{(1-y_n)}\right] = 0. \tag{45}$$

Since the term between brackets is well defined and non-zero, we deduce from (45) that $y_2 = y_3$, and more generally subtracting row $(i+1)$ from row i in (44), we have $y_1 = y_2 = \cdots = y_n$. Also from (44), we deduce that the search efforts (for a given track parameter $\{\theta\}$) are either zero for all the periods or zero for *at most* one period. The rest of the derivation is identical to the 3-period case.

VII. SEARCH FOR MARKOVIAN TRACKS

We next consider how to search for a Markovian target. The classical optimization framework we used previously is here useless due to the complexity of elementary events. Instead, we use Brown's approach [11], where a sequence of search plans is generated incrementally. For the sake of simplicity, our approach is restricted to the AND detection rule.

[8] As seen previously (see Section VIA), this assumption does not reduce the generality of our approach.

The target is moving among a finite number of cells. Let the set of cells be C at each time period. The target occupies one cell during each of the time periods so its path is decribed by $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in C^n$. The searcher starts with a function $g : C^n \to [0,1]$, where $g(\omega)$ is the probability that the target takes the path ω. During the ith period, the searcher has L_i units of search efforts which he may divide between the cells of the ith period in arbitrary proportions. Note that, this time, the search effort L_i is fixed[9] at each time period. Thus, the search effort distribution at time i may be described by a vector $X_i \in \mathbb{R}^m$, $m \stackrel{\Delta}{=} \mathrm{Card}(C)$, with components $x(c,i)$, giving the search effort placed in cell c at time i. The search plan denoted $\mathbf{X}$ is the vector obtained by collecting the vectors X_i. Again, we assume that the probability that this plan will find the target at time i is $1 - \exp[-w(c,i)x(c,i)]$. We assume that the searches at distinct time periods are statistically independent, so that the probability that the target be detected (for the AND detection detection rule) is

$$P = \sum_{\omega\in\Omega} g(\omega) \prod_{i=1}^{n} [1 - \exp(-w(\omega_i, i)x(\omega_i, i))]. \tag{46}$$

Thus, we must solve the following problem:

$$\mathcal{P} \begin{cases} \min -P \qquad \text{where} \quad P \text{ is given by (46)} \\ \text{subject to:} \\ x(c_i, i) \geq 0 \qquad \text{and:} \qquad \displaystyle\sum_{c_i\in C_i} x(c_i, i) \leq L_i. \end{cases} \tag{47}$$

Necessary conditions may be derived from the results of Stone [23]. However, a direct solution to the optimality conditions seems quite unfeasible. It is therefore worth considering the following factorization of $P(\mathbf{X})$ ($\mathbf{X}$: search plan):

$$\begin{aligned} P(\mathbf{X}) &= \sum_{(c,i)} \underbrace{\left(\sum_{\omega\in\Omega;\ \omega_i=(c,i)} g(\omega) \prod_{j\neq i} [1 - \exp(-w(\omega_j, j)x(\omega_j, j))] \right)}_{P(c,i,\mathbf{X})} \\ &\quad \times [1 - \exp(-w(c,i)x(c,i))] \\ &= \sum_{c,i} P(c,i,\mathbf{X})[1 - \exp(-w(c,i)x(c,i))]. \end{aligned} \tag{48}$$

The problem is thus immersed in a stationary framework, in which $P(c,i,\mathbf{X})$ represents the probability that the search has been successful at all periods different from i, for all the target paths passing by the cell (c,i) at the period i. This corresponds to the reallocation problem [11]. So, the main problem then consists in effectively calculating $P(c,i,\mathbf{X})$. To that end, we consider the following factorization of $P(c,i,\mathbf{X})$ [11]:

$$\begin{aligned} P(c,i,\mathbf{X}) &= \mathrm{reach}(c,i,\mathbf{X})\mathrm{surv}(c,i,\mathbf{X}) \\ \mathrm{reach}(c,i,\mathbf{X}) &= \sum_{\omega\in\Omega;\ \omega_i=(c,i)} r(\omega_1)t(\omega_1,\omega_2)\cdots t(\omega_{i-1},c) \\ &\quad \times \prod_{j=1}^{i-1} [1 - \exp(-w(\omega_j, j)x(\omega_j, j))] \\ \mathrm{surv}(c,i,\mathbf{X}) &= \sum_{\omega\in\Omega;\ \omega_i=(c,i)} t(c,\omega_{i+1})\cdots t(\omega_{n-1},\omega_n)s(\omega_n) \\ &\quad \times \prod_{j=i+1}^{n} [1 - \exp(-w(\omega_j, j)x(\omega_j, j))]. \end{aligned} \tag{49}$$

In (49) $t(\omega_1, \omega_2)$ denotes the probability of transition from ω_1 to ω_2. Previously, Ω was small enough to practically enumerate its elements (*conditionallly deterministic motion*). However, this is not feasible since we must consider all the (Markovian) paths $\omega = (\omega_1, \omega_2, \dots, \omega_n)$. The terms $\mathrm{reach}(c,i,\mathbf{X})$ and $\mathrm{surv}(c,i,\mathbf{X})$ are themselves determined by the following recursion [11]:

$$\mathcal{R} \begin{cases} \mathrm{reach}(c,1,\mathbf{X}) = r(c) \\ \mathrm{reach}(c,j+1,\mathbf{X}) = \displaystyle\sum_{d\in C} \mathrm{reach}(c,j,\mathbf{X}) \\ \qquad [1 - \exp(-w(d,j)\mathbf{X}(d,j))]t(d,c) \\ \mathrm{surv}(c,n,\mathbf{X}) = s(c) \\ \mathrm{surv}(c,j-1,\mathbf{X}) = \displaystyle\sum_{d\in C} t(c,d) \\ \qquad [1 - \exp(-w(d,j)\mathbf{X}(d,j))]\mathrm{surv}(c,j,\mathbf{X}). \end{cases} \tag{50}$$

Note that $\mathrm{reach}(c,j,\mathbf{X})$ has a natural interpretation as the probability that the target reaches cell c at time period i, being detected by search $\mathbf{X}$ throughout period 1 to $j-1$ and $\mathrm{surv}(c,j,\mathbf{X})$ as the probability that a target in cell c at the period j will be detected by search $\mathbf{X}$ throughout periods $j+1$ to n.

Let us denote $x^*(\cdot, i)$ the solution to the (stationary) reallocation problem given by (48). Then the algorithm is given as follows.

1) Make an initial guess for the search plan (e.g., zero) and choose a small positive number ε,
2) performs steps 3 and 4 (the main loop) for $i = \{1, \dots, n\}$,
3) replace $x_k^*(\cdot, i)$ with $x_{k+1}^*(\cdot, i)$, the solution to the reallocation problem for time i (use (49) and (50)),
4) increment i,
5) if $|P_n(\mathbf{X}_{k+1} - P_n(\mathbf{X}_k| \leq \varepsilon$, stop,
6) increment k and go to step 3. (51)

In the first pass through steps 3 and 4, since each solution to the reallocation problem provides the search plan that gives the greatest increase in the probability of detection at that time interval, it is the

[9]This assumption is necessary in order to use the basic recursion of the Brown's algorithm.

myopic plan. This algorithm performs an iterative maximization sequence and may be viewed as a steepest ascent.

VIII. TWO-SIDED SEARCH

Up to now, our efforts have been exclusively devoted to the one-sided search, which means that decisions are only made by the searcher. For the two-sided search, game theory is the natural framework. Here, the following game is considered. The strategy for player 1 (target) is a distribution of $g_1(\theta)$, while for player 2 (searcher) it is the distribution of efforts (i.e., the vectors $\mathbf{X}_\theta$; $\theta \in \Theta$).[10] Let us denote $\mathbf{G}_1$ the vector representing the distribution of $g_1(\theta)$; $\theta \in \Theta$, we are now considering the following problem:

Determine the vectors $\mathbf{G}_1^*$, $\mathbf{X}^*$ such that:

$$P(\mathbf{G}_1^*,\mathbf{X}) \leq P(\mathbf{G}_1^*,\mathbf{X}^*) \leq P(\mathbf{G}_1,\mathbf{X}^*) \qquad \forall\ (\mathbf{X},\mathbf{G}_1). \tag{52}$$

The detection function $P(\mathbf{G}_1,\mathbf{X})$ is given by (2). Note that the vector $\mathbf{X}^*$ is a collection of (elementary) vectors $\mathbf{X}_\theta^*$, $\theta \in \Theta$. Equivalently, $\mathbf{X}^*$ and $\mathbf{G}_1^*$ are the solutions to the min-max problem $\min_{\mathbf{G}_1} \max_{\mathbf{X}} P$. Restricting attention to the AND detection test, we must solve the following optimization problem:

$$\mathcal{P}\begin{cases} \min_{\mathbf{G}_1} \max_{\mathbf{X}} \left\{ -\sum_\theta g_1(\theta) \prod_{i=1}^n \gamma_i (1-e^{-w_{i,\theta}x_{i,\theta}}) \right\} \\ \text{under the constraints:} \\ \sum_i \sum_\theta x_{i,\theta} = \Phi \qquad \{x_{i,\theta} \geq 0,\ \forall\ (k,\theta)\} \\ \sum_{\theta\in\Theta} g_1(\theta) = 1. \end{cases} \tag{53}$$

If, furthermore, the following assumption is made (γ_i a constant, $w_{i,\theta} = w_\theta$), the problem may be explicitely solved.

PROPOSITION 3 *The elements of* $\mathbf{G}_1^*$ *and* $\mathbf{X}^*$ *are determined by the following equation:*

$$\begin{aligned} &\forall \quad \theta \in \Theta : x_{1,\theta} = \cdots = x_{n,\theta}, \qquad \textit{and:} \\ &x_{1,\theta} = \frac{\Phi}{n}\left(\sum_{\theta\in\Theta} w_\theta^{-1}\right)^{-1} \qquad \textit{and} \\ &g_1(\theta) = \left(\sum_{\theta\in\Theta} w_\theta^{-1}\right)^{-1} w_\theta^{-1}. \end{aligned} \tag{54}$$

PROOF We must solve two distinct optimization problems in order to obtain $\mathbf{G}_1^*$ and $\mathbf{X}^*$ such that (52) is satisfied. Let us consider first the right inequality:

$$I\begin{cases} \min_{\theta\in\Theta}\left\{ \sum_\theta g_1(\theta) \left[\prod_{i=1}^n (1-e^{-w_\theta x_{i,\theta}^*}) \right] \right\} \\ \sum_\theta g_1(\theta) = 1; \qquad g_1(\theta) \geq 0. \end{cases} \tag{55}$$

Its corresponding Lagrangian then takes the following form:

$$\mathcal{L}(\lambda) = \sum_\theta g_1(\theta) \left[\prod_{i=1}^n (1-e^{-w_\theta x_{i,\theta}^*}) \right] + \lambda \left(\sum_\theta g_1(\theta) - 1 \right) + \sum_\theta \mu_\theta g_1(\theta). \tag{56}$$

The reasoning now follows the following steps.

a) Let us consider any value of θ such that $g_1^*(\theta)$ be strictly positive, then the corresponding Lagrange multiplier μ_θ is equal to zero so that: $\prod_{i=1}^n (1-e^{-w_\theta x_{i,\theta}^*}) = -\lambda$.

b) Assume now that $g_1^*(\theta)$ is zero, then $\mathbf{X}_\theta^* = 0$; therefore, $\mathbf{X}_\theta^* > 0 \Rightarrow g_1^*(\theta) > 0$.

c) Suppose that there exists a value of θ such that $g_1^*(\theta)$ is zero. Let us denote θ_0 a value for which $g_1^*(\theta_0) > 0$. Such a value necessarily exists. Now, from KKT conditions (applied to (56)), we have ($\mu_\theta \geq 0$):

$$\prod_{i=1}^n (1-e^{-w_\theta x_{i,\theta}^*}) = -\lambda + \mu_\theta \geq \prod_{i=1}^n (1-e^{-w_{\theta_0} x_{i,\theta_0}^*}) \tag{57}$$

so that $\mathbf{X}_\theta^* > \mathbf{0}$. Now, this contradicts our assumption (see b). Therefore, $g_1^*(\theta)$ is *strictly* positive on the *whole* set Θ. We are now in position to examine the left optimization problem, i.e.,

$$II\begin{cases} \min_{\mathbf{X}} -P(\mathbf{G}_1^*,\mathbf{X}) \\ \sum_{i,\theta} x_{i,\theta} = \Phi; \qquad x_{i,\theta} \geq 0, \quad \forall\ i, \quad \forall \quad \theta \in \Theta \end{cases} \tag{58}$$

with associated Lagrangian ($\nu_{i,\theta} \geq 0$):

$$\mathcal{L}(\lambda) = -\sum_\theta g_1(\theta) \left[\prod_{i=1}^n (1-e^{-w_\theta x_{i,\theta}}) \right] + \lambda \left(\sum_i \sum_\theta x_{i,\theta} - \Phi \right) + \sum_i \sum_\theta \nu_{i,\theta} x_{i,\theta}. \tag{59}$$

First, we prove that $x_{k,\theta}^*$ is *strictly* positive whatever k and θ. From the above reasoning we know that $\mathbf{X}_\theta^* > \mathbf{0}$

[10] For the notations, we refer to Section II.

($\forall\theta \in \Theta$), hence there exists an index k_0 such that $x^*_{k_0,\theta}$ is *strictly* positive. Then from the KKT conditions, the following equality is deduced

$$\frac{\prod_{i=1,\neq k}^{n}(1-e^{-w_\theta x^*_{i,\theta}})}{\prod_{i=1,\neq k_0}^{n}(1-e^{-w_\theta x^*_{i,\theta}})} = \frac{(1-e^{-w_\theta x^*_{k_0,\theta}})}{(1-e^{-w_\theta x^*_{k,\theta}})} = \frac{\lambda+\nu_{k,\theta}}{\lambda}. \tag{60}$$

From (60), we thus have

$$1-e^{-w_\theta x^*_{k,\theta}} = \left(\frac{\lambda}{\lambda+\nu_{k,\theta}}\right)(1-e^{-w_\theta x^*_{k_0,\theta}}). \tag{61}$$

Now the multiplier λ is positive (KKT $\Rightarrow \lambda = w_\theta g_1(\theta)\prod_{i=1,\neq k_0}^{n}(1-e^{-w_\theta x^*_{i,\theta}})$), and so is $\nu_{k,\theta}$, so that $0 \le (\lambda/\lambda+\nu_{k,\theta}) \le 1$ and, in turn, $x^*_{k,\theta} \ge x^*_{k,\theta_0} > 0$. Then, we have

$$\begin{aligned} &x^*_{1,\theta} = x^*_{2,\theta} = \cdots = x^*_{n,\theta} = x^*_\theta, \qquad \text{and} \\ &g_1^*(\theta) = \text{cst}, \qquad \forall \quad \theta \in \Theta \end{aligned} \tag{62}$$

which ends the proof.

If the above assumptions are valid, the two-sided search problem has an explicit and simple solution [24]. Furthermore, notice that the optimal searcher and target strategies are proportional. Quite intuitively, this strategy is such that the product $w_\theta g_1(\theta)$ remains constant. In the general case (i.e., $p(x_{k,\theta}) = \gamma_k(1-e^{-w_{k,\theta}x_{k,\theta}})$), a direct resolution to the primal problem (53) is unfeasible; however the problem may be easily solved by the dual approach. Equations (23) and (24) are still valid. Another feasible approach is to consider an enumeration of target tracks (see [26]).

IX. SIMULATION RESULTS

First, we examine the AND track detection rule. The following track density is considered for this example. In the first period, the initial localization density is given by the product of two densities. The first one is relative to the angle (β) and is a truncated normal density; its mean represents the mean direction of the track and its variance determines the localization uncertainty. The second density is related to the distance (r) uncertainty, so that the initial target density takes the following form:

$$p(r,\beta) = \mathcal{N}_{\text{tr}}(\beta;\beta_0,\sigma_\beta)\mathcal{N}_{\text{tr}}(r;r_0,\sigma_r). \tag{63}$$

The track density is described by the function $g_1(\theta)$. When the density of the initial target localization is given by (63), the track velocity distribution is a triangular density in angle (centered around 45 deg), while the norm of the velocity vector $\|v\|$ is constant. It is then possible to compute the target density localization at the successive periods. Target densities for time periods 1, 5, and 10 are plotted on the first row of Fig. 3. Notice the "diffusion" of the target localization. Asymptotic forms of the target spatio-temporal distribution can be found in [25]. For instance, in the case of distribution in heading, precise speed; the asymptotic behavior of $p(r,\theta,t)$ is given by

$$p(r,\theta,t) \propto \frac{p_2(\theta)}{\sigma(2\pi v_0 rt)^{1/2}}\exp\left(-\frac{1}{2\sigma^2}(r-v_0t)^2\right). \tag{64}$$

This describes a probability density "wave" which moves radially at speed v_0.

The search efforts are next computed by using the results obtained in Sections IV and V. For this example, it is assumed that all the visibility coefficients are equal (i.e., $w_{k,\theta} \equiv w$; $\forall\{k,\theta\}$). The dual functional $\psi(\lambda)$ (see Section V) is then computed and maximized relatively to λ (yielding $\underline{\lambda}$). From $\underline{\lambda}$, the values of the search efforts $\underline{x}_{k,\theta}$ are deduced from the optimality equations. The optimization procedure is illustrated by Fig. 1. On the top, the values of $\psi(\lambda)$ are plotted, versus λ, notice the concavity of $\psi(\lambda)$. The middle corresponds to the values of the search effort $\Phi(\lambda)$ (versus λ). Notice that $\Phi(\lambda)$ is a decreasing function of λ, and that the maximum value of $\psi(\lambda)$ (say $\psi(\underline{\lambda})$) corresponds to the value of the total search effort (i.e., $\Phi(\underline{\lambda}) = 1000$). This result is important since it proves that there is no duality gap. Finally, the probability of detection P is plotted on the bottom. Again, it is a monotonic (decreasing) function of λ. Algorithm results are illustrated by Fig. 3. The second row represents the distribution of the search efforts (in the (x,y) plane) for the time periods 1, 5, and 10, for a total search effort $\Phi = 7000$. The third row (respectively fourth) corresponds to a total search effort $\Phi = 20000$ (respectively $\Phi = 80000$). Note that the search efforts are determined in the track parameter space. Indeed, the values of $\underline{x}_{k,\theta}$ (k: index of the time period, $\theta \stackrel{\Delta}{=} (\theta)$) are defined in the track parameter space, *but* are represented in the (x,y) search-space. Hence, the search-space is divided into 64 elementary cells, and the value of $\underline{x}_{k,\theta}$ induces a distribution of the elementary search efforts, directly deduced from the values of k and θ.

From these results, we note that the search efforts are concentrated on the maxima of the target localization density when the search effort is small ($\Phi = 7000$). When it grows, this distribution is spread ($\Phi = 20000$) and becomes closely related to the target localization when Φ is large (e.g., $\Phi = 80000$). This behavior seems quite natural. Furthermore, we note that the total amount of search effort is equally distributed between the various periods. This is due to our assumption about the visibility coefficients $w(k,\theta)$. In the general case, this conclusion is no longer valid and the search efforts may be quite unequally distributed on the successive periods. Of course, the

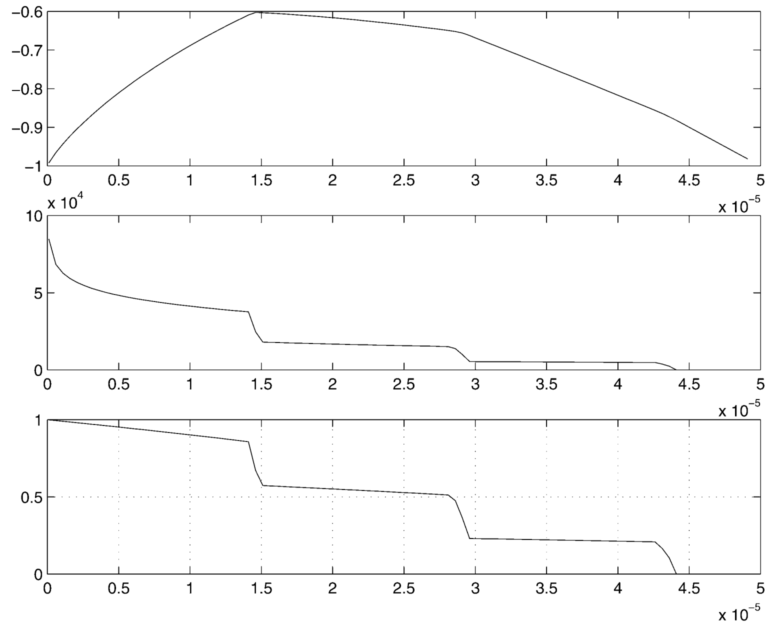


Fig. 1. Top: values of dual function $\psi(\lambda)$, versus λ. Middle: values of total search effort $\Phi(\lambda)$. Bottom: probability of detection, versus λ.

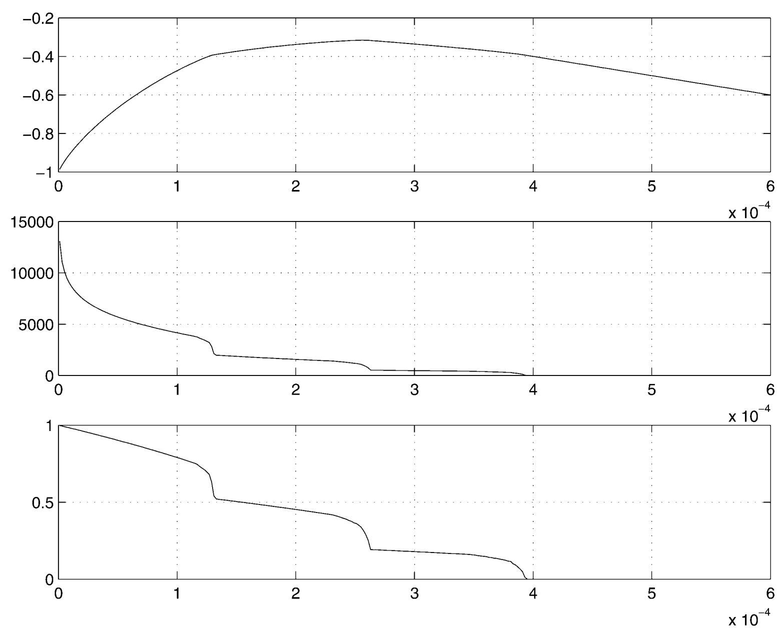


Fig. 2. Top: values of dual function $\psi(\lambda)$, versus λ. Middle: values of total search effort $\Phi(\lambda)$, versus λ. Bottom: probability of detection, versus λ.

values of the probability of track detection are tightly related to the values of the total search effort Φ. This is illustrated by Table I, always for a 10-period search. Roughly, P is an exponential function of Φ.

We now consider the MAJORITY detection rule. We restrict here to a 3-period search, with a total search effort $\Phi = 1000$. Again, we assume that the track density is characterized by the function $g_1(\theta)$. The density of the initial target localization is always described by (63) while the distribution of the target velocity vector is defined by a triangular density in angle (centered around 45 deg) and a discrete distribution of $\|v\|$. The following density $p(v_x, v_y)$ was considered (see Table II).

It is then possible to compute the density of target localization at the successive periods. Results for time periods 1, 2, and 3 are plotted on the first row of Fig. 4. They correspond to a spatio-temporal diffusion.The dual function $\psi(\lambda)$ is computed by means of Section VI results. Equations (39) and (40) are used to compute the optimal values of $X_{1,\theta}$. The result is presented in Fig. 2. On the top, the values of $\psi(\lambda)$ are plotted, versus λ. Again the concavity property (of $\psi(\lambda)$) is verified and the maximum value

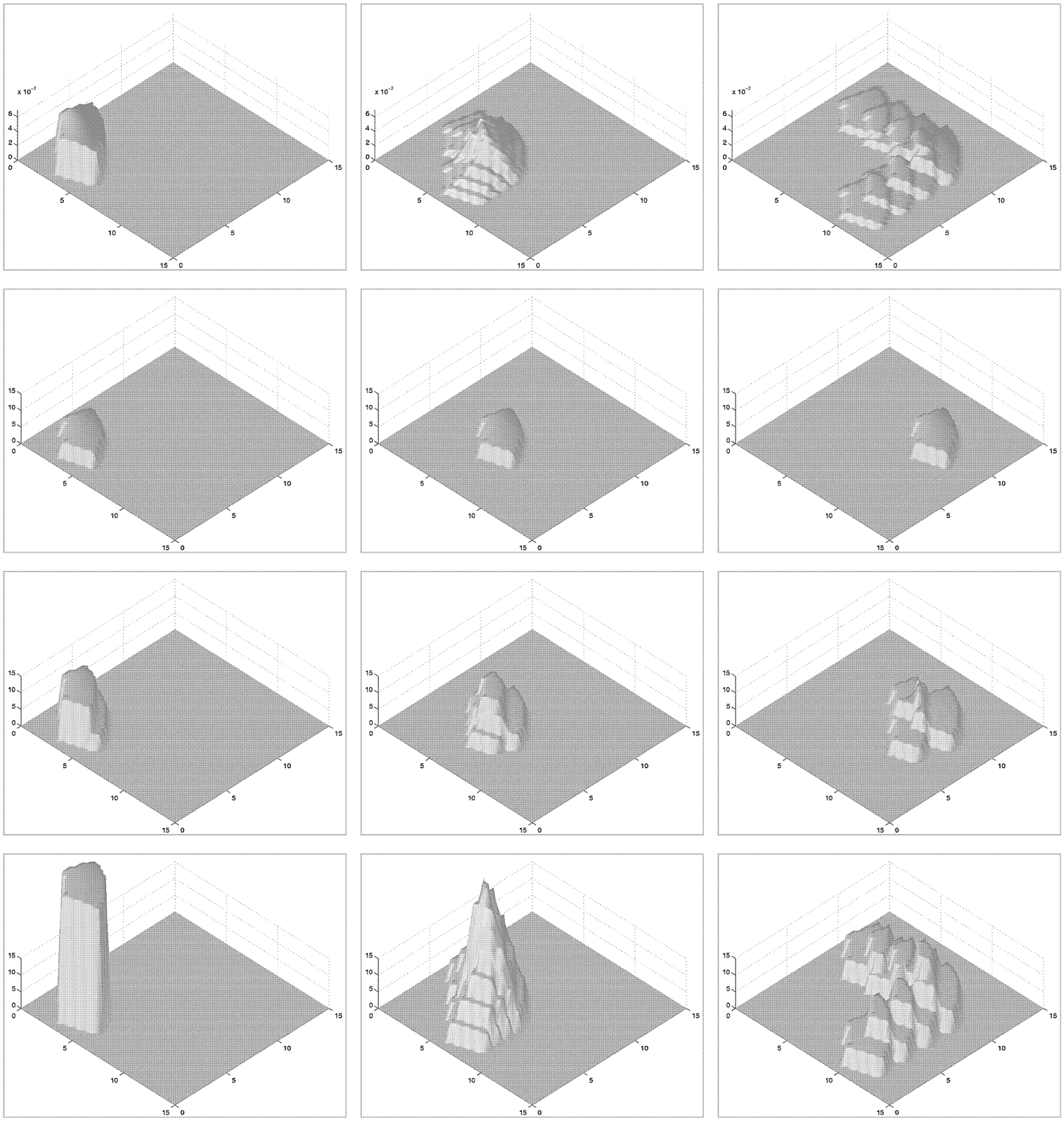


Fig. 3. Distribution of search efforts for AND detection rule and 10-period search. First column: 1st period search. Second column: 5th period search. Third column: 10th period search. First row: density of target localization. Second row: $\Phi = 7000$. Third row: $\Phi = 20000$. Fourth row: $\Phi = 80000$.

TABLE I
Values of Probability of Detection P (Versus Φ), AND Detection Rule, 10 Time Periods

Φ	P	Φ	P
500	0.0139	20 000	0.4325
2000	0.054	40 000	0.6695
7000	0.1858	60 000	0.8438
10 000	0.2497	80 000	0.9548

of $\psi(\lambda)$ (say $\psi(\underline{\lambda})$) corresponds to a value of the total search effort $\Phi(\underline{\lambda}) = 1000$ (no duality gap). Finally, the probability of detection P is plotted on the bottom.

The distribution of the search efforts during successive periods is presented in Fig. 4. The total search effort Φ is equal to 1000. The distribution of the search efforts for the MAJORITY detection rule

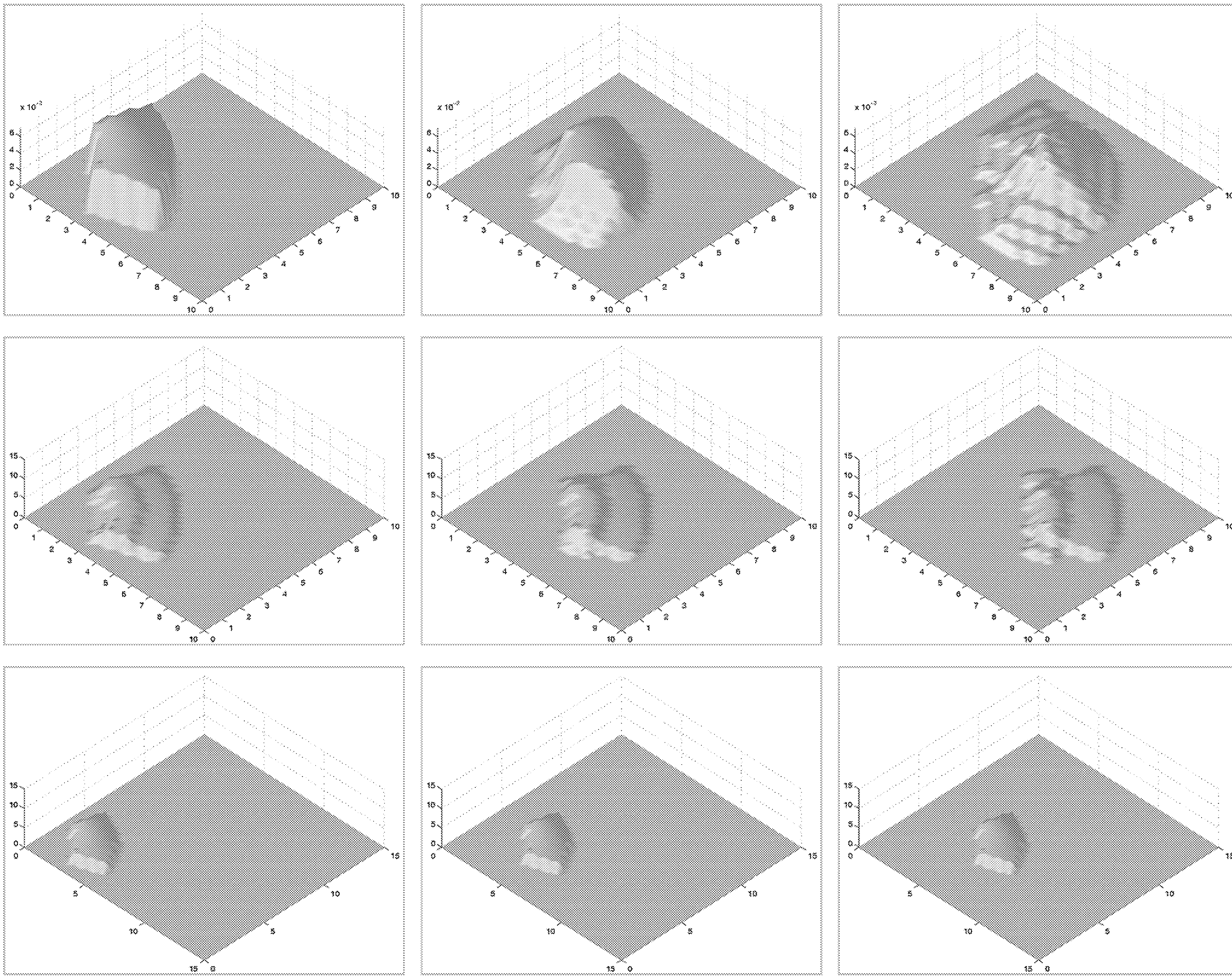


Fig. 4. Distribution of search effort for 3-period search. First column: 1st period search. Second column: 2nd period search. Third column: 3rd period search. First row: density of target localization. Second row: $\Phi = 1000$, MAJORITY detection rule. Third row: $\Phi = 1000$, AND detection rule.

TABLE II
Values of Probability of Target Velocity Vector

v_x	6	6	6	6	4	2	0
v_y	0	2	4	6	6	6	6
$p(v_x, v_y)$	1	1	2	3	2	1	1

TABLE III
Values of Probability of Detection P (Versus Φ), for 3-Period Search and for MAJORITY and AND Detection Rule

Φ	P MAJORITY	P AND	Φ	P MAJORITY	P AND
500	0.1507	0.07	6000	0.8403	0.5522
1000	0.2619	0.1381	10000	0.9692	0.7374
2000	0.4517	0.2475	15000	0.9968	0.9042
4000	0.6729	0.4281	30000	1	0.9961

and 3 consecutive search periods are presented on the second row, while the results obtained for the AND detection rule are plotted on the third row. We can notice the distinctive features characterizing the two detection rules. The distribution of the search efforts for the MAJORITY detection rule is much more widely spreaded than for the AND rule. This result is quite typical of this detection rule. The dependence of the probability of detection P (for MAJORITY and AND), versus the total search effort Φ is illustrated in Table III. We note that the values of P are always greater for the MAJORITY detection rule. This is not surprising since the detection test, for the MAJORITY rule, is less demanding.

Finally, let us consider the optimization of the search effort for a Markovian track (see Section VII). This time, the total amount of search effort is fixed for each time period (denoted L_i). The density of the initial localization of the target is again described by (63), while the distribution of the target velocity vector is identical to the previous one; but, this

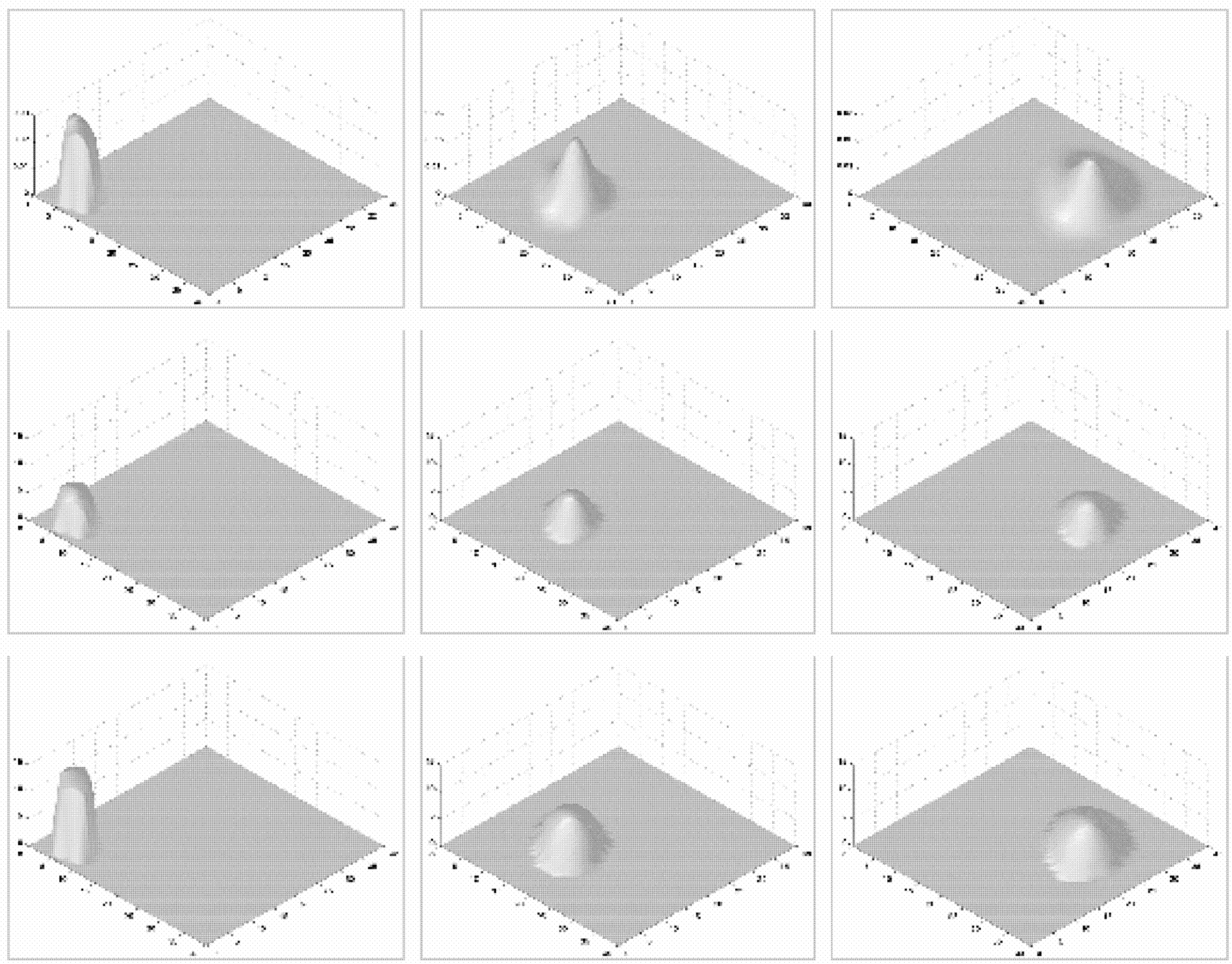


Fig. 5. Distribution of search effort for 10-period search and Markovian target. First column: 1st period search. Second column: 5th period search. Third column: 10th period search. First row: density of target localization. Second row: $\Phi = 200$, AND detection rule. Third row: $\Phi = 500$, AND detection rule.

time, for each time period and with a Markovian hypothesis (i.e., $p(v_i \mid v_{i-1},\ldots,v_1) = p(v_i \mid v_{i-1})$). The target localization density at successive time periods is illustrated by the first row of Fig. 5. The search efforts are then calculated by using the iterative algorithm derived from Brown. The algorithm is initialized by a myopic search and then converges quickly (see [11]). Typically, 4 or 5 iterations of the algorithm described by (48)–(51) are sufficient. The distribution of the search is concentrated on the higher values of the target localization density, for all the time periods. This seems quite natural for an AND detection rule.

X. CONCLUSION

The problem under consideration was the optimization of the search effort for detecting tracks. The problem formulation is closely related to the definition of the track detection criterion. Various definitions have been considered (AND and MAJORITY), focusing on the associated optimization problem. In order to develop tractable methods, we restricted the problem to discrete time and space optimization. Under simple constraints relative to the distribution of the search effort, the dual formalism appears as a feasible and versatile approach, allowing us to derive efficient and relatively simple algorithms.

ACKNOWLEDGMENTS

The authors would like to thank the Sonar and Undersea Systems Editor for his many insightful comments.

APPENDIX A

This Appendix deals with the solution to the elementary search problem [27] (1 time period), by means of duality. Let f be the following *separable* function:

$$\mathbf{X} = (x_1,\ldots,x_n)^* \in \mathbb{R}^n \to f(\mathbf{X}) = \sum_{i=1}^{n} f_i(x_i).$$

The detection functions f_i are assumed differentiable. We then consider the following optimization problem:

$$\mathcal{P}\begin{cases} \min f(\mathbf{X}) \\ \text{Constraints:} \quad x_i \geq 0 \ (i = 1,\ldots,n) \quad \text{and} \quad \displaystyle\sum_{i=1}^{n} x_i = 1. \end{cases} \tag{65}$$

Constraints are necessary qualified since they are affine. First, consider the primal problem $\mathcal{P}$. Let us denote $\underline{\mathbf{X}}$ a solution to $\mathcal{P}$. Then the KKT Theorem implies that there exists Lagrange multipliers $\{\underline{\mu}_1,\ldots,\underline{\mu}_n\} \in (\mathbb{R}^+)^n$ (inequality constraints) and $\underline{\lambda} \in \mathbb{R}$ such that

$$\begin{cases} f_i'(\underline{x}_i) - \underline{\mu}_i + \underline{\lambda} = 0 \\ \underline{\mu}_i \underline{x}_i = 0, \qquad \forall \quad i = 1,\ldots,n. \end{cases} \tag{66}$$

If the index i corresponds to a *strictly* positive value of x_i ($x_i > 0$), then $\underline{\mu}_i = 0$ so that $f_i'(\underline{x}_i) + \underline{\lambda} = 0$. If the index i corresponds to a zero search effort ($x_i = 0$), then $f_i'(\underline{x}_i) + \underline{\lambda} = \underline{\mu}_i \geq 0$. Note that if the functions f_i are convex, the KKT conditions are necessary and sufficient.

Assume now that the (detection) functions f_i are as follows:

$$f_i(x_i) = \gamma_i(e^{-w_i x_i} - 1), \qquad i = 1,\ldots,n$$

and consider the following Lagrangian functional:

$$\mathcal{L}(\mathbf{X},\lambda) = \sum_{i=1}^{n} f_i(x_i) + \lambda\left(\sum_{i=1}^{n} x_i - 1\right)$$

as well as the associated dual function:

$$\psi(\lambda) = \inf_{\mathbf{X}\in(\mathbb{R}^+)^n} \mathcal{L}(\mathbf{X},\lambda) \qquad (\lambda \in \mathbb{R}). \tag{67}$$

In the definition of the dual function ψ, only the equality constraint is "dualized", since the n inequality constraints are included in the definition domain of ψ. We note that $\psi(\lambda) = -\infty$ when λ is negative, we can thus restrict to positive values of λ. In this case, the functional $\mathcal{L}(\mathbf{X},\lambda)$ is smooth and convex on $(\mathbb{R}^+)^n$; hence there exists a unique $\underline{\mathbf{X}}(\lambda)$ minimizing $\mathcal{L}(\mathbf{X},\lambda)$. Let us denote $\mathcal{C}$ the (closed) convex subset of inequalities constraints relative to the positivity of the search efforts ($x_i \geq 0$ for $i = 1,\ldots,n$)). Then this point is defined by the following condition:[11]

$$\begin{aligned} &-\nabla_{\mathbf{X}}\mathcal{L}(\mathbf{X},\lambda) \in \mathcal{N}_{\mathcal{C}}\underline{\mathbf{X}}(\lambda), \\ &\text{where:} \\ &\mathcal{N}_{\mathcal{C}}\underline{\mathbf{X}}(\lambda) \stackrel{\Delta}{=} \{\mathbf{S} \mid \langle \mathbf{S}, \mathbf{C} - \underline{\mathbf{X}}(\lambda)\rangle \leq 0 \quad \forall\, \mathbf{C} \in \mathcal{C}\}. \end{aligned} \tag{68}$$

[11]The symbol $\langle\ ,\ \rangle$ here denotes the scalar product.

In (68), $\mathcal{N}_{\mathcal{C}}\underline{\mathbf{X}}(\lambda)$ is the normal cone [20] to $\mathcal{C}$ in $\underline{\mathbf{X}}(\lambda)$. The above condition is a characterization of the optimum on a (closed) convex subset. In our case, these general conditions simply result in:[12]

$$\begin{cases} -\gamma_i w_i e^{-w_i \underline{x}_i(\lambda)} + \lambda = 0 & \forall\ i \quad \text{s.t.} \quad \underline{x}_i(\lambda) > 0 \\ \gamma_i w_i - \lambda \geq 0 & \forall\ i \quad \text{s.t.} \quad \underline{x}_i(\lambda) = 0. \end{cases} \tag{69}$$

From (69), we deduce that ($\underline{x}_i(\lambda) > 0$) is equivalent to ($\lambda < \gamma_i w_i$), therefore:

$$\begin{aligned} &\underline{x}_i(\lambda) > 0 \Rightarrow \underline{x}_i(\lambda) = \frac{1}{w_i}\ln\left(\frac{\gamma_i w_i}{\lambda}\right) \\ &\text{and more generally:} \\ &\underline{x}_i(\lambda) = \frac{1}{w_i}\left[\ln\left(\frac{\gamma_i w_i}{\lambda}\right)\right]^+. \end{aligned} \tag{70}$$

Consequently, the dual function $\psi(\lambda)$ (see (67)) is

$$\psi(\lambda) = -\sum_{i=1}^{n}\gamma_i\left(1 - \frac{\lambda}{\gamma_i w_i}\right)^+ + \lambda\left(\sum_{i=1}^{n}\frac{1}{w_i}\left[\ln\left(\frac{\gamma_i w_i}{\lambda}\right)\right]^+ - 1\right). \tag{71}$$

As a general result of duality theory [16, 20], we know that the function $\psi(\lambda)$ is concave with respect to λ. This result is valid whatever the primal problem. The dual problem then simply consists in maximizing $\psi(\lambda)$ (relatively to λ). Thanks to the concavity property satisfied by the dual function, this maximum is attained for a unique value of λ (denoted $\underline{\lambda}$). The solution to the primal problem is then

$$\begin{cases} \underline{x}_i = \dfrac{1}{w_i}\ln\left(\dfrac{\gamma_i w_i}{\underline{\lambda}}\right) & \text{if} \quad \gamma_i w_i > \underline{\lambda} \\ \underline{x}_i = 0 & \text{if} \quad \gamma_i w_i \leq \underline{\lambda}. \end{cases} \tag{72}$$

Without any loss of generality, we can assume the following ordering $\gamma_1 w_1 \leq \gamma_2 w_2 \leq \cdots \leq \gamma_n w_n$ and consider $\lambda \in [\gamma_k w_k, \gamma_{k+1} w_{k+1}]$. Then from (71) we obtain

$$\psi(\lambda) = \sum_{i=k+1}^{n}\left(\frac{\lambda}{w_i} - \gamma_i\right) + \lambda\left(\sum_{i=k+1}^{n}\frac{1}{w_i}\ln\left(\frac{\gamma_i w_i}{\lambda}\right) - 1\right). \tag{73}$$

Thus, the function $\psi(\lambda)$ is differentiable on the open interval $]\gamma_k w_k, \gamma_{k+1} w_{k+1}[$, with derivative $\psi'(\lambda) = \sum_{i=k+1}^{n}(1/w_i)\ln(\gamma_i w_i/\lambda) - 1$. From this, it is easily seen that $\lim_{\lambda\to(\gamma_k w_k)_-}\psi'(\lambda) = \lim_{\lambda\to(\gamma_k w_k)_+}\psi'(\lambda)$, so that ψ is differentiable on the whole subset $\mathbb{R}^+$, with derivative:

$$\psi'(\lambda) = \sum_{i=1}^{n}\frac{1}{w_i}\left[\ln\left(\frac{\gamma_i w_i}{\lambda}\right)\right]^+ - 1. \tag{74}$$

[12]These conditions may also be obtained by means of the KKT Theorem (see (66)).

The function ψ' is continuous on $\mathbb{R}+_*$, monotonic (decreasing). Furthermore:

$$\begin{cases} \lim_{\lambda\to 0^+} \psi'(\lambda) = +\infty \\ \psi'(\lambda) = -1 \qquad \text{for} \quad \lambda \geq \gamma_n w_n. \end{cases} \tag{75}$$

Whence, the equation $\psi'(\lambda) = 0$ has a unique solution $\underline{\lambda}$, which is the value of λ maximizing ψ on $\mathbb{R}^+$.

APPENDIX B

In order to prove the absence of a duality gap, it is necessary to prove that there is a *unique* search vector (say $\mathbf{X}$) minimizing the Lagrangian $\mathcal{L}(\mathbf{X},\lambda)$. Here, we consider that λ is fixed and we restrict to the 2-period search for the AND rule. The proof may be generalized in a straightforwardly manner. Let $\mathcal{L}(\mathbf{X},\lambda)$ be defined by (15). Then due to the necessary optimality conditions (16), we have ($\alpha = \lambda/(wg_1(\theta))$):

$$X_{1,\theta}(1 - X_{1,\theta}) = \alpha$$

so that for any candidate search vector (denoted $\tilde{\mathbf{X}}$) minimizing $\mathcal{L}(\mathbf{X},\lambda)$, the Lagrangian takes the following form:

$$\mathcal{L}(\tilde{\mathbf{X}},\lambda) = \sum_\theta [g_1(\theta)e^{-w\tilde{x}_{1,\theta}} + 2\tilde{x}_{1,\theta}] + \text{cst}. \tag{76}$$

Since the optimization problem is now separated in the variables $\tilde{x}_{1,\theta}$, it reduces to separated minimizations of the functions $g_1(\theta)e^{-w\tilde{x}_{1,\theta}} + 2\tilde{x}_{1,\theta}$. These functions have the following form:

$$f(x) = ge^{-wx} + 2\lambda x. \tag{77}$$

Now, this function is differentiable and convex so that its minimum (on the $[0,1]$ interval) is unique. In turn, the search vector (say $\underline{\mathbf{X}}$) minimizing $\mathcal{L}(\mathbf{X},\lambda)$ is also unique and the function $\psi(\lambda)$ is a differentiable function of λ.

APPENDIX C

The object of this Appendix is to provide an example of calculating for the weights $\{\beta_{0,2,3},\beta_{1,2,0},\beta_{1,0,3},\beta_{1,2,3}\}$ (see e.g., (30)). Assume that the target motion is rectilinear and uniform and that the detection system yields an observation vector $\hat{\mathbf{Z}}$ with the following structure:

$$\begin{aligned} &\hat{\mathbf{Z}} \text{ is } \mathcal{N}(\mathbf{Z},\Gamma) \\ &\mathbf{Z} = \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} \\ &\mathbf{X} = [\delta_{1,d}x_0,\ldots,\delta_{n,d}(x_0 + nv_x)]^*, \\ &\mathbf{Y} = [\delta_{1,d}y_0,\ldots,\delta_{n,d}(y_0 + nv_y)]^*. \end{aligned} \tag{78}$$

In (78), the vector $(\delta_{1,d},\ldots,\delta_{n,d})^*$ denotes the vector of detections, i.e., $\delta_{i,d}$ is equal to 1 if an elementary detection occurs at the instant i, to 0 else. Denoting $\mathbf{D}$ this vector and $\odot$ the Schur product [28], we thus have

$$\begin{aligned} &\mathbf{X} = \mathbf{D} \odot (x_0\mathbf{1} + v_x\mathbf{1}') \\ &\text{where:} \\ &\mathbf{1} = (1,1,\ldots,1)^*, \qquad \mathbf{1}' = (0,1,\ldots,n)^*. \end{aligned} \tag{79}$$

Since the target trajectory is characterized by the 4-dimensionnal vector $\mathbf{X}_0$ ($\mathbf{X}_0 = (x_0,y_0,v_x,v_y)^*$), the weighting terms we consider are deduced from the Fisher information matrix (FIM) which is given by ($\Gamma = \sigma^2 I$):

$$\begin{aligned} \text{FIM}(x_0,x_0) &= \frac{1}{2\sigma^2}\|\mathbf{D}\odot\mathbf{1}\|^2, \\ \text{FIM}(x_0,v_x) &= \frac{1}{2\sigma^2}(\mathbf{D}\odot\mathbf{1})^*(\mathbf{D}\odot\mathbf{1}'), \quad \text{etc.} \end{aligned} \tag{80}$$

Then, a convenient weighting may be the determinant of the FIM.

REFERENCES

[1] Koopman, B. O. (1999)
Search and Screening: General Principle with Historical Applications.
Alexandria, VA: MORS Heritage Series, 1999.

[2] Benkovski, S. J., Monticino, M. G., and Weisinger, J. R. (1991)
A survey of the search theory literature.
Naval Research Logistics, **38** (1991), 469–491.

[3] Richardson, H. R. (1988)
Search theory.
Encyclopedia of Statistical Science, **8** (1988), 314–321, Kotz, Johnson and Read eds.

[4] Stone, L. D. (1989)
What's happened in search theory since the 1975 Lanchester prize?
Operations Research, **37**, 3 (May–June 1989), 501–506.

[5] Wagner, D. H., Mylander, W. C., and Sanders, T. J. (Eds.) (1999)
Naval Operations Analysis (3rd ed.).
Annapolis, MD: Naval Institute Press, 1999.

[6] Stone, L. D. (1989)
Theory of Optimal Search (2nd ed.).
Arlington, VA: Operations Research Society of America, ORSA Books, 1989.

[7] Iida, K. (1992)
Studies on the optimal search plan.
In *Lecture Notes in Statistics*, vol. 70.
New York: Springer-Verlag, 1992.

[8] Washburn, A. R. (1989)
Search and detection (2nd ed.).
Arlington, VA: Operations Research Society of America, ORSA Books, 1989.

[9] Washburn, A. R. (1994)
Two Persons Zero-Sum Games (2nd ed.).
Topics in Operations Research, INFORMS, 1994.

[10] Gal, S. (1980)
Search Games.
New York: Academic Press, 1980.

[11] Brown, S. S. (1980)
Optimal search for a moving target in discrete time and space.
Operations Research, **28**, 6 (Nov.–Dec. 1980), 1275–1289.

[12] Washburn, A. R. (1983)
Search for a moving target: The FAB algorithm.
Operations Research, **31**, 4 (July–Aug. 1983), 739–751.

[13] Dobbie, J. M. (1966)
Transfer of detection contacts to tracking contacts in surveillance.
Operations Research, **14** (1966), 791–800.

[14] Van Keuk, G. (1998)
Sequential track extraction.
IEEE Transactions on Aerospace and Electronic Systems, **34**, 4 (Oct. 1998), 1135–1148.

[15] Stromquist, W. R., and Stone, L. D. (1981)
Constrained optimization of functionals with search theory applications.
Mathematics of Operations Research, **6**, 4 (Nov. 1981), 518–529.

[16] Minoux, M. (1983)
Programmation Mathématique: Théorie et Algorithmes, tome 1.
Collection Technique et Scientifique des Télécommunications, Dunod, Paris, 1983.

[17] Stone, L. D., and Richardson, H. R. (1974)
Search for targets with conditionally deterministic motion.
SIAM Journal of Applied Mathematics, **27**, 2 (Sept. 1974), 239–255.

[18] Iida, K. (1989)
Optimal search plan minimizing the expected risk of the search for a target with conditionally deterministic motion.
Naval Research Logistics, **36** (1989), 596–613.

[19] Pursiheimo, U. (1980)
Conjugate duality in the optimization of the search for a target with generalized conditionally deterministic motion.
Journal of Optimization Theory and Applications, **31**, 3 (June 1980), 261–275.

[20] Hiriart-Urruty, J. B., and Lemaréchal, C. (1993)
Convex Analysis and Minimization Algorithms; Vol I: *Fundamentals*.
New York: Springer-Verlag, 1993.

[21] Elias-Fust, A. R., Broquetas-Ibars, A., Antequera, J. P., and Marin Yuste, J. C. (1992)
CFAR data fusion with inhomogeneous receivers.
IEEE Transactions on Aerospace and Electronic Systems, **28**, 1 (Jan. 1992), 276–285.

[22] Varshney, P. K. (1997)
Distributed Detection and Data Fusion.
New York: Springer-Verlag, 1997.

[23] Stone, L. D. (1979)
Necessary and sufficient conditions for optimal search plans for moving targets.
Mathematics of Operations Research, **4**, 4 (Nov. 1979), 431–440.

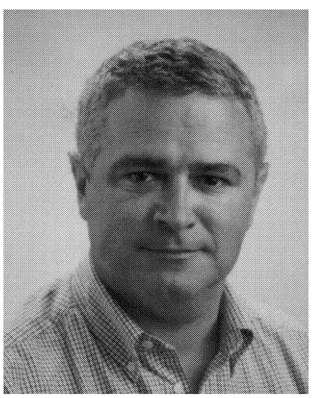
[24] Nakai, T. (1988)
Search models with continuous effort under various criteria.
Journal of the Operations Research Society of Japan, **31**, 3 (Sept. 1988), 335–351.

[25] Moskowitz, I., and Simmen, J. (1989)
Asymptotic results in search theory.
Naval Research Logistics, **36** (1989), 577–596.

[26] Iida, K., Hohzaki, R., and Furui, S. (1996)
A search game for a mobile target with the conditionally deterministic motion defined by paths.
Journal of the Operations Research Society of Japan, **39**, 4 (Dec. 1996), 501–511.

[27] Charnes, A., and Cooper, W. W. (1958)
The theory of search: Optimum distribution of search effort.
Management Science, **5** (1958), 44–50.

[28] Horn, R. A., and Johnson, C. R. (1990)
Matrix Analysis.
New York: Cambridge University Press, 1990.



J.-P. Le Cadre (M'93) received the M.S. degree in mathematics in 1977, the Doctorat de 3^{-eme} cycle in 1982 and the Doctorat d'Etat in 1987, all from INPG, Grenoble, France.

From 1980 to 1989, he worked at the GERDSM (Groupe d'Etudes et de Recherche en Detection Sous-Marines), a laboratory of the DCN (Direction des Constructions Navales), mainly on array processing. Since 1989, he has been with IRISA/CNRS, where he is a CNRS "Directeur de recherche". His interests have now moved toward topics like system analysis, detection, data association multitarget tracking and operations research.

He received (with O. Zugmeyer) the Eurasip *Signal Processing* best paper award in 1993. He is a member of various societies of IEEE.

Guillaume Souris graduated from ENSAI (Ecole Nationale de la Statistique et de l'Analyse de l'Information), in Rennes (France). After general studies in mathematics, probability , he specialized in the applications of statistics to industrial processes (e.g. reliability, quality analysis, signal processing). In IRISA, he worked about some optimization problems related to search theory.

Presently, he is an engineer-consultant within the society Ligeron S.A. (safety analysis, reliabilty, programm management).