Searching Tracks *

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Abstract

Search theory is the discipline which treats the problem of how best to find the optimal distribution of the total search effort which maximizes the probability of detection. In the "classical" search theory, the target is said detected if a detection occurs at any time of the time frame. Here, the target track will be said detected if elementary detections occur at various times. That means that there is a test for acceptation (or detection) of a target track and that the problem is to optimize the allocation of the search effort for track detection. Keywords: Search theory, optimization, duality, detection

1 Introduction

Search theory is the discipline which treats the problem of how best to search for an object when the amount of searching efforts is limited and only probabilities of the object's possible position are given. An important literature has been devoted to this subject, including surveys [1] and books [2], [3], [4]. The situation is characterized by three data: (i) the probabilities of the searched object (the "target") being in various possible locations; (ii) the local *detection probability* that a particular amount of local search effort should detect the target: (iii) the total amount of searching effort available.

However, we shall consider here a radically different problem. The problem is to detect target tracks. In the "classical" search theory, the target is said detected if a detection occurs at any time of the time frame. Here, the target track will be said detected if elementary detections occur at various times. That means that there is a test for acceptation (or detection) of a target track; associated with a spatio-temporal modelling of the target track. Moreover, we shall not consider (in general) bounds relative to the search effort at each period. The bound is relative to the global search effort.

The paper is organized as follows. In section 2, the optimization framework is presented; followed by the general formulation of the search problem (see section 3). In section 4, we deal with the 2-period search problem, for the "AND" detection rule. Then, the optimization problems are detailed and solved, while they are extended to the n-period search in section 5. Another detection rule is considered in section 6, the "MAJORITY" detection rule. For a more extensive presentation (including simulation results), we refer to [5].

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2 The optimization framework

The major part of this paper is centered around the following (primal) optimization problem :

$$\mathcal{P}\left\{\begin{array}{l} \min\ -P \quad \text{with} : P=\sum_\theta\ F(x_{1,\theta},x_{2,\theta},\cdots,x_{n,\theta})\ , \\ \text{where :} \\ F(x_{1,\theta},x_{2,\theta},\cdots,x_{n,\theta}) \triangleq f\left(p(x_{1,\theta})p(x_{2,\theta})\cdots p(x_{n,\theta})\right)\ , \\ \text{under the resource constraints :} \\ \sum_\theta\ x_{1,\theta}+x_{2,\theta}\cdots+x_{n,\theta}=\Phi\ , \\ x_{1,\theta}\geq 0\ ,x_{2,\theta}\geq 0\ ,\cdots,x_{n,\theta}\geq 0\ \ \forall(\theta)\ . \end{array}\right. \tag{2.1}$$

In 2.1, $x_{k,\theta}$ represents a research effort, affected to the cell indexed by the parameter θ, at the search period k. The index k takes its values in the subset $\{1,\cdots,n\}$. The parameter θ takes its values in a multidimensional space, characterizing the target trajectory (e.g. initial position and velocity) and the n-dimensional vector $\mathbf{X}_\theta \triangleq (x_{1,\theta},x_{2,\theta},\cdots,x_{n,\theta})^*$ represents the effort vector associated with the target trajectory (or track) indexed by θ. Furthermore, $p(x_{k,\theta})$ is the elementary probability of detection in the cell (k,θ), for a search effort $x_{k,\theta}$; while f is a given differentiable function. The following simple remarks are then fundamental :

- the functional $F(x_{1,\theta},\cdots,x_{n,\theta})$ is a differentiable functional of the variables $x_{k,\theta}$,
- the constraints are qualified since they are linear,
- the "hard constraint" is the equality constraint (i.e. $\sum_\theta x_{1,\theta}+x_{2,\theta}\cdots+x_{n,\theta}=\Phi$), the inequality constraints being *implicitely* taken into account.

These considerations lead us to consider and use basically the dual formalism. The following dual function is considered :

$$\left\{\begin{array}{l} \psi(\lambda)=\inf_{x_{1,\theta},\cdots,x_{n,\theta}}\ \mathcal{L}(\lambda)\ , \\ \text{where :} \\ \mathcal{L}(\lambda)=-P+\lambda\left(\sum_\theta\ x_{1,\theta}+x_{2,\theta}\cdots+x_{n,\theta}-\Phi\right)\ . \end{array}\right. \tag{2.2}$$

We stress that, in our framework, the function $\psi(\lambda)$ may be explicitely determined on the subset defined by the inequality constraints. The dual problem $(\mathcal{D})$ then takes the following form :

$$\mathcal{D} \; : \; \max_\lambda \; \psi(\lambda) \; . \tag{2.3}$$

The decisive benefits of this approach are :

- the maximization of $\psi(\lambda)$ is an (unconstrained) *monodimensional* [1] problem,
- the function $\psi(\lambda)$ is differentiable,
- from the solution $\underline{\lambda}$ of the dual problem, the solution $\underline{\mathbf{X}}$ of the primal problem $\mathcal{P}$ is deduced (say $\underline{\mathbf{X}}(\underline{\lambda})$). The couple $(\underline{\lambda}, \underline{\mathbf{X}})$ is a saddle point of the primal-dual problem.

3 Modelling and formulation of the problem

Assuming the target motion rectilinear and uniform, it is completely defined by its initial position vector (i) and a velocity vector (v), i.e. $\theta \equiv (i, v)$. Assumptions of our search problem are as follows :

- A target moves in a search space consisting of a finite number of search cells $C_t = \{c_{\theta,t}\}_\theta$ in discrete time $\mathbf{T} = \{1, 2, \cdots, n\}$. We further assume that the sequence of (searched) cells $\{c_{\theta,t}\}_t$ is completely defined by the parameter (θ) (conditionally deterministic motion). Thus, the mapping $c_{\theta,1} \to c_{\theta,2} \cdots \to c_{\theta,n}$ is a bijection. In the simpler case (rectilinear motion of the target), this function mapping is simply a translation of vector v.
- The search effort applied to cell $c_{\theta,t}$ is denoted $x_{t,\theta}$ $(x_{t,\theta} \geq 0)$.
- The conditional probability of detecting the target given that the target is in the cell $c_{\theta,t}$ and that the search effort applied to this cell is $x_{t,\theta}$ is $p(x_{t,\theta})$. This probability is a classical exponential law, i.e. $p(x_{t,\theta}) = 1 - \exp(-w_{t,\theta}\, x_{t,\theta})$. The term $w_{t,\theta}$ stands for the particular conditions of detection (visibility) for the cell $c_{\theta,t}$.

4 The 2-period search for the "AND" track detection rule

First, we shall deal with the two period search problem (i.e. $n = 2$). More specifically, we shall say that the target track has been detected if the target has been detected at *each* (temporal) period of the search . We then have to solve the following search problem :

$$\mathcal{P} \begin{cases} \min \; -P \quad \text{where: } P = \sum_\theta g_1(\theta)\, p(x_{1,\theta})\, p(x_{2,\theta}) \; , \\ \text{under the constraints :} \\ \sum_\theta \, (x_{1,\theta} + x_{2,\theta}) = \Phi \quad , x_{1,\theta} \geq 0 \; , x_{2,\theta} \geq 0 \; , \; \forall(\theta) \; . \end{cases} \tag{4.4}$$

In the above equation $x_{1,\theta}$ (respectively $x_{2,\theta}$) denotes the search effort applied to the cell $c_{\theta,1}$ (respectively $c_{\theta,2}$). Then, we form the Lagrangian of the primal problem 4.4, i.e. :

$$\begin{aligned} \mathcal{L}(\lambda) \;\; = \;\; & -\sum_\theta g_1(\theta)\left(1 - e^{-w x_{1,\theta}}\right)\left(1 - e^{-w x_{2,\theta}}\right) \; , \\ & + \lambda \left(\sum_\theta x_{1,\theta} + \sum_\theta x_{2,\theta} - \Phi\right) \; , \\ & - \sum_\theta \mu_{1,\theta} x_{1,\theta} - \sum_\theta \mu_{2,\theta} x_{2,\theta} \; , \\ & \mu_{1,\theta} \geq 0, \quad \mu_{2,\theta} \geq 0 \; . \end{aligned}$$

In order to apply the Karush-Kuhn-Tucker conditions of optimality (KKT for the sequel), we must consider two cases.

4.1 KKT optimality conditions and their consequences

case 1 $(x_{1,\theta} > 0)$

In this case, the KKT condition $\{\mu_{1,\theta}\, x_{1,\theta} = 0\}$ implies $\{\mu_{1,\theta} = 0\}$. Then, the KKT stationarity condition (for the Lagrangian) simply results in :

$$\frac{\partial}{\partial x_{1,\theta}} \mathcal{L}(\lambda) = -w\, g_1(\theta)\, e^{-w\, x_{1,\theta}}\left(1 - e^{-w\, x_{2,\theta}}\right) + \lambda = 0 \; . \tag{4.5}$$

From 4.5, we note that the assumption $x_{1,\theta} > 0$ implies $x_{2,\theta} > 0$, otherwise the multiplier λ should be zero. Indeed, if $\lambda = 0$ then it is easily seen (see 4.5) that the value of the dual function $\psi(\lambda) = \inf_{(x_{1,\theta}, x_{2,\theta})} \mathcal{L}(\lambda)$ is $-\infty$. Since, we have to maximize $\psi(\lambda)$, we see that λ is necessarily *strictly* positive (see 4.5 for the sign). Thus, 4.5 implies the validity of the following equation :

$$\frac{\partial}{\partial x_{2,\theta}} \mathcal{L}(\lambda) = -w\, g_1(\theta)\, e^{-w x_{2,\theta}}\left(1 - e^{-w\, x_{1,\theta}}\right) + \lambda = 0 \; . \tag{4.6}$$

By collecting 4.5 and 4.6, and denoting $X_{1,\theta} = e^{-w\, x_{1,\theta}}$,$X_{2\theta} = e^{-w\, x_{2,\theta}}$, we obtain :

$$\begin{aligned} & X_{1,\theta}\left(1 - X_{2,\theta}\right) = X_{2,\theta}\left(1 - X_{1,\theta}\right) \; , \\ & \text{so, that :} \\ & X_{1,\theta} = X_{2,\theta} \quad \text{i.e. } x_{1,\theta} = x_{2,\theta} \; . \end{aligned} \tag{4.7}$$

[1] In the case of a unique "hard" resource constraint

The above equality is fundamental for solving the problem.

case 2 $(x_{1,\theta}=0)$

Assume now that $x_{2,\theta}>0$, then the KKT condition (relative to $x_{2,\theta}$) should imply (see 4.6, with $x_{1,\theta}=0$) :

$$\frac{\partial}{\partial x_{2,\theta}}\,\mathcal{L}(\lambda)=\lambda=0\,, \tag{4.8}$$

and, in turn, that the multiplier λ should be zero. Under this assumption, the value of $\psi(\lambda)$ is $-\infty$. Hence, we can restrict to the *strictly* positive values of λ, which means that the assumption $x_{1,\theta}=0$ implies $x_{2,\theta}=0$. Indeed, the hypothesis $x_{2,\theta}>0$ should imply the validity of 4.6 and, in turn, the multiplier λ should be zero since we assume the nullity of $x_{1,\theta}$, which contradicts the fact that λ is strictly positive.

4.2 Solving the dual problem

In conclusion, the following result has been stated : $x_{1,\theta}=x_{2,\theta}$. So that, we have now to deal with the following (simplified) optimization problem :

$$\mathcal{P}\begin{cases}\min\ -P \quad \text{where} : P=\sum_\theta g_1(\theta)\,(p(x_{1,\theta}))^2\,,\\ \text{under the constraints :}\\ \sum_\theta x_{1,\theta}=\Phi/2 \quad ,x_{1,\theta}\geq 0\,,\ \forall(\theta)\,.\end{cases} \tag{4.9}$$

Again, we examine the necessary conditions induced by the KKT theorem. Now, we consider the reduced Lagrangian functional $\mathcal{L}(\lambda)$ given by :

$$\mathcal{L}(\lambda)=-\sum_\theta g_1(\theta)\,(1-e^{-wx_{1,\theta}})^2+\lambda\,\left(2\sum_\theta x_{1,\theta}-\Phi\right)\,. \tag{4.10}$$

The positivity constraints relative to the search variables $\{x_{1,\theta}\}$ are *implicitely* taken into account by restricting our search to positive values of the variables $x_{1,\theta}$. Under the assumption that $x_{1,\theta}$ is *strictly* positive and differentiating $\mathcal{L}(\lambda)$ relatively to $x_{1,\theta}$, we then obtain :

$$\begin{array}{l}\frac{\partial\mathcal{L}(\lambda)}{\partial x_{1,\theta}}=-2\,w\,g_1(\theta)\,e^{-w\,x_{1,\theta}}\,(1-e^{-w\,x_{1,\theta}})+2\,\lambda=0\,,\\ \text{or, equivalently :}\\ X_{1,\theta}\,(1-X_{1,\theta})=\frac{\lambda}{w\,g_1(\theta)}\,.\end{array} \tag{4.11}$$

Equation 4.11 is a second order equation (in $X_{1,\theta}$), allowing us to determine $\underline{x}_{1,\theta}$, for a given value of λ. Note that we restrict to the roots (0 or 2) of 4.11 lying inside the interval $[0,1]$, and select the root (denoted $\underline{X}_{1,\theta}(\lambda)$) which minimizes the reduced Lagrangian functional $\mathcal{L}(\lambda)$ [2]).

[2]Note that we must test and compare the value of $\mathcal{L}(\lambda)$ not only for the roots of 4.11, but also for its lower bound (i.e. $X_{1,\theta}=1\Leftrightarrow x_{1,\theta}=0$

We have now to deal with the maximization of the dual functional defined by :

$$\begin{array}{ll}\psi(\lambda) & =-\sum_{(\theta)_+} g_1(\theta)\,\left(1-\underline{X}_{1,\theta}(\lambda)\right)^2\,,\\ & +\lambda\,\left(2\,\sum_{(\theta)_+}\,\underline{x}_{1,\theta}(\lambda)-\Phi\right)\,,\\ \underline{x}_{1,\theta}(\lambda) & =-\frac{1}{w}\,\ln\left(\underline{X}_{1,\theta}(\lambda)\right)\ \text{if} : \underline{x}_{1,\theta}(\lambda)>0\,,\end{array} \tag{4.12}$$

where the symbol $(\theta)_+$ denotes the values of the index for which 4.11 has a root inside $[0,1]$. The maximization of $\psi(\lambda)$ is rather easy since it corresponds to an unidimensional search for a concave and differentiable function. In turn, the is no duality gap.

Notation 1 *The (spatio-temporal) index* (θ,t) *for which the research efforts are strictly positive are denoted* $(\theta,t)_+$ *(t : index of the search period);* $(\theta)_+$ *for the first search period.*

5 The n-period search for the "AND" track detection rule

Quite similarly to the 2-period search, we assume that the probability of detection of the track is the product of elementary detection probability of detection (i.e. at each period) and is thus given by [3] :

$$\begin{cases}P=\sum_\theta\ g_1(\theta)\,p(x_{1,\theta})\,p(x_{2,\theta})\cdots p(x_{n,\theta})\,,\\ p(x_{k,\theta})=\gamma_k\ (1-e^{-w_{k,\theta}\,x_{k,\theta}}) \quad k=1,\cdots,n\,,\end{cases} \tag{5.13}$$

and the optimization problem is again :

$$\mathcal{P}\begin{cases}\min\ -P\,,\\ \text{under the constraints :}\\ \sum_\theta\ [x_{1,\theta}+\cdots+x_{n,\theta}]=\Phi\,,\\ x_{1,\theta}\geq 0\,,\cdots,x_{n,\theta}\geq 0\,,\ \forall(\theta)\,.\end{cases} \tag{5.14}$$

Assume $x_{1,\theta}\neq 0$, then by a reasoning strictly identical to the 2-period case, we deduce that $x_{2,\theta}\neq 0,\cdots,x_{n,\theta}\neq 0$. The optimality equations deduced from the KKT conditions then yield the following (non-linear) system of n equations :

$$\begin{cases}\gamma_1\ X_{1,\theta}\,(1-\gamma_2\ X_{2,\theta})\cdots(1-\gamma_n\ X_{n,\theta})=\frac{\lambda}{w_{1,\theta}\ g_1(\theta)}=\alpha_1\,,\\ \gamma_2\ X_{2,\theta}\,(1-\gamma_1\ X_{1,\theta})\cdots(1-\gamma_n\ X_{n,\theta})=\frac{\lambda}{w_{2,\theta}\ g_1(\theta)}=\alpha_2\,,\\ \vdots\\ \gamma_n\ X_{n,\theta}\,(1-\gamma_1\ X_{1,\theta})\cdots(1-\gamma_{n-1}\ X_{n-1,\theta})=\frac{\lambda}{w_{n,\theta}\ g_1(\theta)}=c\end{cases} \tag{5.15}$$

[3]The scalar $w_{k,\theta}$ stands for the possibly changing visibility conditions from one period to another one

Consider now the above system, dividing row (1) by row (p) and denoting $Y_{1,\theta} \triangleq \gamma_1 \; X_{1,\theta} \;, \cdots, Y_{p,\theta} \triangleq \gamma_p \; X_{p,\theta}$, we obtain :

$$\begin{aligned} &\frac{Y_{1,\theta}\;(1-Y_{p,\theta})}{Y_{p,\theta}\;(1-Y_{1,\theta})} = \frac{\alpha_1}{\alpha_p} \;, \\ &\text{i.e. : } Y_{p,\theta} = \frac{\alpha_p Y_{1,\theta}}{Y_{1,\theta}(\alpha_p-\alpha_1)+\alpha_1} \;. \end{aligned} \tag{5.16}$$

Consequently, $x_{p,\theta}$ is deduced from $x_{1,\theta}$, itself given by:

$$x_{1,\theta} = \frac{1}{w_{1,\theta}} \left[ln \left(\frac{\gamma_1}{Y_{1,\theta}} \right) \right]^+ .$$

The problem is thus reduced to the determination of $\underline{x}_{1,\theta}$. From 5.16 we have $1 - X_{p,\theta} = [\alpha_1 \,(1 - X_{1,\theta})] \,/\, (X_{1,\theta}(\alpha_p - \alpha_1) + \alpha_1)$. Inserting this equality in 5.15, we see that $\underline{X}_{1,\theta}$ is a root of the following n-th order polynomial equation :

$$\alpha_1{}^{n-2}\; X_{1,\theta}\; (1 - X_{1,\theta})^{n-1} - \prod_{i=2}^{n} (X_{1,\theta}\; (\alpha_i - \alpha_1) + \alpha_1) = 0 \,. \tag{5.17}$$

The value of $\underline{X}_{1,\theta}(\lambda)$ is the root of 5.17 which minimizes the Lagrangian, deduced from 5.13; where $\underline{x}_{2,\theta}, \cdots, \underline{x}_{n,\theta}$ are determined (from $\underline{x}_{1,\theta}$) by 5.16. The computation load is relatively modest. From $\underline{x}_{1,\theta}$, the dual function $\psi(\lambda)$ is deduced, i.e. :

$$\psi(\lambda) = - \sum_{(\theta)_+} \prod_{k_+} \gamma_k \; (1 - X_{k,\theta}) + \lambda \left(\sum_{(\theta,k)_+} x_{k,\theta} - \Phi \right) . \tag{5.18}$$

The problem is simply to determine the value of λ which maximizes the concave function $\psi(\lambda)$.

So far, the problem has been considered in its full generality. To illustrate the previous calculations, assume now that the visibility coefficients $\{w_{1,\theta}, \cdots, w_{n,\theta}\}$ are equal altogether, i.e. $p(x_{k,\theta}) = \gamma \; (1 - e^{-w \, x_{k,\theta}}) \quad k = 1, \cdots, n$ then the optimality equations 5.15 and 5.16 simply reduce to $Y_{1,\theta} = \cdots = Y_{n,\theta}$, so that $X_{1,\theta} = \cdots = X_{n,\theta}$ and the probability of track detection as well as the dual function $\psi(\lambda)$ become :

$$\left\{ \begin{array}{ll} P & = \sum_\theta \; g_1(\theta) \; [\gamma \; (1 - e^{-w \, x_{k,\theta}})]^n \,, \\ \psi(\lambda) & = -\sum_{(\theta)_+} \; g_1(\theta) \left[\gamma \; \left(1 - \underline{X}_{1,\theta}(\lambda)\right) \right]^n \\ & \quad + \lambda \left(n \, \sum_{(\theta)_+} \; \underline{x}_{1,\theta}(\lambda) - \Phi \right) . \end{array} \right. \tag{5.19}$$

Again, we have to deal now with a simple monodimensional optimization problem, involving a concave functional.

Let us denote $\Phi(\lambda)$ the optimal value of the (total) search effort for a given λ; then the following result holds :

Proposition 1 *$\Phi(\lambda)$ is a decreasing function of λ .*

Proof : Denoting θ, the track parameter, the Lagrangian $\mathcal{L}(\lambda)$ of the constrained problem is $\mathcal{L}(\lambda, \theta) = -P + \lambda (\sum_{i=1}^{n} x_{i,\theta} - \Phi)$ $(P = \sum_\theta \; g_1(\theta) \, p(x_{1,\theta}) \cdots p(x_{n,\theta})$); so, that : $\frac{\partial \mathcal{L}(\lambda)}{\partial x_{i,\theta}} = -\frac{\partial P}{\partial x_{i,\theta}} + \lambda$, and consequently :

$$\lambda_2 > \lambda_1 \Rightarrow \frac{\partial \mathcal{L}(\lambda_2)}{\partial x_{i,\theta}} \geq \frac{\partial \mathcal{L}(\lambda_2)}{\partial x_{i,\theta}} \,, \tag{5.20}$$

hence $\underline{x}_{i,\theta}(\lambda_1) \geq \underline{x}_{i,\theta}(\lambda_2)$ $(\forall i, \; \theta)$; and in turn $\Phi(\lambda_2) \leq \Phi(\lambda_1)$.

6 The "MAJORITY" rule for track detection :

Up to now, our analysis has been restricted to an "AND" rule for track detection. For numerous applications, a MAJORITY rule is also quite realistic. This means that a track is said detected if a "sufficient" number of elementary detections occur "along" the track. We have now to face specific problems. First, it is difficult to give a general formulation (for the general n-period search) of the detection rule. Second, the optimization problems become far more intricated.

6.1 The 3-period case and the "MAJORITY" track detection rule

The detection function is modified in order to take into account a majority rule ("MAJORITY") for decision. More precisely, the track is said to be detected if the target is detected *at least* at 2 periods. With this rule, the probability of detection becomes :

$$\begin{aligned} P \;\; = & \;\; \sum_\theta \; g_1(\theta) \; [\beta_{0,2,3} \; P_{0,2,3} \,, \\ & + \;\; \beta_{1,2,0} \; P_{1,2,0} + \beta_{1,0,3} \; P_{1,0,3} + \beta_{1,2,3} \; P_{1,2,3}] \,. \end{aligned} \tag{6.21}$$

In 6.21, the notation $P_{0,2,3}$ corresponds to the following hypothesis: no detection at period 1, detection at periods 2 and 3, idem for $P_{1,2,0}$ and $P_{1,0,3}$. The notation $P_{1,2,3}$ corresponds to a detection at each period. Finally, the weights $\beta_{0,2,3}, \cdots, \beta_{1,2,3}$ are related to the information "gain" associated with an elementary event. Thus, the elementary detection terms $P_{0,2,3}, \cdots, P_{1,2,3}$ have the following form :

$$\begin{cases} P_{0,2,3} = & e^{-w\, x_{1,\theta}}\ (1-e^{-w\, x_{2,\theta}})\ (1-e^{-w\, x_{3,\theta}})\ , \\ P_{1,2,0} = & e^{-w\, x_{3,\theta}}\ (1-e^{-w\, x_{1,\theta}})\ (1-e^{-w\, x_{2,\theta}})\ , \\ P_{1,0,3} = & e^{-w\, x_{2,\theta}}\ (1-e^{-w\, x_{1,\theta}})\ (1-e^{-w\, x_{3,\theta}})\ , \\ P_{1,2,3} = & (1-e^{-w\, x_{1,\theta}})\ (1-e^{-w\, x_{2,\theta}})\ (1-e^{-w\, x_{3,\theta}})\ . \end{cases} \tag{6.22}$$

Defining the reduced Lagrangian as $\mathcal{L}(\lambda) = -P + \lambda\ (\sum_\theta\ (x_{1,\theta} + x_{2,\theta} + x_{3,\theta}) - \Phi)$, we adopt the following notations for the sake of simplicity [4] :

$$\begin{cases} \beta_{0,2,3} \equiv \delta_1,\ \beta_{1,2,0} \equiv \delta_3,\ \beta_{1,0,3} \equiv \delta_2,\ \beta_{1,2,3} \equiv \delta^*\ , \\ X_{1,\theta} = e^{-w\, x_{1,\theta}} \equiv y_1, \cdots, X_{3,\theta} = e^{-w\, x_{3,\theta}} \equiv y_3\ . \end{cases} \tag{6.23}$$

Assuming that y_1, y_2, y_3 differ altogether from 1, the KKT conditions yield :

$$y_3 = \left[\frac{y_1\,(\delta^* - \delta_1 - \delta_2) + \delta_2 - \delta^*}{y_1\,(\delta^* - \delta_1 - \delta_3) + \delta_3 - \delta^*}\right]\ y_2\ . \tag{6.24}$$

Then, inserting $y_3 = f(y_1)\ y_2$ (see 6.24) in 6.22, we obtain the following 2-th order equation :

$$(a - b\, y_1)\ y_2^2 + (c - d\, y_1^2)\ y_2 + (e\, y_1^2 + f\, y_1) = 0\ ,$$

where the coefficients (a, b, c, d) are easily calculated. In this case $(x_{k,\theta} \neq 0\ ;\ k = 1, 2, 3)$, the distribution of the search efforts is now completely determined by the optimality equations.

Also from the optimality equations, we see that the nullity of the search effort at two periods (i.e. $y_k = y_{k'} = 1$ for $k \neq k'$) results in the nullity of the total search effort (i.e. $y_1 = y_2 = y_3 = 1$). So, we must consider the cases where the search effort is null at one period. In this case, only two optimality equations are valid. Consider for instance (other cases are completely similar), the case $x_{2,\theta} = 0$, then we obtain $\delta_2\ y_1\,(1 - y_3) = \frac{\lambda}{g_1}$.

6.2 The n-period search and the "MAJORITY" track detection rule

We shall now restrict to the following track detection rule. The track is said detected if, at least, $(n-1)$ elementary detections occur (for a n-period search). Thus, the probabilities of the following events are considered :

$$\begin{cases} P_1 & \equiv P_{0,2,\cdots,n} & = y_1\ \prod_{i=2}^{n} (1 - y_i)\ , \\ P_2 & \equiv P_{1,0,2,\cdots,n} & = y_1\ \prod_{i=1,\neq 2}^{n} (1 - y_i)\ , \\ & \vdots & \\ P_n & \equiv P_{1,2,\cdots,n-1,0} & = y_n\ \prod_{i=1}^{n-1} (1 - y_i)\ , \\ P_* & \equiv P_{1,2,\cdots,n} & = \prod_{i=1}^{n-1} (1 - y_i)\ . \end{cases} \tag{6.25}$$

For the sake of simplicity, the following assumptions are made: the (detection) coefficients (i.e. $\beta_{0,2,\cdots,n}, \beta_{1,0,2,\cdots,n}, \cdots, \beta_{1,2,\cdots,n}$, see 6.23) are equal [5]. Let us first assume that the search efforts are non-zero for all the periods (i.e. : $x_1 \neq 0, \cdots, x_n \neq 0$), then the KKT conditions result in :

$$(y_2 - y_3)\ \prod_{i \neq 2,3}^{n}\ \left[\frac{y_1}{(1-y_1)} + \frac{y_4}{(1-y_4)} + \cdots + \frac{y_n}{(1-y_n)}\right] = 0\ . \tag{6.26}$$

Since the term between brackets is well defined and non-zero, we deduce from 6.26 that $y_2 = y_3$, and more generally considering the difference equations obtained by substracting row $(i+1)$ to row i in the optimality equations , we have $y_1 = y_2 = \cdots = y_n$. Moreover, we can prove that the search efforts (for a given track parameter $\{\theta\}$) is either zero for all the periods or zero for *at most* one period. The rest of the derivation is identical to the 3-period case.

7 Conclusion

The problem under consideration was the optimization of the search effort for detecting tracks. In order to develop feasible methods, we focused on discrete (both in time and space) optimization . Under simple constraints (relative to the distribution of the search effort), the dual formalism appears as a feasible and versatile approach.

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[4] The index of missed detection is the index of δ

[5] As seen previously (see section 6.1), this assumption does not reduce the generality of our approach.