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Brief Paper

Optimizing the receiver maneuvers for bearings-only tracking[1]

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Abstract

This paper deals with the optimization of the observer trajectory for target motion analysis. The observations are made of estimated bearings. The problem consists in determining the sequence of controls (e.g. the observer headings) which maximizes a cost functional. This cost functional is generally a functional of the FIM matrix associated with the estimation of the source trajectory parameters. Further, note that these parameters are only *partially* observed. The determinant of the Fisher information matrix (FIM) has all the desirable properties, the monotonicity property excepted. This is a fundamental difference with "classical" optimal control. The analysis is thus greatly complicated. So, a large part of this paper is centered around a direct analysis of the FIM determinants. Using them, it is shown that, under the long-range and bounded controls hypotheses, the sequence of controls lies in the general class of bang-bang ones. These results demonstrate the interest of maneuver diversity. More generally, they provide a general framework for devising a strategy for optimizing the observer trajectory, only based on the observations (i.e. the bearing rates).

Keywords: Target tracking; Optimal control; Optimization; Bang-bang control

1. Introduction

Passive bearings-only tracking (BOT for the sequel) techniques are used in a variety of applications (Nardone et al., 1984; Aidala and Hammel, 1983) like sonar, infrared tracking (Barniv, 1985), optronic or electronic warfare. In the passive sonar context, the basic problem of target motion analysis (TMA) is to estimate the *trajectory* of a moving object (e.g. position and velocity for a rectilinear motion) from noise-corrupted sensor data. Classical bearings-only TMA methods are restricted to constant motion parameters (usually position and velocity) (Nardone et al., 1984) even if extensions to maneuvering sources constitute an important area of present research (Chang and Tabaczinski, 1984; Blom and Bar-Shalom, 1988). In the case of a maneuvering source, a leg-by-leg[2] hypothesis for the source trajectory is quite common in the sonar context.

In the BOT context, the source trajectory is only partially observed through noisy non-linear measurements (estimated bearings). Here, a moving observer passively monitors noisy bearings to estimate a source trajectory. A great deal of work has yet been devoted to the analysis of the observability (Nardone and Aidala, 1981; Le Cadre and Jauffret, 1997). However, this is a binary (yes/no) analysis and a practical fundamental question remains: if the system is observable what is the accuracy of the source trajectory estimate? The TMA methods are now well understood and their performance has been carefully investigated, at least for simple scenarios (Nardone et al., 1984; Trémois and Le Cadre, 1996). It is quite evident that the performance of any TMA

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[2] A leg is a portion of the trajectory where the velocity vector is constant

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method is highly depending on the observer maneuvers Mc Cabe, 1985; Fawcet, 1988). In the past, the importance of this problem has been recognized even if efforts were, for a large extent, devoted to the considerably simpler case of bearings-only localization (i.e. the source is *fixed*). For instance, Liu (1988) and then Hammel et al. (1989) have considered a lower bound of the determinant of the two-dimensional Fisher information matrix (FIM), allowing to approximate the problem within the general framework of optimal control (see Section 3).

The general problem consists in optimizing controls, which are kinematic parameters of the observer, in order to optimize the statistical behavior of the TMA. A classical approach consists then in considering the (FIM) and more precisely its determinant. The choice of the determinant functional is reasonable (Helferty and Mudgett, 1993; Jauffret and Musso, 1991). This is a common cost functional in the estimation literature. The relations between the criteria derived from various functionals have been thoroughly studied in the optimal design literature. Furthermore, we shall show that, under hypotheses reasonable in the BOT context, the maximum of det(FIM) is attained when the sphericity criterion is maximum. More generally, note that this problem is, of course, not specific of TMA and arises in a great variety of applications, mixing estimation and control. A good example is the optimisation of tracking performance (Kershaw and Evans, 1994). The absence of any a priori knowledge about the source trajectory is also a great source of difficulty. This is solved by considering the problem in its natural framework, i.e. that of modified polar coordinates, thus separating the observable and unobservable parameters as well as their respective effects.

Another reasonable assumption consists in modelling the observer trajectory by a sequence of legs. A leg is a part of the trajectory where the motion is rectilinear and uniform (constant velocity vector) (Nardone et al., 1984). Then, the problem consists in determining the sequence of observer headings which optimize a cost functional (Holtsberg, 1992). For instance, Fawcett (1988) has considered a two-leg observer trajectory and has determined the heading of the second leg which maximizes the accuracy of the source range estimation. Jauffret and Musso (1991) and Hammel et al. (1989) have extended this work to an arbitrary number of observer legs (up to 20 for Hammel, 1988). Even if this approach is mainly computational, interesting and thorough insights have been obtained by this way. Another view of the problem has been provided by Olsder (1984) and Passerieux and Van Cappel (1991, 1998). In their approaches, the FIM components are included in the source state. The corresponding methods lie in the class of optimum design approaches, thus assuming that the source trajectory is known. In Olsder's work (1984), the idea is that, given a certain maneuver, a better maneuver is found and this procedure is repeated up to convergence, possibly to a local extremum. However, a system of 18 nonlinear coupled differential equations is needed which leads to a certain numerical burden. The approach of Passerieux and Van Cappel (1991, 1998) is also quite brilliant but seems less demanding. The aim of such approaches is mainly to provide a catalogue of recommended maneuvers (see Section 4).

In fact, the optimal control approaches are very attractive. Unfortunately, as we shall see later, the det functional does not have the monotonicity property (see Sections 2 and 4) so it is not at all true that adding a control optimal for the time t to a control sequence optimal up to time $t+1$ will yield an optimal control sequence up to time t. This explains, for a large part, the complexity of the analysis. Practically, this means that our problem cannot be treated (*in its full generality*) with the methods of optimal control. So, a large part of the paper is centered a direct analysis of the FIM determinants which will play *the* central role. More precisely, we shall show that using elementary multilinear algebra, accurate approximations of det(FIM) may be obtained. More specifically, we shall prove that det(FIM) may be approximated by a functional involving only the successive source bearing rates, thus yielding the general form of the optimal controls (observer maneuvers). In particular, it will be shown that, under the long-range and bounded controls hypotheses, the sequence of optimal controls is a bang-bang one. These results demonstrate the interest of maneuver diversity. More generally, they provide a general framework for optimizing the observer trajectory and constitute the major result of this paper.

First, a general presentation of the problem will be given in Section 2. Connections with optimal design approaches and optimal control are also included. The applications of optimal control theory are considered in Section 3. Section 4 constitutes the core of the paper. It relies upon a direct analysis of the FIM determinant. First, a constant source bearing rate is considered. Then, using the same formalism, these results are extended to the case of time-varying (piecewise constant) source bearing rates. The optimisation problems are presented in detail. This is followed by a geometric interpretation and general conclusions.

Standard notations will be used throughout this paper:

- a bold letter denotes a vector while a capital standard letter denotes a matrix,
- a capital calligraphic letter generally denotes a subspace or a subset of vectors,
- the symbol $(*)$ means transposition,
- r_x and r_y represent the relative x and y coordinates, v_x and v_y denote the relative x and y velocities,
- t or k is the time variable, r is the distance (range), θ is a bearing and u a heading,

- the symbol $\stackrel{i}{=}$ means approximation at the ith order,
- det represents the determinant, tr stands for the trace, diag denotes a diagonal matrix.
- Id_n is the n-dimensional identity matrix, the symbol cov stands for the covariance matrix,
- BOT: bearings-only tracking; RUN: rectilinear uniform motion; TMA: target motion analysis; FIM: Fisher information matrix; PMP: Pontryagin maximum principle.

2. Problem formulation

The general notations are identical to those of the reference paper (Nardone et al., 1984). The physical parameters are depicted in Fig. 1. The source, located at the coordinates (r_{xs}, r_{ys}) moves with a constant velocity vector $\mathbf{v}$ (v_{xs}, v_{ys}) and is thus defined to have the state vector:

$$\mathbf{X}_s \triangleq [r_{xs}, r_{ys}, v_{xs}, v_{ys}]^*. \tag{2.1}$$

The observer state is similarly defined as

$$\mathbf{X}_{rec} \triangleq [r_{x\,rec}, r_{y\,rec}, v_{x\,rec}, v_{y\,rec}]^*,$$

so that, in terms of the relative state vector $\mathbf{X}$ defined by

$$\mathbf{X} = \mathbf{X}_s - \mathbf{X}_{rec} \triangleq [r_x, r_y, v_x, v_y]^*,$$

the discrete-time equation of the system (i.e. the equation of the relative motion) takes the following form ($\alpha \triangleq t_{k+1} - t_k = \text{cst}$):

$$\mathbf{X}_{k+1} = F\mathbf{X}_k + \mathbf{u}_k, \tag{2.2}$$

where

$$F = \Phi(k, k+1) = \begin{pmatrix} Id_2 & \alpha Id_2 \\ 0 & Id_2 \end{pmatrix}, \qquad Id_2 \triangleq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

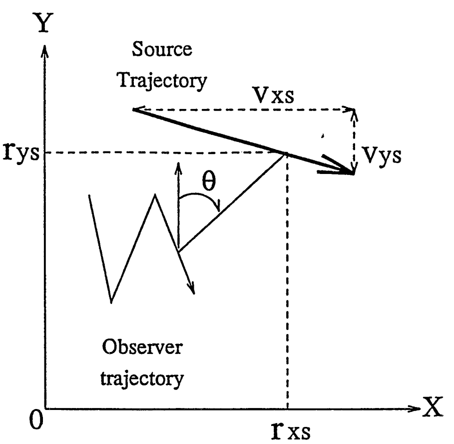

Fig. 1. Typical TMA scenario.

In the above formula t_k is the time at the kth sample, while the vector $\mathbf{u}_k = (0, 0, u_x(k), u_y(k))^*$ accounts for the effects of observer accelerations (or controls). We denote $\mathcal{U}_n \triangleq (\mathbf{u}_1, \dots, \mathbf{u}_n)$, the sequence of observer controls up to time n. Eq. (2.2) assumes that between t_k and t_{k+1} the source motion is rectilinear and uniform, this hypothesis will be taken along the whole paper. All along the paper, the vector $\mathbf{X}$ denotes the *relative* state vector at a *given* instant. An usual choice for the reference time is 0, in this case $\mathbf{X}$ is $\mathbf{X}_0$. For the sake of brevity, the reference time will be generally omitted. Indeed, it is a fundamental result that our analysis is independent of the reference time (cf Proposition 2).

As usual in passive TMA (Nardone et al., 1984), the available measurements are the estimated angles $\hat{\theta}_k$ (bearings) from the observer to the source, so that the observation equation stands as follows (w_k is the measurement noise):

$$\hat{\theta}_k = \theta_k + w_k, \tag{2.3}$$

with

$$\theta_k = \tan^{-1}\left(\frac{r_x(k)}{r_y(k)}\right).$$

The measurement noise w_k is usually modelled by an i.i.d. zero-mean, gaussian process with a given variance σ^2. The four-dimensional state equation (2.2) and the non-linear measurement equation (2.3) define the bearings-only motion analysis process (TMA). Given the history of measured bearings $\hat{\Theta} \triangleq \{\hat{\theta}_i\}_{i=1}^n$ the likelihood function is (Nardone et al., 1984)

$$P(\hat{\Theta}|\mathbf{X}) = \text{cst} \exp\left[-\tfrac{1}{2}\|\hat{\Theta} - \Theta(\mathbf{X})\|_\Sigma^2\right],$$

$$\begin{aligned} &\Theta(\mathbf{X}) \text{ defined by Eqs. (2.2) and (2.3)}, \\ &\|\hat{\Theta} - \Theta(\mathbf{X})\|_\Sigma^2 \triangleq (\hat{\Theta} - \Theta(\mathbf{X}))^* \Sigma^{-1} (\hat{\Theta} - \Theta(\mathbf{X})), \\ &\Sigma = \text{diag}(\sigma^2). \end{aligned} \tag{2.4}$$

The maximum likelihood estimate (MLE) $\hat{\mathbf{X}}$ is the solution to the likelihood equation

$$\hat{\mathbf{X}} = \arg\max_{\mathbf{X}} \log[P(\hat{\Theta}|\mathbf{X})]. \tag{2.5}$$

The above equation does not have explicit solution. So, the following Gauss–Newton algorithm is usually considered (i is the iteration index):

$$\hat{\mathbf{X}}_{i+1} = \hat{\mathbf{X}}_i - \rho_i \left[\left(\frac{\partial\Theta(\mathbf{X})}{\partial\mathbf{X}}\right)^* \Sigma^{-1} \frac{\partial\Theta(\mathbf{X})}{\partial\mathbf{X}}\right]^{-1}_{\mathbf{X}=\hat{\mathbf{X}}_i} \times \left(\frac{\partial\Theta(\mathbf{X})}{\partial\mathbf{X}}\right)^* \Sigma^{-1} (\hat{\Theta} - \Theta(\hat{\mathbf{X}}_i)) \tag{2.6}$$

with ρ_i being the step size of the algorithm.

The calculation of the gradient vector of the components of $\Theta(\mathbf{X})$ is easily derived from Eqs. (2.2) and

(2.3), yielding

$$\tan(\theta_k) = \frac{r_x(k)}{r_y(k)} = \frac{r_x(0) + kv_x + \sum_l u_{l,x}}{r_y(0) + kv_y + \sum_l u_{l,y}},$$

and consequently

$$\frac{\partial}{\partial r_x(0)} \tan(\theta_k) = \frac{1}{\cos^2(\theta_k)} \frac{\partial \theta_k}{\partial r_x(0)} = \frac{1}{r_y(k)},$$

so that

$$\frac{\partial \theta_k}{\partial r_x(0)} \sim \frac{\cos^2 \theta_k}{r_y(k)} = \frac{r_y(k)}{r_k} \frac{\cos \theta_k}{r_y(k)} = \frac{\cos \theta_k}{r_k}. \quad (2.7)$$

The other components of the partial derivatives are obtained by the same way, giving

$$\frac{\partial \theta_k}{\partial r_y(0)} = -\frac{\sin \theta_k}{r_k},$$
$$\frac{\partial \theta_k}{\partial v_x} = \frac{k}{r_y(k)} \cos^2 \theta_k = \frac{k}{r_k} \cos \theta_k, \quad (2.8)$$
$$\frac{\partial \theta_k}{\partial v_y} = -\frac{k}{r_k} \sin \theta_k.$$

Collecting the previous results, the partial derivative matrix of the bearing vector $\Theta(\mathbf{X})$ with respect to the state components is deduced (Nardone et al., 1984), yielding

$$\frac{\partial \Theta(\mathbf{X})}{\partial \mathbf{X}} = \begin{pmatrix} \frac{\cos \theta_1}{r_1} & -\frac{\sin \theta_1}{r_1} & \frac{\cos \theta_1}{r_1} & -\frac{\sin \theta_1}{r_1} \\ \vdots & & & \\ \frac{\cos \theta_n}{r_n} & -\frac{\sin \theta_n}{r_n} & \frac{n \cos \theta_1}{r_n} & -\frac{n \sin \theta_n}{r_n} \end{pmatrix}, \quad (2.9)$$

where $\{\theta_i\}_{i=1}^n$ represent the source bearing at the instant i and $\{r_i\}$ the source-observer distance. In Eq. (2.9) the reference time is the instant 0 (see Proposition 2). Consider now the case of a non-maneuvering source (constant velocity vector), then the calculation of the FIM is a routine exercise yielding, under the Gaussian assumption (Nardone et al., 1984)

$$\text{FIM}(\mathbf{X}, \mathscr{U}) = \left(\frac{\partial \Theta(\mathbf{X})}{\partial \mathbf{X}}\right)^* \Sigma^{-1} \left(\frac{\partial \Theta(\mathbf{X})}{\partial \mathbf{X}}\right). \quad (2.10)$$

We shall use this formulation of the FIM along the whole paper. Actually, the performance of any TMA algorithm is dramatically related to the observer maneuvers. So, optimizing these maneuvers represents the main problem in TMA. It is therefore not surprising that a great deal of work has been devoted to this subject. The problem then consists in determining a sequence of controls maximizing a cost function. Since we deal with the estimation of the source trajectory (the vector $\mathbf{X}$, in fact) it seems reasonable to consider that the cost function is a functional of the FIM, leading to the following problem:

Denoting $\mathbf{G}_k$[3] *the gradient vector of the log-likelihood functional* (2.4), *i.e.* .

$$\mathbf{G}_k = \frac{\mathbf{1}}{\sigma_k r_k} (\cos(\theta_k), -\sin(\theta_k), k\cos(\theta_k), -k\sin(\theta_k))^*,$$

the problem is to determine the sequence of controls $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ *(denoted* $\mathscr{U}_n$*) such that*

$$\mathscr{U}_n \to \arg\max \left[\det\left(\sum_{k=1}^{n} \mathbf{G}_k \mathbf{G}_k^*\right)\right], \quad (2.11)$$

$$\tan(\theta_k) = \frac{r_x(0) + kv_x + \sum_{l=1}^k u_{l,x}}{r_y(0) + kv_y + \sum_{l=1}^k u_{l,y}}.$$

In Eq. (2.11), the gradient vector $\mathbf{G}_k$ denotes, in fact, the vector $\mathbf{G}_{\mathbf{X}_k}$ ($\mathbf{X}_k = f_k(\mathbf{X}, \mathscr{U}_k)$, see Eq. (2.2)). The difficulty and the originality of the above problem stem from the two following facts. First, the source motion is unknown which means that the reference state vector $\mathbf{X}$ is *unknown.*[4] The problem is to optimize its estimation. So, practically, the sequence of bearings is unpredictable. Second, we deal with a global optimization problem which means that we seek for an optimal sequence of controls maximizing a cost function based on the *whole* FIM matrix. Of course, the problem is drastically eased by considering an additive (matrix) cost functional, but (as we shall see later) if the problem becomes far simpler the associated solutions optimized observer trajectories) perform quite poorly. Moreover, in this context, our problem can be treated by means of the dynamic programming principle. Unfortunately, it *necessarily* requires that the cost functional f (from $\mathscr{H}_n$, the vector space of n-dimensional Hermitian matrix to $\mathbb{R}$) satisfies the following motonicity property, denoted MDP (Matrix Dynamic Programming Property) and defined below:

Definition 1. The function f ($\mathscr{H}_n \to \mathbb{R}$, differentiable, $\mathscr{C}^2$) has the MDP if the following implication holds, whatever $C \in \mathscr{H}_n$:

$$f(B) > f(A) \Rightarrow f(B + C) > f(A + C).$$

An interpretation of this definition in terms of dynamic programming (maximization) is the following type of inequality (Le Cadre and Trémois, 1997):[5]

$$\max_{\bar{\mathscr{U}}_k^*} f\left(\sum_{i=n}^{k} \mathbf{G}_{\mathbf{X}_i} \mathbf{G}_{\mathbf{X}_i}^*\right) \le \max_{u_k} [f\{\mathbf{G}_{\mathbf{X}_k} \mathbf{G}_{\mathbf{X}_k}^* + F(\mathbf{X}_0, \bar{\mathscr{U}}_{k+1}^*\}]$$

which must be valid for the strategy $\bar{\mathscr{U}}_k^*$, optimal up to time k, and for $k = n-1, \ldots, 0$. Roughly, the MDP

[3] $\partial\Theta(\mathbf{X})/\partial\mathbf{X} = (\mathbf{G}_1, \ldots, \mathbf{G}_n)^*$.

[4] $\mathbf{X}$ is $\mathbf{X}_0$ if the reference time is 0.

[5] F denotes a FIM matrix, $\bar{\mathscr{U}}_k$ the optimal sequence of controls from n to k.

appears as a material form of a "comparison" principle. So, it plays a fundamental role in dynamic programming. A fundamental question consists in determining the functionals f having the MDP property. An answer is provided with the following result.

Proposition 1. *Let f satisfy the* MDP *property, then*

$f(A) = g(tr(AR)).$

where g is any monotonic increasing function and R is a fixed matrix.

We refer to Le Cadre and Trémois (1997) for the proof of Proposition 1. It is simply based on the fact that if $\nabla f(A)$ and $\nabla f(B)$ are not colinear, then their respective orthogonal subspaces are distinct which implies that there exists a matrix C for which the MDP is not satisfied. Therefore, $\nabla f(A)$ and $\nabla f(B)$ must be colinear, whatever A and B. This is a very strong property. In turn, this yields the general form of f. Consider for instance $f(A) = \log\det A$, then (Lancaster and Tismenetsky) for a non-singular matrix A,

$Df_A(C) = tr(A^{-1}C) = (\nabla^* f(A), C),$

and we see immediately that f does not have the MDP property. The same remark is valid for functionals as simple as $f(A) = tr(A^{-1})$.

Actually, our problem presents strong similarities with the theory of optimal experiment design (Whittle, 1973). More precisely, this general problem consists in determining the values of the experiments $\mathbf{X}_k$, or more generally a continuous distribution $\xi(d\mathbf{X})$ in a *design space* which optimize the estimation of an unknown parameter conditioning the observation ($\mathbf{X}_0$, here). So, the following matrix $M(\xi)$ is considered:[6]

$$M(\xi) \approx (1/n) \sum_{k=1}^{n} \mathbf{G}_{\mathbf{X}_k}\mathbf{G}^*_{\mathbf{X}_k},$$
$$= \int \mathbf{G}_\mathbf{X}\,\mathbf{G}^*_\mathbf{X}\,\xi(d\mathbf{X}). \tag{2.12}$$

The design problem is to determine ξ so as to maximize some functional Φ of $M(\xi)$. Various choices for Φ have been proposed in the literature, the two commonest ones being those of C-optimality and D-optimality, for which, effectively:[6]

$$\Phi(\xi) = -\,\mathbf{C}^* M(\xi)^{-1}\mathbf{C},$$
$$\Phi(\xi) = \log(\det(M(\xi)). \tag{2.13}$$

Now, for the case of D-optimality in particular, considerable use has been made of the *equivalence theorem*, stating that the three following characterizations of a D-optimal design ξ^* are equivalent:

(i) ξ^* maximizes $\det(M(\xi))$,
(ii) ξ^* minimizes $\bar{d}(\xi) \triangleq \sup_{\mathbf{X}\in\chi} d(\mathbf{X},\xi)$, where

$$d(\mathbf{X},\xi) = \mathbf{G}^*_\mathbf{X} M(\xi)^{-1}\mathbf{G}_\mathbf{X}, \tag{2.14}$$

(iii) $\bar{d}(\xi^*) = \dim(\mathbf{X})$.

These equivalences were first proved by Kiefer using game-theoretic methods. A general version of this theorem (with a simpler proof) was later given by Whittle (1973). Moreover, a characterization of the D-optimality in terms of invariance of ranking has been obtained. All these considerations plead for considering D-optimal design despite the fundamental difficulties mentioned above. Various algorithms for the calculation of D-optimal design have been proposed (Atwood, 1973), a general formulation stands as follows (i is the iteration index, $\delta_\mathbf{X}$ the measure concentrated on $\mathbf{X}$ and the Φ design functional):

$$\xi(i+1) = \frac{i\xi(i) + \delta_{\mathbf{X}(i)}}{i+1},$$

where

$$\mathbf{X}(i) = \arg\max D(\mathbf{X},\xi_i),$$

and

$$D(\mathbf{X},\xi) = \Phi(\xi,\delta_\mathbf{X}), \tag{2.15}$$

$$\Phi(\xi,\eta) = \lim_{\varepsilon\to 0}\varepsilon^{-1}[\Phi\{(1-\varepsilon)\xi + \varepsilon\eta\} - \Phi(\xi)].$$

Let us now consider the trace of the FIM; direct calculations yield

$$tr(\mathbf{G}_k\mathbf{G}^*_k) = \|\mathbf{G}_k\|^2 = \frac{1+k^2}{\sigma_k^2 r_k^2},$$

whence

$$tr(\text{FIM}) = \sum_1^N \frac{(1+k^2)}{\sigma^2 r_k^2}. \tag{2.16}$$

Thus, since tr(FIM) is independent of the bearings sequence $\{\Theta\}$, functionals involving the trace functional are quite simple. However, the associated optimization problems are restricted to minimizing a functional of the source-observer distance. Fortunately, for our problem (see Eq. (2.11)), the above result means that maximizing det(FIM) reverts to optimize (as a by-product) the FIM conditioning.

In this spirit, an elegant approach (Olsder, 1984; Passerieux and Van Cappel, 1991) for this particular optimal design problem consists in considering it as an optimal control problem where the cost is reduced to $Q = \det[F(T)]$. Then, the state vector comprises the FIM elements. However, we note that, basically, the cost functional does not have an integral part. So, it is easily shown that the Pontryagin Maximum Principle (PMP

[6] **C** is a given vector.

for the rest) simply reverts to a direct maximization of det (FIM) as described below.

Actually, a simple and intuitive approach, to this problem, is to model the observer trajectory by a certain number of legs and to optimize the various headings ($\{u_k\}$) of the observer (Hammel, 1988). Rather surprisingly, it may be easily shown that this intuitive approach and the optimal control based method (Passerieux and van Cappel, 1991) (discrete time formulation) leads to an identical optimization problem. However, we stress that this requires the knowledge of the gradient vectors $\{\mathbf{G}_k\}$ and thus implicitly assumes that the source trajectory is known. Furthermore, local extrema are quite likely. So, the object of this approach is mainly reduced to provide a databank of recommended maneuvers.

In order to remedy for the previous problems, we shall first consider *integral* approximations of the FIM determinant. It will then be possible to use efficiently the PMP formalism (see Section 3). However, the interest of such approximations is relatively limited, essentially because the integral cost, derived in this way, are very poor approximations of the exact determinant. This is particularly true when the observer is *maneuvering*. So, a direct analysis of the FIM determinant is required (see Section 4). This constitutes the conerstone of this paper.

3. Optimal control theory approaches

This section is divided in two parts. The first one deals, in the localization context, with the maximization of a lower bound of the FIM determinant. The basic tool is the Minkowski's inequality which is again used, in the second part, to derive a lower bound for the moving source case.

3.1. Optimal observer trajectory for localization

In fact, an important part of previous works has been devoted to the localization problem, i.e. the restriction of the TMA problem to the *fixed source* case. First, we stress that this case is considerably simpler in comparison with the moving source one. However, the interesting feature of this approach is that an integral approximation of the cost functional has been considered. Let us present briefly this approximation.

At first, the analysis is *restricted* to the estimation of the source position (r_x, r_y). The FIM matrix is thus *two-dimensional*. Let us consider the temporal evolution of the FIM, i.e.

$$F_{k+1} = F_{k-1} + M_k \quad \text{and} \quad M_k = \mathbf{G}_k\mathbf{G}_k^* + \mathbf{G}_{k+1}\mathbf{G}_{k+1}^*,$$

where

$$\mathbf{G}_k = \frac{1}{\sigma r_k}\begin{pmatrix} \cos\theta_k \\ -\sin\theta_k \end{pmatrix}. \tag{3.1}$$

Invoking the Minkowski's inequality, which is valid for n-dimensional positive-definite matrices A and B, i.e. $[\det(A+B)]^{1/n} \geq (\det A)^{1/n} + (\det B)^{1/n}$, we obtain

$$[\det(F_{k-1} + M_k)]^{1/2} \geq (\det F_{k-1})^{1/2} + (\det M_k)^{1/2}$$

so that

$$[\det(F_{k+1})]^{1/2} \geq (\det F_0)^{1/2} + \frac{1}{2}\sum_{j=1}^{k} [\det(M_j)]^{1/2}. \tag{3.2}$$

It now remains to calculate det (M_j). Elementary calculations yield[7]

$$\begin{aligned}\det(M_j) &= \det[(\mathbf{G}_j, \mathbf{G}_{j+1})(\mathbf{G}_j, \mathbf{G}_{j+1})^*] \\ &= \det[\mathrm{Gram}(\mathbf{G}_j, \mathbf{G}_{j+1})] \\ &= \frac{1}{\sigma^4 r_j^2 r_{j+1}^2}\sin^2(\theta_j - \theta_{j+1}).\end{aligned} \tag{3.3}$$

A *lower bound* of the FIM determinant has thus been derived. An attractive approach consists then in considering the maximization of this lower bound instead of the direct maximization of det (FIM). The above calculations (see Eq. (3.2) and (3.3)) suggest to consider the following integral cost[8] (Liu et al., 1988):

$$\mathscr{C}_T = \int_0^T \frac{\dot{\theta}(t)}{2\sigma^2 r^2(t)}\,\mathrm{d}t. \tag{3.4}$$

Using the transversality and stationarity conditions (for the Hamiltonian), the costates are first determined, leading to consider a rather intricated system of differential equations involving $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, $\ddot{\theta}$ and u. Rather surprisingly however, assuming v (the observer velocity modulus) constant, an explicit resolution of the optimality conditions can be obtained, simply yielding (Hammel et al., 1989) (u is the optimal control)

$$\dot{u}_* = -2\dot{\theta}. \tag{3.5}$$

This results in a feasible control since an estimation of $\dot{\theta}$ can be obtained from the estimated bearings. The value of the Hamiltonian along an optimal trajectory is constant (Hocking, 1997) and stands as follows:

$$H_{\text{opt}} = -\frac{v^2}{r^4\dot{\theta}}, \tag{3.6}$$

which means that the product $r^4\dot{\theta}$ is constant (v constant). An optimized trajectory thus represents a trade-off between the range and the bearing rate. So, we can consider that the observer heading is, at first, approximately equal to the bearing. Then, as the source-observer range decreases, the bearing rate increases. The observer is thus "spiralling" around the source.

[7] The symbol "Gram" denotes a Grammian matrix (Lancaster and Tismenetsky, 1985)

[8] $(\sin^2(\theta_j - \theta_{j+1}) \sim \dot{\theta}_j^2)$.

3.2. Optimizing the observer trajectory for TMA

Obviously, this *lower bound* approach may be extended to a moving source. The only changes (*for the cost functional*) and the FIM structure and dimension which result in (see Section 4):

$$\det (M_j) \approx 16\left(\frac{\sin(\dot{\theta}_j)}{r_j\sigma_j}\right)^8,$$

so that, $\mathscr{C}_T$ becomes

$$\mathscr{C}_T = \int_0^T \frac{\dot{\theta}(t)^2}{2\sigma^2 r^2(t)}\,\mathrm{d}t. \tag{3.7}$$

We note that the only change in Eq. (3.7) (versus Eq. (3.4)) is the exponent of $\dot{\theta}$. Again, this problem can be investigated by the optimal control approach. However, a main difficulty appears. The source is moving, so we cannot assume that the modulus of the (*relative*) observer velocity is constant. The analysis of optimality conditions is considerably complicated by the presence of additional terms (related to source motions) in the expressions of bearing and range-rates, thus requiring the use of numerical methods (Teo et al., 1991). In order to simplify the calculations, a reasonable assumption is that the observer velocity is far greater than the source one.

Then, the system Hamiltonian is

$$H = -\left(\frac{\dot{\theta}}{r}\right)^2 + \lambda_1 v \sin u + \lambda_2 v \cos u. \tag{3.8}$$

The costates λ_1 and λ_2 are explicited by the method presented in Appendix B, the optimal control u_* is then deduced:

$$4\left(\frac{\dot{r}}{r}\right)^2 + \dot{u}_*\dot{\theta} = -1. \tag{3.9}$$

The (constant) value of the system Hamiltonian along an optimal trajectory is

$$H_{\mathrm{opt}} = -\frac{1}{r^2}\left[\dot{\theta}^2 + 2\left(\frac{\dot{r}}{r}\right)^2\right]. \tag{3.10}$$

An optimized observer trajectory is thus a trade-off between the three terms r, $\dot{r}/r$, $\dot{\theta}$. The term $(\dot{\theta}^2 + 2(\dot{r}/r)^2)$ decreases as the square of the source-observer distance.

Numerous integral criteria may be considered but not any seems really suitable for TMA. Actually, as it will be shown in the next section, it is impossible to approximate conveniently a relevant cost functional (e.g. $\det F$ or $tr(F^{-1})$) by an integral one.

4. A direct analysis of the FIM determinant

For the sake of simplicity, the following assumptions are made along this section. First, the distance will be assumed to be *constant*. Further, we consider that the diagonal noise matrix Σ is proportional to the identity (i.e. $\Sigma = \sigma^2 Id$). Even if the first hypothesis seems rather restrictive, we shall see later (see Section 4.4) that the effects of range and bearing-rate variations are decoupled, allowing us to analyze separately their effects. Furthermore, the effects of range variations are concentrated in a multiplicative term, factor of the determinant.

A direct analysis of the effect of the observer controls, based on the discrete-time equation of motion (Eqs. (2.2)–(2.12)) seems unfeasible. We shall thus consider a *simplified* model of the source motion:

$$\theta_{i+j} = \theta_i + j\dot{\theta} + \sum_k u_k, \tag{4.1}$$

where $\dot{\theta}$ is the bearing rate (for a given reference time), and u is the bearing rate change corresponding to an observer maneuver (control). For the sequel, the *controls will be the observer bearing-rate changes u_k.*

The aim of this modelling is to obtain an explicit analysis of the FIM determinant. Obviously, the effects of the observer maneuver are only indirectly analyzed. However, the bearing rates are directly related to the cartesian parameters (see Appendix A). Overall, the bearing rates being estimable from the sensor outputs, we can thus derive feasible methods for the observer trajectory optimization. *The fundamental interest of this approach lies in the fact that no a priori knowledge of the source trajectory is assumed.* More generally also, the problem is only partially observable. For the Modified Polar Coordinate coordinates (Aidala and Hammel, 1983), only $\{\theta, \dot{\theta}, \dot{r}/r\}$ are available measurements. Further, we shall prove that det(FIM) is independent of θ (cf. Proposition 2). Then, since the effects of bearing-rate changes and range variations are decoupled, it is quite natural to concentrate our efforts on the effects of bearing rate changes.

We shall denote by $F_{k_0,4}$ the FIM corresponding to an arbitrary reference time k_0 and four consecutive measurements, $\theta_{k0}, \ldots, \theta_{k_0+3}$. Then the FIM $F_{k_0,4}$ takes the following form:

$$F_{k_0,4} = (\sigma r)^{-2}\mathscr{G}_{k_0,4}\,\mathscr{G}^*_{k_0,4},$$

where

$$\mathscr{G}_{k_0,4} = (\mathbf{G}_{k_0}, \mathbf{G}_{k_0+1}, \mathbf{G}_{k_0+2}, \mathbf{G}_{k_0+3}).$$

and $\mathbf{G}_k$ is the gradient vector of the observation θ_k w.r.t. $\mathbf{X}_0$, i.e.

$$\mathbf{G}_k = (\cos\theta_k, -\sin\theta_k, k\cos\theta_k, -k\sin\theta_k)^*. \tag{4.2}$$

Assuming $\mathscr{G}_{k_0,4}$ invertible, we have

$$\det(F_{k_0,4}) = (\sigma r)^{-8}(\det\mathscr{G}_{k_0,4})^2.$$

Of course, our attention is not limited to four measurements per leg. So, the previous calculations will now be extended to any number of measurements. Let ℓ be the number of consecutive measurements and consider now the (4×4) FIM $F_{k_0,\ell}$ $(l \geq 4)$ defined as in Eq. (4.2) by[9]

$$F_{k_0,\ell} = (\sigma r)^{-2}\mathcal{G}_{k_0,\ell}\,\mathcal{G}^*_{k_0,\ell},$$

where

$$\mathcal{G}_{k_0,\ell} = (\mathbf{G}_{k_0}, \mathbf{G}_{k_0+1}, \dots, \mathbf{G}_{k_0+\ell}), \quad \ell \geq 0. \tag{4.3}$$

Using classical properties of multilinear algebra, namely the Cauchy–Binet formula (Lancaster and Tismenetsky, 1985), det $(F_{k_0,\ell})$ is given by the following formula:

$$\det(F_{k_0,\ell}) = (\sigma r)^{-8}\sum_E [\det(\mathcal{G}_E)]^2,$$

where E is the index subset defined by

$$E = \{i_1, i_2, i_3, i_4\} \quad \text{s.t. } 1 \leq i_1 < i_2 < i_3 < i_4 \leq \ell,$$

and

$$\mathcal{G}_E = (\mathbf{C}_{i_1}, C_{i_2}, \mathbf{C}_{i_3}, \mathbf{C}_{i_4}), \quad \mathbf{C}_{i_j} \triangleq \mathbf{G}_{k_0+i_j}. \tag{4.4}$$

We stress that the above formula plays a central role in the analysis of the FIM determinant.

4.1. The case of constant bearing rate

In Eq. (4.4) $\mathbf{C}_{i_j}$ stands for the i_jth column of the matrix $\mathcal{G}$. Considering, for instance, a first-order expansion of the bearings θ_{k_0+i} (i.e. $\theta_{k_0+i} \stackrel{1}{=} \theta_{k_0} + i\dot{\theta}$), the calculation of $\det(F_{k_0,\ell})$ is reduced to the calculation of the determinants det $(\mathcal{G}_E)$. Now each of these determinants is the determinant of a 4×4 matrix. Its calculation is greatly eased by using the following basic result.

Proposition 2. *Let* $\mathbf{E}$ be the first vector of the canonical basis of $\mathbb{R}^4$, $(\mathbf{E} = (1,0,0,0)^*)$, *then the following equality holds*:

$$\det\mathcal{G}_E = \det(R_1^{i_1}\mathbf{E}, R_1^{i_2}\mathbf{E}, R_1^{i_3}\mathbf{E}, R_1^{i_4}\mathbf{E}). \tag{4.5}$$

Proof. Consider the determinant $\det\mathcal{G}_E$ (see Eq. (4.4)) where as previously, $E = \{i_1, i_2, i_3, i_4\}$ and $i_1 < i_2 < i_3 < i_4$, then

$$\begin{aligned}\det\mathcal{G}_E &= \det(\mathbf{G}_{i_1}, \dots, \mathbf{G}_{i_4})\\ &= \det(R_1^{i_1}\mathbf{G}_{k_0}, R_1^{i_2}\mathbf{G}_{k_0}, R_1^{i_3}\mathbf{G}_{k_0}, R_1^{i_4}\mathbf{G}_{k_0}),\end{aligned}$$

where

$$R_1 \triangleq \begin{pmatrix} R_0 & 0 \\ R_0 & R_0 \end{pmatrix} \quad\text{and}\quad R_0 \triangleq \begin{pmatrix} \cos\dot{\theta} & \sin\dot{\theta} \\ -\sin\dot{\theta} & \cos\dot{\theta} \end{pmatrix} \tag{4.6}$$

In the same spirit, the vector $\mathbf{G}_{k_0}$ may be written as

$$\mathbf{G}_{k_0} = T_1^{k_0}\mathbf{E},$$

where

$$T_1 \triangleq \begin{pmatrix} T_0 & 0 \\ T_0 & T_0 \end{pmatrix} \quad\text{and}\quad T_0 \triangleq \begin{pmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

$$\theta \triangleq \theta_{k_0}/k_0, \quad \mathbf{E} = (1,0,0,0)^* \tag{4.7}$$

Now the following properties are instrumental:

- the matrices R_0 and T_0 are rotation matrices, hence they commute,
- $\det(R_1) = \det^2(R_0) = 1$.
- $\det(T_1) = \det^2(T_0) = 1$.

The matrices R_1 and T_1 then also commute and using this property $\det\mathcal{G}_E$ becomes

$$\begin{aligned}\det\mathcal{G}_E &= \det(T_1^{k_0}R_1^{i_1}\mathbf{E}, T_1^{k_0}R_1^{i_2}\mathbf{E}, T_1^{k_0}R_1^{i_3}\mathbf{E}, T_1^{k_0}R_1^{i_4}\mathbf{E})\\ &= \det(T_1^{k_0})\det(R_1^{i_1}\mathbf{E}, R_1^{i_2}\mathbf{E}, R_1^{i_3}\mathbf{E}, R_1^{i_4}\mathbf{E})\\ &= \det(\mathbf{E}, R_1^{i_2-i_1}\mathbf{E}, R_1^{i_3-i_1}\mathbf{E}, R_1^{i_4-i_1}\mathbf{E}).\end{aligned} \tag{4.8}$$

Furthermore, the following property has thus been proved in passing: $\det\mathcal{G}_E$ is independent of k_0 and θ_{k_0}. This remarkable property is due to the basic properties of the determinant and the structures of the matrices R_1 and T_1. □

The above determinant itself (i.e. $\det\mathcal{G}_E = \det(\mathbf{E}, R_1^i\mathbf{E}, R_1^j, \mathbf{E}, R_1^k\mathbf{E})$) can now be easily calculated by means of exterior algebra (Yokonuma, 1992; Darling, 1994), yielding the following simple and general result:

Proposition 3. *The following equality holds*:

$$\det\mathcal{G}_E = j(k-i)\sin((k-j)x)\sin(ix) + i(k-j)\sin(jx)\sin((i-k)x),$$

where

$$i = i_2 - i_1, \quad j = i_3 - i_1, \quad k = i_4 - i_1. \tag{4.9}$$

Proof. The calculation of $\det\mathcal{G}_E$ is greatly eased by using exterior algebra. First, let us briefly recall the definition of the exterior powers of a vector space. Let V be an n-dimensional vector space over $\mathbb{R}$, then $\Lambda^2 V$ consists of all formal sums $\sum_i \alpha_i(\mathbf{U}_i \wedge \mathbf{V}_j)$, where the "wedge product" $\mathbf{U}\wedge\mathbf{V}$ is bilinear and alternate. This definition is straightforwardly extended to higher exterior powers (Darling, 1994). For any basis $\{\mathbf{V}_1, \dots, \mathbf{V}_n\}$ of V, the set of p-vectors $\{\mathbf{V}_{i_1}\wedge\cdots\wedge\mathbf{V}_{i_p},\ i_1 < \cdots < i_p \leq n\}$ forms a basis of the $n!/(n-p)!p!$-dimensional vector space $\Lambda^p V$. In particular, $\Lambda^4\mathbb{R}^4$ is *one-dimensional*, and throughout the paper we make intensive use of the isomorphism $\Lambda^4\mathbb{R}^4 \equiv \Lambda^2\mathbb{R}^4 \wedge \Lambda^2\mathbb{R}^4$. The exterior algebra formalism thus appears as an economical way to conduct determinant calculations.

[9] Note that in Eq. (4.3) the source-observer distance is again assumed to be constant.

The canonical basis of $\mathbb{R}^4$ is denoted $\{\mathbf{E}_1, \dots, \mathbf{E}_4\}$. For the coherence of notations, the vector $\mathbf{E}$ (Eqs. (4.4)–(4.8)) is identified with $\mathbf{E}_1$. Then, the components (denoted α_0, β_0, γ_0) of the exterior products $\mathbf{E}_1 \wedge R_1^i \mathbf{E}_1$, in the "reduced" basis $\{\mathbf{E}_1 \wedge \mathbf{E}_2, \mathbf{E}_1 \wedge \mathbf{E}_3, \mathbf{E}_1 \wedge \mathbf{E}_4\}$ of $\Lambda^2(\mathbb{R}^4)$ are straightforwardly calculated and given below:

$$\begin{aligned} \alpha_0 &= -\sin(ix) \leftarrow \mathbf{E}_1 \wedge \mathbf{E}_2, \\ \beta_0 &= i\cos(ix) \leftarrow \mathbf{E}_1 \wedge \mathbf{E}_3, \\ \gamma_0 &= -i\sin(ix) \leftarrow \mathbf{E}_1 \wedge \mathbf{E}_4. \end{aligned} \tag{4.10}$$

Similarly, the components of $R_1^j \mathbf{E}_1 \wedge R_1^k \mathbf{E}_1$, in the "reduced" basis $\{\mathbf{E}_3 \wedge \mathbf{E}_4, \mathbf{E}_2 \wedge \mathbf{E}_4, \mathbf{E}_2 \wedge \mathbf{E}_3\}$ are

$$\begin{aligned} \alpha_1 &= jk\sin((j-k)x) \leftarrow \mathbf{E}_3 \wedge \mathbf{E}_4, \\ \beta_1 &= (k-j)\sin(jx)\sin(kx) \leftarrow \mathbf{E}_2 \wedge \mathbf{E}_4, \\ \gamma_1 &= j\sin((k-j)x) - (k-j)\sin(jx) \leftarrow \mathbf{E}_2 \wedge \mathbf{E}_3. \end{aligned} \tag{4.11}$$

The determinant $\det \mathscr{G}_E$ is deduced from the above calculations, by considering the sum of the coefficients of the vector $\mathbf{E}_1 \wedge \mathbf{E}_2 \wedge \mathbf{E}_3 \wedge \mathbf{E}_4$ which spans the one-dimensional space $\Lambda^4(\mathbb{R}^4)$, i.e.

$$\begin{aligned} \det \mathscr{G}_E &= \alpha_0\alpha_1 - \beta_0\beta_1 + \gamma_0\gamma_1, \\ &= j(k-i)\sin((k-j)x)\sin(ix) \\ &\quad + i(k-j)\sin(jx)\sin((i-k)x). \quad \square \end{aligned} \tag{4.12}$$

Using Proposition 3 and the Cauchy–Binet formula, a general formulation of det (FIM) stands as follows:

$$\begin{aligned} \det(\mathrm{FIM}) = \textstyle\sum_{1 \le i < j < l \le l} [\, & j(k-i)\sin((k-j)x)\sin(ix) \\ & + i(k-j)\sin(jx)\sin((i-k)x)]^2. \end{aligned}$$

Practically, the following approximations are easily deduced from the above property.

Result 1. The following approximation of $\det \mathscr{G}_E$ holds ($x = \dot{\theta} \ll 1$):

$$\det \mathscr{G}_E \overset{6}{=} \frac{(ijk)}{3}(k-i)(k-j)(j-i)\,x^4,$$

and therefore

$$\det(F_{k_0,l}) \approx \sum_{1 \le i < j < k \le l} \left[\frac{(ijk)}{3}(k-i)(k-j)(j-i)\right]^2 \left(\frac{\dot{\theta}}{\sigma r}\right)^8. \tag{4.13}$$

From Result 1, the following approximation is deduced:

$$\begin{aligned} \det(F_{k_0,l}) &\approx \alpha^{-1}[\ell^3(1+\ell)^4(\ell^2+2\ell-8)(\ell^2+2\ell-3)^2 \\ &\qquad \times (2+\ell)^3] \left(\frac{\dot{\theta}}{\sigma r}\right)^8, \\ &\propto \ell^{16}\left(\frac{\dot{\theta}}{\sigma r}\right)^8. \end{aligned} \tag{4.14}$$

Using the previous formalism, an extension to higher-order expansions of θ_{k_0+i} is quite straightforward but not truly enlightening.

Remarks. (1) If a third-order expansion of θ_{k_0+i} is considered instead of the first-order one, then the value of det (FIM) is exactly zero.. This corroborates the fact that the TMA problem is not observable when the observer does not maneuver.

(2) However, the BOT problem is observable if multiple measurements are available (at each time). In this case, bounds derived from Eq. (4.13) are accurate.

(3) In fact, a small variation model for the bearing rate, i.e. $\dot{\theta}_{k+1} = \dot{\theta}_k + \eta_k$ (η_k w.g.n. $\mathcal{N}(0,\tau^2)$) yields a value of the type (4.13), Eq. (4.13) roughly appears as an upper bound of det (FIM). More precisely, we obtain[10] $\mathbb{E}[\det(\mathrm{FIM}_{t,t+3})] \approx (16c_3/r^8)\exp(-\frac{3}{2}\tau^2)\,(\sin(\dot{\theta}))^8$.

It has thus been shown that $\det(F_{k_0,l})$ is proportional to $\ell^{16}(\sin\dot{\theta}/\sigma r)^8$. As practically, $\dot{\theta}$ is very small (see Appendix A), this means that $\det(F_{k_0,l})$ remains very small as far as no real observer maneuver occurs. So, we shall now investigate the effects of a bearing-rate change.

4.2. The case of bearing-rate change

We shall now quantify the effects of observer maneuvers. First, the following property is an extension of the previous one to this case.

Consider that the temporal evolutions of the source bearings on two successive legs are described by the following two linear models ($k_0' \triangleq k_0 + j$; observer maneuvering instant):

$$\begin{aligned} \theta_{k_0+i} &\overset{1}{=} \theta_{k_0} + i\dot{\theta}_1 \text{ on the 1st obs. leg } 0 \le i \le j, \\ \theta_{k_0'+m} &\overset{1}{=} \theta_{k_0'} + m\dot{\theta}_2 \text{ on the 2nd obs. leg } m \ge 0. \end{aligned} \tag{4.15}$$

Then the following property holds and extends the previous result (Proposition 3):

Proposition 4. *Consider the case of two consecutive bearing rates x and y; then we have* ($j' = i - j$)

$$\begin{aligned} &\det(R_1^i\mathbf{E}_1, R_1^j\mathbf{E}_1, R_1^jR_2^k\mathbf{E}_1, R_1^jR_2^l\mathbf{E}_1) \\ &\quad = (k)(j'-l)\sin(j'x)\sin((k-l)y), \\ &\quad + j'(l-k)\sin(ky)\sin(j'x-ly). \end{aligned} \tag{4.16}$$

Furthermore, the following approximation holds:

$$\frac{\partial}{\partial y}\det(R_1^i\mathbf{E}_1, R_1^j\mathbf{E}_1, R_1^jR_2^k\mathbf{E}_1, R_1^jR_2^l\mathbf{E}_1) \approx kl(l-k)(j-i)y.$$

Proof. In Eq. (4.16), the matrices R_1 and R_2 are the bearing-rate matrices (cf. Eq. (4.6) associated with the bearing-rates x and y. Mimicking the proof of

[10] The symbol $\mathbb{E}$ denotes the math. expectation.

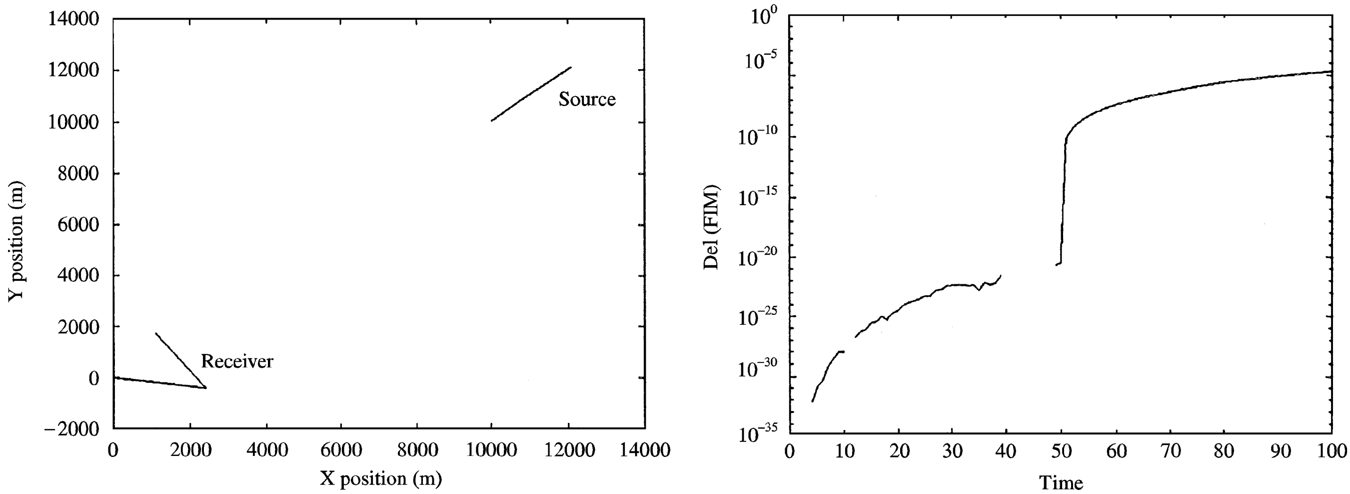

Fig. 2. Evolution of the FIM determinant det $F_{l_1 l_2}$.

Proposition 3, we calculate the components $(\alpha_0, \beta_0, \gamma_0)$ of $\mathbf{E}_1 \wedge R_1^{j'}\mathbf{E}_1$ in the "reduced" basis $\{\mathbf{E}_1\wedge\mathbf{E}_2, \mathbf{E}_1\wedge\mathbf{E}_3, \mathbf{E}_1\wedge\mathbf{E}_4\}$ of $\Lambda^2(\mathbb{R}^4)$ as well as that of $R_2^k\mathbf{E}_1 \wedge R_2^l\mathbf{E}_1$ in $\{\mathbf{E}_3\wedge\mathbf{E}_4, \mathbf{E}_2\wedge\mathbf{E}_4, \mathbf{E}_2\wedge\mathbf{E}_3\}$ (denoted $(\alpha_1, \beta_1, \gamma_1)$), yielding

$$\alpha_0 = -\sin(j'x), \quad \alpha_1 = kl\sin((k-l)y),$$
$$\beta_0 = j'\cos(j'x), \quad \beta_1 = (l-k)\sin(ky)\sin(ly),$$
$$\gamma_0 = -j'\sin(j'x), \gamma_1 = k\sin((l-k)y) - (l-k)\sin(ky)\cos(ly).$$

The first part of Proposition 4 is obtained by calculating the scalar $\alpha_0\alpha_1 - \beta_0\beta_1 + \gamma_0\gamma_1$. Expliciting the scalar

$$\alpha_0\frac{\partial\alpha_1}{\partial y} - \beta_0\frac{\partial\beta_1}{\partial y} + \gamma_0\frac{\partial\gamma_1}{\partial y},$$

yields the second part.

From Eq. (4.16), the following approximations are easily deduced:

$$\det(R_1^i\mathbf{E}_1, R_1^j\mathbf{E}_1, R_1^j R_2^k\mathbf{E}_1, R_1^j R_2^l\mathbf{E}_1)$$
$$\approx kl(l-k)(j-i)y(x-y+cx^2y). \quad (4.17)$$

The above property allows us to approximate det (FIM) in the case of a maneuvering observer and thus to investigate the effects of the observer maneuvers. In particular, the role of the bearing-rate changes then clearly appears. Indeed, since the parameters $\dot\theta_1$ and $\dot\theta_2$ are usually small (both are proportional to $1/r^2$), we shall examine an expansion of $\det(\mathscr{G}_E)$ w.r.t $\dot\theta_1$ and $\dot\theta_2$ around the point (0, 0). Then, we obtain the following types[11] of fourth-order expansions (in $\dot\theta_1$ and $\dot\theta_2$) of $\det(\mathscr{G}_E)$, given by Eqs. (4.16) and (4.17):

$$(\det\mathscr{G}_E)^2 \simeq K(y(x-y))^2, \quad K(x(x-y))^2 \quad (4.18)$$

with

$$K>0, \quad x \triangleq \dot\theta_1, \quad y \triangleq \dot\theta_2.$$

This result is quite fundamental for TMA (see Proposition 5) and will be clarified by a geometric interpretation (see Section 4.3).

The effects of the observer maneuvers on the FIM determinant is now illustrated. Thus, in Fig. 2, $\det(F_{l_1 l_2})$ is plotted versus the measurement index. The scenario parameters are described in the Fig. 2. The observer maneuver induces a dramatic increase of the FIM determinant which is precisely located at the maneuver instant.

For the two bearing-rate case, the expansion of $\det F_{k_0,\ell_1,\ell_2}$[12] is

$$F_{k_0,\ell_1,\ell_2} \simeq \frac{1}{(\sigma r)^8}\left[\sum_{i=1}^{5} P_i(\ell_1,\ell_2)y^{5-i}x^{i-1}\right], \quad (4.19)$$

where the polynomials $\{P_i(\ell_1,\ell_2)\}_{i=1}^5$ are detailed in Appendix C. From Eqs. (4.18) and (4.19), we note that the maximum value of $\det F_{k_0,\ell_1,\ell_2}$ is proportional to $\ell^{12}\dot\theta^4$ ($\ell_1 \simeq \ell_2$, $\dot\theta_1 \simeq -\dot\theta_2$). The result must be compared with Eq. (4.14). In fact, denoting by $F_\ell(x)$ the FIM associated with a constant bearing-rate x and $F_{\ell/2,\ell/2}(x,-x)$ the FIM associated with a two-leg observer trajectory (leg 1: $\ell/2$ meas., bear. rate x; leg 2: $\ell/2$ meas., bear. rate $-x$), Eqs. (4.13) and (4.17) yield the following important result:

Result 2. The gain of the maximal bearing-rate change (mid-course) is

$$\frac{\det[F_{\ell/2,\ell/2}(x,-x)]}{\det(F_\ell(x))} \simeq 134\,\ell^{-4}x^{-4}$$
$$\simeq 134\,(\Delta x)^{-4}, \quad (4.20)$$

[11] The type of the expansion only depends on the relative values of i_1, i_2, i_3, i_4.

[12] ℓ_i measurements associated with $\dot\theta_i$, $i = 1, 2$.

where Δx denotes the total bearing variation (i.e. $\Delta x = \ell x$). For usual scenarios, Δx is (quite) small with regard to 1 and, therefore, the increase in the FIM determinant gained by optimized observer maneuvers may be rather impressive. Further, note that this gain is proportional to $(\Delta x)^{-4}$. The above calculation is easily extended to the case of a maneuvering source. The dimension of the state vector is then equal to 6, while the gain of a bearing-rate change is, this time, proportional to $(\Delta x)^{-8}$.

The following example may be rather enlightening. We present in Fig. 3, the values of $\det F_{\ell_1,\ell_2}(x, -x) \times (x = 10^{-4})$ as calculated as a function of the first leg length l_1 (see Appendix C). We can remark that the optimum is attained for very unequal leg lengths. Actually, it seems that the optimum corresponds to a "long" first leg in order to maximize the observer baseline, followed by a "short" second leg.

Formula (4.19) may be easily extended to the three-leg case (i.e. $\{x, y, z\}$). As previously, $\det F_{k,\ell_1,\ell_2,\ell_3}$ is an homogeneous polynomial in (x, y, z), i.e.

$$\det F_{\ell_1,\ell_2,\ell_3} \simeq \frac{1}{(\sigma r)^8}\left[\sum_{i,j,k} P_{i,j,k}(\ell_1,\ell_2,\ell_3)\, x^i y^j z^k\right], \tag{4.21}$$

with

$$0 \le \{i,j,k\} \le 4 \quad \text{and} \quad i+j+k=4.$$

For the sake of brevity, the analytical expressions of the $P_{i,j,k}$ are not detailed. Practically, for equal legs (i.e. $\ell_1 = \ell_2 = \ell_3 = \ell$), the maximum value $(x = -y = z)$ of $\det F_{k,\ell_1,\ell_2,\ell_3}$ is approximately $45\,\ell^{12}\dot{\theta}^4$.

Let us now illustrate the above results. The source-observer scenario is depicted in Fig. 4. The source is in rectilinear and uniform motion. The observer motion is modelled by a three-leg path. Furthermore, it is assumed that the modulus of the observer velocity is constant. The controls are the successive observer headings. They are optimized by means of a standard numerical optimization procedure. The optimized observer trajectories are presented in Fig. 5. Note that the general shape is a "Z" which is stretched when the source-observer distance decreases. This result is quite general, whatever the relative source-observer position and the number of observer legs.

The computation of the bearing rates is also quite illustrative (see Fig. 6). Actually, we see that the optimal headings u_k induce a bang-bang behavior of $\dot{\theta}$. More precisely, the sequence of optimized controls (i.e. the observer headings) leads to choose alternatively the two bearing-rate bounds (i.e. $-\dot{\theta}_{\max}$, $\dot{\theta}_{\max}$, see Appendices A, B and D). This quite agrees with the theoretical results.

The previous results are more formally summarized by the following property.

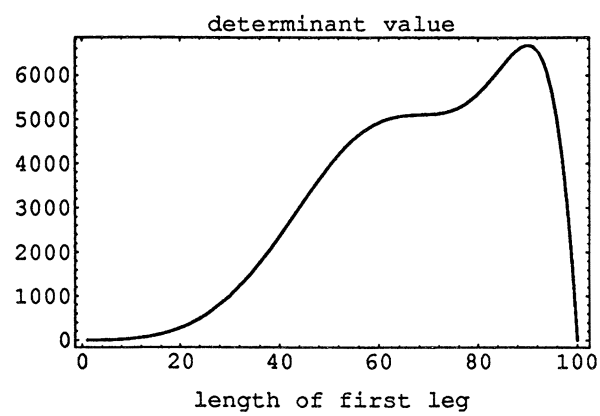

Fig. 3. Evolution of the FIM determinant as a function of the first leg length for a two-leg receiver trajectory.

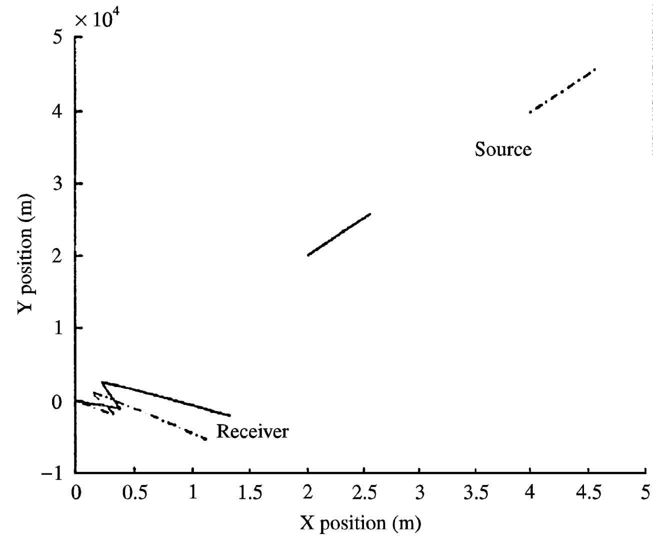

Fig. 4. Optimized receiver trajectory for two source-receiver ranges.

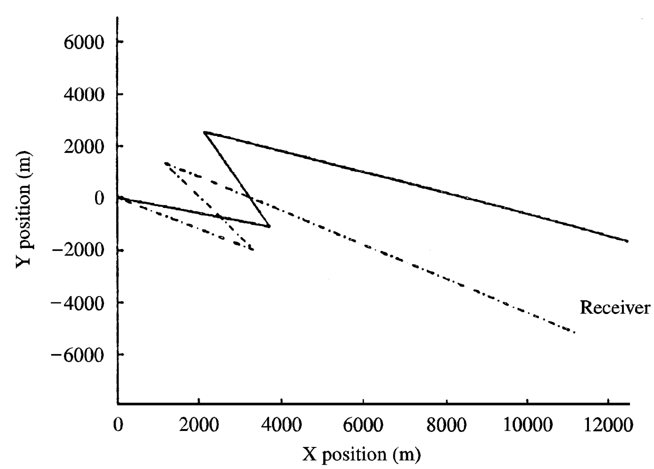

Fig. 5. Details of optimized receiver trajectories.

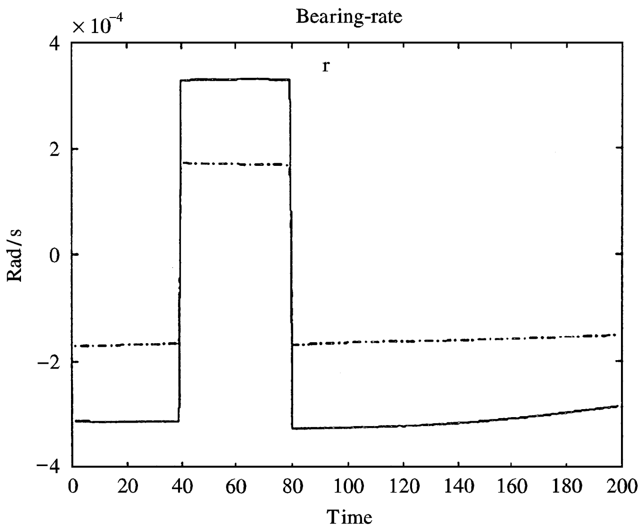

Fig. 6. Values of $\dot{\theta}$ for the two scenarios.

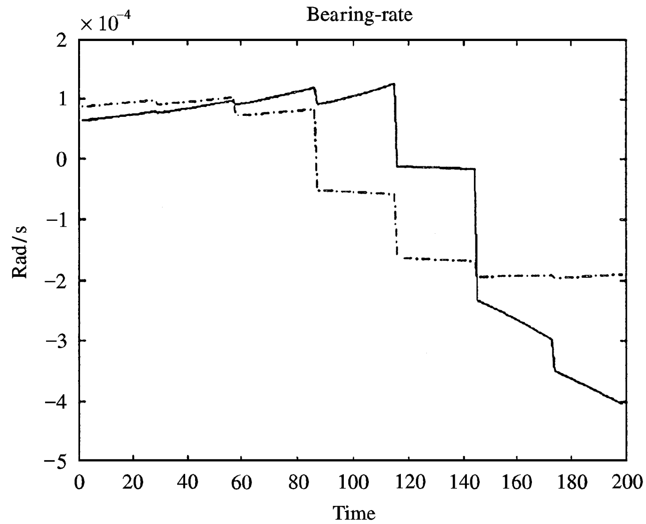

Fig. 7. Values of $\dot{\theta}$ for the two scenarios.

Proposition 5. *Let* $x_1, x_2, \dots, x_n$ *be the consecutive bearing* rates, *then*

$$\arg_{x_1,\dots,x_n} \max \det(\mathrm{FIM}) = \varepsilon(\dot{\theta}_{\max}, -\dot{\theta}_{\max}, \dot{\theta}_{\max}, \dots),$$
$$\varepsilon = \pm 1. \tag{4.22}$$

We refer to Appendix D for a proof of Proposition 5. Since the values of $(-\dot{\theta}_{\max}, \dot{\theta}_{\max})$ can be *estimated* by the observer (e.g. from the estimated bearings), it remains to determine the optimal number of switching (Sussmann, 1979) (from $\dot{\theta}_{\max}$ to $-\dot{\theta}_{\max}$) as well as their locations. Using the previous results, the problem may be formulated as follows. Consider a multileg observer trajectory, then the problem consists in maximizing $\det(F_{l_1,l_2,l_3,\dots})$ (l_i is the length of the ith leg) given below:

$$(\sigma r)^8 \det(F_{l_1,l_2,l_3}) \approx P(l_1, \dot{\theta}_{\max})\dot{\theta}^8 + P(l_1, l_2, \dot{\theta}_{\max})(2\dot{\theta}_{\max})^4$$
$$+ \mathrm{P}(l_1, l_3, \dot{\theta}_{\max})(2\dot{\theta}_{\max})^4 + \cdots. \tag{4.23}$$

The polynomials $P(l_i, l_j)$ have the form given in Appendix C. The corresponding optimization problem may be solved by numerical methods. However, it seems rather impossible to derive a general bound relative to the number of switching.

Finally, if the cost functional is replaced by the trace, the (optimized) observer trajectory is quite different. Actually, minimizing the distance becomes quite predominant. Thus, the evolutions of the bearing rates corresponding to the scenario depicted in Fig. 4 are presented in Fig. 7. We may note that the optimized bearing rates no longer exhibit a bang-bang behavior. A comparison of the statistical bounds, associated with the observer trajectories corresponding to the optimization of the two functionals (i.e. det and tr), shows a very clear supremacy of the det.

4.3. Geometric interpretations of the properties of the FIM determinant

Since we are especially interested in the effects of observer maneuvers, we shall investigate them by means of the previous results and differential calculus. Consider, for instance, the following determinant ($\mathbf{E} = \mathbf{E}_1$):

$$f(y) = \det(\mathbf{E}, R^i_{1,x}\mathbf{E}, R^j_{1,x}\mathbf{E}, R^k_{1,x}R^l_{1,y}\mathbf{E}), \tag{4.24}$$

where $x = \dot{\theta}_1$, $y = \dot{\theta}_2$.

Let us now calculate the partial derivatives $\partial f/\partial y(x)$; we obtain

$$\frac{\partial f}{\partial y}(x) = l \det(\mathbf{E}, R^i_{1,x}\mathbf{E}, R^j_{1,x}\mathbf{E}, R^{k+l-1}_{1,x} S_{1,x}\mathbf{E}), \tag{4.25}$$

where $S_{1,x} = (\partial/\partial y) R_{1,y})_{(y=x)}$, or, explicitly

$$S_{1,x} = \begin{pmatrix} S_{0,x} & 0 \\ S_{0,x} & S_{0,x} \end{pmatrix}, \tag{4.26}$$

with

$$S_{0,x} = \begin{pmatrix} -\sin x & -\cos x \\ \cos x & -\sin x \end{pmatrix}.$$

Using the definitions of $R_{1,x}$ and $S_{1,x}$, and denoting J the two-dimensional $\pi/2$ rotation matrix, elementary calculations yield:

Proposition 6. *The following properties hold*:

$$S_{0,x} = J R_{0,x}, \qquad \frac{\partial}{\partial x} S_{0,x} = J^2 R_{0,x} = -R_{0,x},$$
$$R^k_{1,x} S^{k'}_{1,x} = S^{k'}_{1,x} R^k_{1,x} = S^{k+k'}_{1,x}, \qquad R^k_{1,x} S^{k'}_{1,y} = S^{k+k'}_{1,x+y}. \tag{4.27}$$

The above results give the algebraic rule for analyzing the partial derivatives of f. Thus, from Proposition 6, we directly deduce

$$\frac{\partial f}{\partial y}(x) = l \det(\mathbf{E}, R_{1,x}^{i}\mathbf{E}, R_{1,x}^{j}\mathbf{E}, S_{1,x}^{k+1}\mathbf{E}),$$

$$\frac{\partial^2 f}{\partial y^2}(x,x) = -l^2 \det(\mathbf{E}, R_{1,x}^{i}\mathbf{E}, R_{1,x}^{j}\mathbf{E}, R_{1,x}^{k+l}\mathbf{E}), \text{ etc.} \tag{4.28}$$

At this point, it is worth noting that the vector $S_{1,x}^{m}\mathbf{E} = (-\sin mx, \cos mx, -m\sin mx, m\cos mx)^*$ (Eqs. (4.25) and (4.26)) is approximately *orthogonal* to the vectors $\{\mathbf{E}, R_{1,x}^{i}\mathbf{E}, R_{1,x}^{j}\mathbf{E}\}$. This fact is typical of a four-dimensional state vector and corresponds to a diversity in maneuvers. Thus Eq. (4.28), $(\partial f/\partial y)(x)$ is proportional to x, while $(\partial^2/\partial y^2)(x,x)$ is proportional to x^4, so that

$$f(y) \approx \gamma x(y-x). \tag{4.29}$$

From Eq. (4.28) it is clear that the increase of $\det(F_{x,y})$ is maximized when the terms $(x(x-y))^2$ are maximized. Since $\dot{\theta}$ is bounded, an optimal sequence of controls is necessarily a bang-bang one (see Appendix D).

4.4. The effects of range variations

Up to now, the effects of range variations have not been considered. However, the analysis is greatly simplified if we remark that the effects of range and bearing-rate variations are *geometrically* decoupled. This follows easily by considering $\det(\mathscr{G}_E)$. Indeed, including the range, the elementary determinant $\det(\mathscr{G}_E)$ becomes

$$\det \mathscr{G}_E = \det\left(\frac{1}{r_{i_1}} R_1^{i_1}\mathbf{E}, \ldots, \frac{1}{r_{i_4}} R_1^{i_4}\mathbf{E}\right),$$

$$= \frac{1}{r_{i_1}} \cdots \frac{1}{r_{i_4}} \det(R_1^{i_1}\mathbf{E}, \ldots, R_1^{i_4}\mathbf{E}), \tag{4.30}$$

so that

$$\det(F_{k,\ell}) = (1/\sigma^8)\sum_E \left[\left(\frac{1}{r_{i_1}} \cdots \frac{1}{r_{i_4}}\right) \det(R_1^{i_1}\mathbf{E}, \ldots, R_1^{i_4}\mathbf{E})\right]^2.$$

More precise calculations can be achieved if we consider (for instance) a first-order expansion of the source-observer distance (i.e. $r_{k+i} \overset{1}{=} r_k + i\dot{r}$). Roughly, the matrix R_1 is then replaced by $(1+\dot{r}/r)^{-1}R_1$. For instance, the basic result then becomes

$$\det(F_{k,4}) \propto \frac{1}{\sigma^8 r^{12}} P_l(\dot{\theta}) Q_l(\dot{r}).$$

with (4.31)

$$Q_l(\dot{r}) = r_0^4 + \eta_1 r_0^3 \dot{r} + \eta_2 r_0^2 \dot{r}^2 + \eta_3 r_0 \dot{r}^3 + \eta_4 \dot{r}^4.$$

This approximation is valid as long as $\dot{r}/r \ll 1$.[13] If this assumption is valid and if the total duration is "reasonable", the conclusions of the previous sections still hold, since the effects of bearing-rate changes are preeminent.

5. Conclusion

The optimization of the observer maneuvers has been considered along this paper. This problem is not relevant of classical optimal control. So, a large part of this paper is centered around the approximation of the cost functional. Using basic tools of multilinear algebra, it has been proved that this functional may be accurately approximated by a functional involving only the successive bearing-range rates of the source. The approach is thus quite indirect but has the great advantage to involve only observed data. In particular, it has been shown that under the long-range and bounded controls hypotheses, the sequence of optimal control lies in the general class of bang-bang (relatively to the bearing-rates) one. These results have been illustrated by simulation results. They demonstrate the interest of maneuver diversity. More generally, they provide us with a simple and feasible approach for optimizing the observer trajectory.

Acknowledgements

The authors would like to thank the reviewers and the editor for their many helpful comments.

Appendix A: Calculation of the bearing-rate bounds

This appendix deals with the calculation of the bounds for the bearings rate $\dot{\theta}$ under the following hypothesis:

1. The source velocity vector $\mathbf{V}_s$ is fixed.
2. The modulus v_0 of the observer velocity is fixed ($\|\mathbf{V}_0\|^2 \triangleq v_0^2$).

First, recall the basic expression for the bearing-rate $\dot{\theta}$:

$$\dot{\theta} = \frac{1}{r^2}\det(\mathbf{V}, \mathbf{R}),$$

where $\mathbf{V}$ and $\mathbf{R}$ are the relative velocity and position vectors.

From the above formula $\dot{\theta}$ is maximum when $\mathbf{V}$ is orthogonal to $\mathbf{R}$, i.e.

$$\mathbf{V} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \frac{\alpha}{r}\begin{pmatrix} r_y \\ -r_x \end{pmatrix}. \tag{A.1}$$

[13] On a leg, the following formula holds: $\ddot{\theta} = -2(\dot{r}/r)\dot{\theta}$.

We thus have

$$\|\mathbf{V}\|^2 = \alpha^2 = \|\mathbf{V}_0\|^2 + v_s^2 - 2\mathbf{V}_0^*\mathbf{V}_s$$
$$= -v_0^2 + v_s^2 + 2\mathbf{V}_0^*\mathbf{V}$$
$$= (v_s^2 - v_0^2) + 2\frac{\alpha}{r}\mathbf{V}_0^* J\mathbf{R},$$

where J is the $\pi/2$ 2D-rotation matrix. The scalar α is thus defined as one of the solutions of the second-order equation

$$\alpha^2 - \frac{2}{r}(\mathbf{V}_0^* J\mathbf{R})\alpha + (v_0^2 - v_s^2) = 0. \tag{A.2}$$

Expliciting the scalar product $\mathbf{V}_0^* J\mathbf{R}$, i.e. ($\mathbf{V}_0^* J\mathbf{R} = \alpha r + \mathbf{V}_s^* J\mathbf{R}$), the previous equation becomes

$$-\alpha^2 - 2\alpha\frac{\mathbf{V}_0^* J\mathbf{R}}{r} + (v_0^2 - v_s^2) = 0. \tag{A.3}$$

If the modulus of the observer velocity (v_0 is superior to the source one (v_s), the above equation has two roots, α_1 and α_2, of opposed signs and the following inequality holds:

$$-\dot{\theta}_{\max} \le \dot{\theta} \le \dot{\theta}_{\max},$$

where (A.4)

$$\dot{\theta}_{\max} = \alpha_1, \qquad -\dot{\theta}_{\max} = \alpha_2.$$

Appendix B: An optimal control approach

This appendix deals with the optimal control problem of the Section 3.2 and, more precisely, with the solution of the optimization problem associated with (3.7).

Expliciting the PMP optimality conditions, we have:

$$\frac{\partial H}{\partial u} = -\frac{\partial}{\partial u}\left(\frac{\dot{\theta}^2}{r^2}\right) + \lambda_1 v\cos u - \lambda_2 v \sin u,$$
$$\frac{\partial H}{\partial r_x} = -\frac{1}{r^4}\left[\frac{\partial \dot{\theta}^2}{\partial r_x} r^2 - \frac{\partial r^2}{\partial r_x}\dot{\theta}^2\right]. \tag{B.1}$$

Owing to the classical relations $(v_0 \gg v_s)\dot{\theta} = v/r\sin(u-\theta)$, $\dot{r} = v\cos(u-\theta)$, $\partial H/\partial u$ is easily calculated yielding:

$$\frac{\partial H}{\partial u} = -2\frac{\dot{r}\dot{\theta}}{r^3} + \lambda_1 v\cos u - \lambda_2 v\sin u. \tag{B.2}$$

Using classical differential calculus, $\partial\dot{\theta}/\partial r_x$ and $\partial\dot{\theta}/\partial r_y$ are explicited:

$$\frac{\partial\dot{\theta}}{\partial r_x} = -\frac{v}{r^2}\cos(2\theta - u),$$
$$\frac{\partial\dot{\theta}}{\partial r_y} = \frac{v}{r^2}\sin(2\theta - u). \tag{B.3}$$

From Eqs (B.1) and (B.3), the following equalities are easily deduced:

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial r_x} = -\frac{2v}{r^4}\sin(u-\theta)\left[\frac{v}{r}\cos(2\theta - u) + \sin(\theta)\right],$$
$$\dot{\lambda}_1 = -\frac{\partial H}{\partial r_y} = -\frac{2v}{r^4}\sin(u-\theta)\left[\frac{-v}{r}\sin(2\theta - u) + \cos(\theta)\right]. \tag{B.4}$$

Thus, the time derivatives of the costates λ_1 and λ_2 satisfy the following equation:

$$\dot{\lambda}_1 r_y - \dot{\lambda}_2 r_x = -2\frac{\dot{\theta}\dot{r}}{r^3}. \tag{B.5}$$

Then, integrating by parts the optimality condition $\partial H/\partial u^* = 0$, we obtain

$$\lambda_1\cos\theta - \lambda_2\sin\theta = 0. \tag{B.6}$$

It remains to determine the costates λ_1 and λ_2. Differentiating (versus time) Eq. (B.6) yields

$$\dot{\lambda}_1\cos\theta - \dot{\lambda}_2\sin\theta - (\lambda_1\sin\theta + \lambda_2\cos\theta)\dot{\theta} = 0.$$

Now

$$\dot{\lambda}_1\cos\theta - \dot{\lambda}_2\sin\theta = -2\frac{\dot{\theta}\dot{r}}{r^4},$$

so that, (B.6):

$$\lambda_1 = -2\frac{\dot{r}}{r^4}\sin\theta \quad \text{and} \quad \lambda_2 = -2\frac{\dot{r}}{r^4}\cos\theta. \tag{B.7}$$

From Eq. (B.7), we immediately deduce that

$$\dot{\lambda}_1 = -2\frac{v\theta}{r^4}\cos(2\theta - u) + 8\frac{\dot{r}^2}{r^5}\sin\theta + \frac{2}{r^3}\dot{u}\dot{\theta}\sin\theta. \tag{B.8}$$

Collecting Eqs. (B.8) with (B.4), yields

$$4\left(\frac{\dot{r}}{r}\right)^2 + \dot{u}\dot{\theta} = -1. \tag{B.9}$$

Appendix C: The bearing-rate change polynomials

Explicit expressions of the polynomials $\{P_i(\ell_1, \ell_2)\}_{i=1}^5$ are given below:[14,15]

$$P_1 = (\alpha^{-1})\ell_1^2(n-\ell_1)^2[5\ell_1^8 + 46\ell_1^7 n - 57\ell_1^6 n^2 + 32\ell_1^5 n^3 - 113\ell_1^4 n^4 + 102\ell_1^3 n^5 - 19\ell_1^2 n^6 + 4\ell_1 n^7],$$
$$P_2 = (\beta^{-1})\ell_1^2(n-\ell_1)^2[-10\ell_1^8 - 56\ell_1^7 n + 87\ell_1^6 n^2 - 42\ell_1^5 n^3 + 108\ell_1^4 n^4 - 102\ell_1^3 n^5 + 19\ell_1^2 n^6 - 4\ell_1 n^7],$$
$$P_3 = (\alpha^{-1})\ell_1^2[30\ell_1^{10} - 270\ell_1^8 n^2 + 480\ell_1^7 n^3 - 405\ell_1^6 n^4 + 360\ell_1^5 n^5 - 316\ell_1^4 n^6 + 144\ell_1^3 n^7 - 27\ell_1^2 n^8 + 4\ell_1 n^9],$$

[14] These results are obtained by means of symbolic computations.
[15] n is the *total* number of measurements.

$$P_4 = (\beta^{-1})\ell_1^6[-10\ell_1^6 + 36\ell_1^5 n - 9\ell_1^4 n^2 - 48\ell_1^3 n^3 + 36\ell_1^2 n^4 + 5\ell_1^6],$$

$$P_5 = (\alpha^{-1})\ell_1^6[5\ell_1^6 - 36\ell_1^5 n + 54\ell_1^4 n^2 - 32\ell_1^3 n^3 + 9\ell_1^2 n^4], \tag{C.1}$$

where

$$\alpha = 25920, \quad \beta = 12960, \quad n = \ell_1 + \ell_2.$$

Using Eq. (C.1), the following equality is easily shown:

$$\sum_{i=1}^{5} P_i(\ell_1, \ell_2) x^{5-i} y^{i-1} = 0 \quad \forall x, \quad y = x, \ell_1, \ell_2. \tag{C.2}$$

Appendix D: A proof of the bang-bang property

The aim of this section is to prove Proposition 5. For the clarity of presentation, the analysis will be first restricted to the case of a unique bearing-rate change. Then, it will be extended to the general case.

In the unique bearing-rate change case (x and y are the consecutive bearing rates), the maximization of the FIM determinant reverts to the following optimization problem (see Eqs. (4.15)–(4.17)):

$$\begin{aligned} &\min f(x, y) \\ &f(x, y) = \alpha x^2 (y - x)^2 + \beta y^2 (y - x)^2, \\ &x - 1 \le 0, \quad y - 1 \le 0, \\ &-x - 1 \le 0, \quad -y - 1 \le 0 \\ &\alpha \text{ and } \beta < 0. \end{aligned} \tag{D.1}$$

The Lagrangian of the above problem is then

$$L(x, y) = f(x, y) + u_1(x - 1) + u_2(-x - 1) + v_1(y - 1) + v_2(-y - 1).$$

Assume that the solution to the problem lies (strictly) inside the square of constraints, then the stationary (Kuhn–Tucker) conditions imply

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2\alpha x(y - x)^2 - 2(\alpha x^2 + \beta y^2)(y - x) = 0, \\ \frac{\partial f}{\partial y} &= 2\beta y(y - x)^2 + 2(\alpha x^2 + \beta y^2)(y - x) = 0. \end{aligned} \tag{D.2}$$

In particular, the optimal values of x and y must satisfy

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 2(y - x)^2[\alpha x + \beta y] = 0,$$

so, that ($y \ne x$):

$$\alpha x + \beta y = 0,$$

whence

$$f(x, y) = \left(\frac{\alpha + \beta}{\beta}\right)^2 \frac{\alpha}{\beta}(\beta + \alpha) x^4. \tag{D.3}$$

Now, both α and β are negative, so is the term $(\alpha/\beta)(\beta + \alpha)$. Thus, the minimum of $f(x, y)$ cannot be inside the square of constraints which implies that the minimum of $f(x, y)$ can be achieved only on the boundary of the constraint domain and more precisely for an extreme point. Thus, the function $f(x, y)$ is minimum for $x = 1$, $y = -1$. Proposition 5 is thus proved in this case.

Extension to the general case is a bit more intricated, even if the basic idea is very similar. Consider, for instance, the case of three bearing-rates (x, y, z) and the following functional $f(x, y, z)$:

$$f(x, y, z) = \alpha x^2 (y - x)^2 + \alpha' y^2 (y - x)^2 + \beta y^2 (z - y)^2, + \beta' z^2 (z - y)^2 + \gamma z^2 (z - x)^2 + \gamma' x^2 (z - x)^2.$$

We are now dealing with the following optimization problem:

$$\begin{aligned} &\min f(x, y, z) \\ &x - 1 \le 0, \ y - 1, \quad z - 1 \le 0, \\ &-x - 1 \le 0, \quad -y - 1, \quad -z - 1 \le 0 \\ &\alpha, \alpha', \dots, \gamma, \gamma' < 0. \end{aligned} \tag{D.4}$$

For an *interior point*, the Lagrangian stationarity then implies

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0,$$

or, explicitly

$$[\alpha(y - x)^2 + \gamma'(z - x)^2]x + [\alpha'(y - x)^2 + \beta(z - y)^2]y + [\gamma(z - x)^2 + \beta'(z - y)^2]z = 0,$$

whence

$$z = \delta_1 x + \delta_2 y,$$

with

$$\delta_1 = -\left(\frac{\alpha(y - x)^2 + \gamma'(z - x)^2}{\gamma(z - x)^2 + \beta'(z - y)^2}\right), \quad \delta_2 = -\left(\frac{\alpha'(y - x)^2 + \beta(z - y)^2}{\gamma(z - x)^2 + \beta'(z - y)^2}\right). \tag{D.5}$$

From Eq. (D.5), we immediately deduce that δ_1 and δ_2 are negative. The following expressions of the elementary terms of $f(x, y, z)$ are then obtained:

$$\begin{aligned} \beta y^2 (z - y)^2 &= \beta y^2 [\delta_1 x + (\delta_2 - 1) y]^2 = f_1(x, y), \\ \gamma' x^2 (z - x)^2 &= \gamma' x^2 [(\delta_1 - 1) x + \delta_2 y]^2 = f_2(x, y), \\ \beta' z^2 (z - y)^2 &= \beta' (\delta_1 x + \delta_2 y)^2 [\delta_1 x + (\delta_2 - 1) y]^2 = f_3(x, y), \\ \gamma z^2 (z - y)^2 &= \gamma (\delta_1 x + \delta_2 y)^2 [(\delta_1 - 1) x + \delta_2 y]^2 = f_4(x, y). \end{aligned} \tag{D.6}$$

The following simple remarks are then instrumental: $\beta, \beta', \gamma, \gamma'$ are altogether negative, as well as $\delta_1, \delta_2, \delta_1 - 1,$

$\delta_2 - 1$. Using the previous reasoning, we thus see that the minimum of $\{f_i(x, y)\}_{i=1}^4$ is attained for $x = 1$, $y = -1$. Since the same (minimum) property is also verified by the functions $\alpha x^2(y-x)^2$ and $\alpha' x^2(y-x)^2$, the minimum of $f(x, y, z)$ is *necessary* attained on the boundary of the cube of constraints. Hence we easily deduce that the minimum of f is attained for an extreme point of the boundary, i.e. $x = 1$, $y = -1$, $z = 1$. Note that the values of $f(x, y, z)$ are identical on the points $x = 1$, $y = -1$, $z = 1$ and $x = -1$, $y = 1$, $z = -1$.

By recursion, this reasoning is easily extended to higher dimension.

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