Properties of estimability criteria for target motion analysis

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Abstract: A significant amount of study has been devoted to the observability analysis for passive target motion analysis. A unified presentation of observability and estimability is provided. Using a common formalism, explicit results are thus obtained.

1 Introduction

This study concerns target motion analysis (TMA) when the system state is not directly observed. A classical example is that of passive TMA where measurements are only made of estimated bearings. Such systems are employed in passive sonar, infrared tracking or electronic warfare.

The primary aim of this paper is to show that, in the TMA context, strong similarities exist between the (classical) observability concepts and the estimability. A classical observability criterion is a nonlinear differential equation which can be obtained by 'brute force' calculations (or symbolic computation). We refer to [1–4] for a general presentation of the problem. For the sake of clarity we shall apply the notation used in [1].

We shall prove that this criterion may be explicitly obtained by means of elementary multilinear algebra thus revealing its basic nature and interest. As a by-product of this result, we shall see that a local estimability criterion may be deduced. In contrast with the analysis of the observability, the problem here consists in determining the system design (controls, multiple receivers etc.) which maximises the related cost function. The associated optimisation problems are considered within a common framework for various applications: planar TMA, multiple observers, manoeuvring source, utilisation of various measurements (TDOA, Doppler) and (local) optimisation of the receiver manoeuvres. The connections between estimation and control then become evident.

2 Problem formulation and main result

The equations of motion for a constant velocity target may be expressed in the form [1]:

$$\mathbf{r}(t) = \mathbf{r}(0) + t\mathbf{v}(0) - \int_0^t (t-\tau)\mathbf{a}_o(\tau)d\tau \qquad (1)$$

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where the reference time is 0, $\mathbf{r}$ and $\mathbf{v}$ are, respectively, the relative target range and velocity vectors, and $\mathbf{a}_0$ describes own-ship accelerations.

If we are concerned with observability, only noise-free data measurements will be considered. For bearings-only (planar problem) TMA, such measurements consist of line-of-sight angles which satisfy the following relation:

$$\beta(t) = \tan^{-1}[r_x(t)/r_y(t)] \qquad (2)$$

Note that eqn. 2 takes the following form eqn. 1:

$$\mathbf{M}_t^* \mathbf{X} = y(t)$$

where, using matrix (vector) transposition (denoted by *),

$$\mathbf{X} = \begin{pmatrix} \mathbf{r}(0) \\ \mathbf{v}(0) \end{pmatrix}$$

$$\mathbf{M}_t^* = (\cos\beta_t, -\sin\beta_t, t\cos\beta_t, -t\sin\beta_t)$$

$$y(t) = \int_{t_0}^t (t-\tau)[a_{ox}(\tau)\cos\beta_t - a_{oy}(\tau)\sin\beta_t]d\tau \qquad (3)$$

Since the problem consists in determining the (initial) state vector $\mathbf{X}$ for the measurements β_t the following observability criterion has been considered.

Let $A(t)$ be the (4×4) matrix defined as

$$A(t) = (\mathbf{M}_t, \mathbf{M}_t^{(1)}, \mathbf{M}_t^{(2)}, \mathbf{M}_t^{(3)})$$

where $\mathbf{M}_t^{(i)}$ denotes the ith time derivative of the vector $\mathbf{M}_t$. Then, a classical analysis [1–3] of the observability of the system, defined by eqns. 1 and 2, consists in examining the rank of the matrix $A(t)$ for a given t. More precisely, we say that the TMA system is observable if the matrix $A(t)$ is full rank [1, 5] for at least one value of t. Note that this observability criterion appears only as a local one at a first glance. Even if more sophisticated approaches are considered, we shall see that this simple approach can yield interesting results.

As we shall now see, a 'measure' of the system estimability can be easily derived from this observability criterion. For that purpose consider eqn. 2, and assume that the measurements are noisy (additive Gaussian white noise with variance σ^2), then under this hypothesis and assuming that the relative range r (approximately) is constant (note: The aim of prop. 2 (below) is precisely to show that this assumption is, in fact, not restrictive), the Fisher information matrix (FIM) relative to the estimation of the $\mathbf{X}$ vector takes the following classical form [5]:

$$\mathrm{FIM}(t, t+k) = (\sigma r)^{-2}\mathcal{M}(t, t+k)\mathcal{M}^*(t, t+k)$$

where

$$\mathcal{M}(t, t+k) = (\mathbf{M}_t, \mathbf{M}_{t+1}, \ldots, \mathbf{M}_{t+k}), \quad k \geq 3 \quad (4)$$

The determinant of the FIM is a convenient 'measure' for the estimability of the problem. We refer, for instance, to the numerous papers devoted to D-optimal design [6, 7]. Using eqn. 4, the calculation of det(FIM(t, t + 3)) is direct, yielding:

$$\det(\mathrm{FIM}(t, t+3)) = (\sigma r)^{-8}[\det(\mathcal{M}(t, t+3))]^2$$

A local approximation of this determinant may be calculated by considering an expansion of the vectors $\mathbf{M}_{t+i}$. For instance, let us consider the following third order expansion of $\mathbf{M}_{t+i}$:

$$\mathbf{M}_{t+i} \stackrel{3}{=} \mathbf{M}_t + i\mathbf{M}_t^{(1)} + \frac{i^2}{2}\mathbf{M}_t^{(2)} + \frac{i^3}{6}\mathbf{M}_t^{(3)} \qquad (5)$$

Note that the above expansion must be considered as an expansion of $\mathbf{M}_{t+i}$, relative to the arbitrary small sampling time τ_e (i.e. the time separating two consecutive measurements). For the sake of brevity, τ_e is omitted ($\tau_e \equiv 1$).

Using exterior algebra and the above expansion, the following basic result is obtained:

Property 1

Consider a third order expansion of the vectors $\mathbf{M}_{t+i}$, then

$$\det(\mathcal{M}(t, t+3)) \stackrel{3}{=} \det(\mathbf{M}_t, \mathbf{M}_t^{(1)}, \mathbf{M}_t^{(2)}, \mathbf{M}_t^{(3)})$$

Proof: First, we briefly recall the definition of the exterior powers of a vector space. (For a complete presentation, we refer, for example to [8, 9].) Let V be an n-dimensional vector space over $\mathbf{R}$, then $\Lambda^2 V$ consists of all formal sums $\Sigma_i \alpha_i$ ($\mathbf{U}_i \wedge \mathbf{V}_j$), where the 'wedge product' $\mathbf{U} \wedge \mathbf{V}$ is bilinear and alternate. This definition is straightforwardly extended to higher exterior powers [8]. For any basis $\{\mathbf{V}_1, ..., \mathbf{V}_n\}$ of V, the set of p-vectors $\{\mathbf{V}_{i1} \wedge ... \wedge \mathbf{V}_{ip}, i_1 < ... < i_p \leq n\}$ forms a basis of the $n!/(n-p)!p!$-dimensional vector space $\Lambda^p V$. In particular, $\Lambda^4\mathbf{R}^4$ is one-dimensional, and throughout the paper we make intensive use of the isomorphism $\Lambda^4\mathbf{R}^4 \equiv \Lambda^2\mathbf{R}^4 \wedge \Lambda^2\mathbf{R}^4$. The exterior algebra formalism thus appears to be an economical way to conduct determinant calculations.

Denoting $\mathbf{M} \wedge \mathbf{N}$ the elements of the exterior product $\Lambda^2\mathbf{R}^4$, we have:

$$\det(\mathcal{M}(t, t+3)) = (\mathbf{M}_t \wedge \mathbf{M}_{t+1}) \wedge (\mathbf{M}_{t+2} \wedge \mathbf{M}_{t+3}) \qquad (6)$$

It remains to calculate the two vectors $\mathbf{M}_t \wedge \mathbf{M}_{t+1}$ and $\mathbf{M}_{t+2} \wedge \mathbf{M}_{t+3}$ of the exterior power [8] $\Lambda^2\mathbf{R}^4$. Invoking the basic properties of exterior algebra, we obtain straightforwardly [8, 9]:

$$\begin{aligned}\mathbf{M}_t \wedge \mathbf{M}_{t+1} &= \mathbf{M}_t \wedge \mathbf{M}_t^{(1)} + \frac{1}{2}\mathbf{M}_t \wedge \mathbf{M}_t^{(2)} \\ &\quad + \frac{1}{6}\mathbf{M}_t \wedge \mathbf{M}_t^{(3)} \\ \mathbf{M}_{t+2} \wedge \mathbf{M}_{t+3} &= 3\mathbf{M}_t^{(1)} \wedge \mathbf{M}_t^{(2)} + 3\mathbf{M}_t^{(2)} \wedge \mathbf{M}_t^{(3)} \\ &\quad + 5\mathbf{M}_t^{(1)} \wedge \mathbf{M}_t^{(3)} \qquad (7)\end{aligned}$$

Note that the terms involving $\mathbf{M}_t$ in $\mathbf{M}_{t+2} \wedge \mathbf{M}_{t+3}$ are not considered since their contribution in det($\mathcal{M}$(t, t + 3)) is null. Then from eqn. 7, we deduce det($\mathcal{M}$(t, t + 3)):

$$\begin{aligned}\det(\mathcal{M}(t, t+3)) &= 3\mathbf{M}_t \wedge \mathbf{M}_t^{(1)} \wedge \mathbf{M}_t^{(2)} \wedge \mathbf{M}_t^{(3)} \\ &\quad + \frac{5}{2}\mathbf{M}_t \wedge \mathbf{M}_t^{(2)} \wedge \mathbf{M}_t^{(1)} \wedge \mathbf{M}_t^{(3)} \\ &\quad + \frac{1}{2}\mathbf{M}_t \wedge \mathbf{M}_t^{(3)} \wedge \mathbf{M}_t^{(1)} \wedge \mathbf{M}_t^{(2)}\end{aligned}$$

so that, finally:

$$\begin{aligned}\det(\mathcal{M}(t, t+3)) &= \det(\mathbf{M}_t, \mathbf{M}_t^{(1)}, \mathbf{M}_t^{(2)}, \mathbf{M}_t^{(3)}) \\ &= \det(A(t)) \qquad (8)\end{aligned}$$

This result is surprisingly simple. The observability criterion of Nardone and Aidala has thus received another interpretation. Furthermore, we shall prove (see the main result) that this determinant is independent of t and β_t. Practically, this means that its calculation is quite direct.

In terms of observability, this result proves that if det($A(t)$) is non-null then $\mathcal{M}_{(t,t+3)}$ is invertible since its determinant is non-null which means that the problem is observable. The observability criterion of Nardone and Aidala is, therefore, quite justified.

Considering a third order expansion of the relative range r_t, we shall prove that the calculation of det($\mathrm{FIM}_{t,t+k}$) is unchanged. The hypothesis of (approximately) constant relative range can thus be removed.

Property 2

Consider a third order expansion of the vectors $\mathbf{M}_{t+i}$ and of the relative range r_{t+i}, then

$$\det(\mathrm{FIM}_{t,t+3}) \stackrel{3}{=} (\sigma r_t)^{-8}[\det(\mathbf{M}_t, \mathbf{M}_t^{(1)}, \mathbf{M}_t^{(2)}, \mathbf{M}_t^{(3)})]^2$$

Proof: The FIM takes the following general form:

$$\mathrm{FIM} = (\mathbf{G}_t, \ldots, \mathbf{G}_{t+3})(\mathbf{G}_t, \ldots, \mathbf{G}_{t+3})^*$$

where:

$$\mathbf{G}_{t+i} = \frac{1}{r_{t+1}}\mathbf{M}_{t+i}$$

(In fact, $\mathbf{G}_{t+i} = 1/\sigma r_{t+i}\mathbf{M}_{t+i}$, but the coefficient σ is omitted for the sake of brevity.)

We can then invoke property 1, thus obtaining

$$\det(\mathrm{FIM}) = [\det(\mathbf{G}_t, \ldots, \mathbf{G}_t^{(3)})]^2$$

The calculation of the derivative vector $\mathbf{G}_t^{(i)}$ (i = 1, 2, 3)) is straightforward, yielding:

$$\begin{aligned}\mathbf{G}^{(1)} &= \frac{1}{r}\mathbf{M}^{(1)} - \frac{g}{r}\mathbf{M} \\ \mathbf{G}^{(2)} &= \frac{1}{r}\mathbf{M}^{(2)} - 2\frac{g}{r}\mathbf{M}^{(1)} + \left(\frac{g^2}{r^2} - \frac{g^{(1)}}{r}\right)\mathbf{M} \\ \mathbf{G}^{(3)} &= \frac{1}{r}\mathbf{M}^{(3)} - 3\frac{g}{r}\mathbf{M}^{(2)} + 3\left(\frac{g^2}{r^2} - \frac{g^{(1)}}{r}\right)\mathbf{M}^{(1)} \\ &\quad + \left(3\frac{g^{(1)}g}{r^2} - 2\frac{g^3}{r^2} - \frac{g^{(2)}}{r}\right)\mathbf{M}, \quad g = \dot{r}/r \qquad (9)\end{aligned}$$

(For the sake of simplicity the time index t is omitted in the above formula.) Now, the following equalities are direct consequences of exterior algebra properties:

$$\begin{aligned}\mathbf{G} \wedge \mathbf{G}^{(1)} &= \frac{1}{r^2}\mathbf{M} \wedge \mathbf{M}^{(1)} \\ \mathbf{G}^{(2)} \wedge \mathbf{G}^{(3)} &= \frac{1}{r^2}\mathbf{M}^{(2)} \wedge \mathbf{M}^{(3)} + \text{other terms} \qquad (10)\end{aligned}$$

The 'other terms' is only formed of exterior products of either the vector $\mathbf{M}$ or $\mathbf{M}^{(1)}$ with another vector ($\mathbf{M}^{(i)}$, i = 0, 1, 2). Since $\mathbf{G} \wedge \mathbf{G}^{(1)} = (1/r^2)\mathbf{M} \wedge \mathbf{M}^{(1)}$, the contribution of the 'other terms' in the calculation of det(FIM) is null. Property 2 is thus proved. We stress that this property is essentially due to the fact that the term $1/r$ appears as a multiplicative factor. As we shall see in Section 3.4, this is not valid for Doppler measurements.

The calculation of the approximation of $\det(\mathrm{FIM}_{t,t+k})$, where $k \geq 4$ is also surprisingly simple. Using exterior algebra and more precisely the Binet–Cauchy formula, (assuming that σ and r are constant for the duration of the analysis, i.e. $\{t, ..., t+k\}$) [10], we obtain

$$\det(\mathrm{FIM}_{t,t+k}) = (\sigma r_t)^{-8} \sum_{0 \leq i_1 < i_2 < i_3 < i_4 \leq k} [\det(\mathbf{M}_{t+i_1}, \ldots, \mathbf{M}_{t+i_4})]^2$$

with

$$\det(\mathbf{M}_{t+i_1}, \ldots, \mathbf{M}_{t+i_4}) = P_{(i_1,i_2,i_3,i_4)}(\det A(t))$$

so that

$$\begin{aligned}\det(\mathrm{FIM}_{t,t+k}) &= (\sigma r_t)^{-8} \left[\sum_{0 \leq i_1 < i_2 < i_3 < i_4 \leq k} P^2_{(i_1,i_2,i_3,i_4)} \right] (\det(A(t)))^2 \\ &= c_k(\det(A(t)))^2 \end{aligned} \quad (11)$$

In eqn. 11, $P(_{i1}, _{i2}, _{i3}, _{i4})$ is a polynomial in i_1, i_2, i_3, i_4 of degree homogeneously equal to 12.

The problem we deal with is to obtain an explicit formula of $\det A(t)$. For that purpose the following result is instrumental:

Property 3

$\det A(t)$ is independent of the value of β_t, furthermore we can consider that the reference time is zero.

Proof:

Using the Kronecker product ($\otimes$), define the matrix $\mathcal{R}_t$ as follows:

$$\mathcal{R}_t = \begin{pmatrix} R_t & 0 \\ tR_t & R_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \otimes R_t$$

where

$$R_t = \begin{pmatrix} \cos\beta_t & \sin\beta_t \\ -\sin\beta_t & \cos\beta_t \end{pmatrix} \quad (12)$$

Then, the following equality is straightforwardly verified:

$$\mathbf{M}_t = \mathcal{R}_t \mathbf{E}_1, \quad \mathbf{E}_1 = (1,0,0,0)^*$$

hence,

$$\mathbf{M}_t^{(1)} = \mathcal{R}_t^{(1)} \mathbf{E}_1, \ldots, \mathbf{M}_t^{(3)} = \mathcal{R}_t^{(3)} \mathbf{E}_1 \quad (13)$$

Furthermore, the following equalities are straightforwardly deduced from eqn. 12:

$$\begin{aligned} \mathcal{R}_t &= C_t \otimes R_t \\ \mathcal{R}_t^{(1)} &= \beta_t^{(1)}(C_t \otimes R_t^{(1)}) + D \otimes R_t \\ \mathcal{R}_t^{(2)} &= \beta_t^{(2)}(C_t \otimes R_t^{(1)}) - \beta_t^{(1)}\mathcal{R}_t + (\beta_t^{(1)}+1)(D \otimes R_t^{(1)}) \\ \mathcal{R}_t^{(3)} &= \beta_t^{(3)}(C_t \otimes R_t^{(1)}) - 2\beta_t^{(2)}\mathcal{R}_t + \cdots \end{aligned}$$

where

$$C_t \triangleq \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad D \triangleq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (14)$$

The above expression may be somewhat simplified by means of the following remark:

$$R_t^{(1)} = R_t J \text{ where } J \triangleq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (15)$$

Using the multilinearity property of the determinant we deduce, from eqns. 8, 13 and 14, that $\det A(t)$ is a sum of elementary expressions of the type

$$\det[\mathcal{R}_t\mathbf{E}_1, (C_t \otimes R_t J)\mathbf{E}_1, (D \otimes R_t J)\mathbf{E}_1, (C_t \otimes R_t J)\mathbf{E}_1] \text{ etc.} \quad (16)$$

The following classical property of the tensor product is then instrumental [9]:

$$(H_1F_1) \otimes (H_2F_2) = (H_1 \otimes H_2)(F_1 \otimes F_2) \quad (17)$$

where H and F are endomorphisms of the state space.

Applying this general property to expr. 16, yields:

$$\begin{aligned} \mathcal{R}_t &= C_t \otimes R_t \\ &= (IdC_t) \otimes (R_t Id) \\ &= (Id \otimes R_t)(C_t \otimes Id) \end{aligned} \quad (18)$$

where Id is the identity matrix. Similarly,

$$\begin{aligned} C_t \otimes R_t J &= (IdC_t) \otimes (R_t J) \\ &= (Id \otimes R_t)(C_t \otimes J) \ldots \end{aligned} \quad (19)$$

Thus, each of the terms (expr. 16) admits the following factorisation ($\det(A \otimes B) = (\det A)^2 (\det B)^2$, A and B 2×2 matrices):

$$\begin{aligned} \text{Expr.16} \equiv (\det R_t)^2 \det[&(C_t \otimes Id_2)\mathbf{E}_1, (C_t \otimes J)\mathbf{E}_1, \\ &(D \otimes Id_2)\mathbf{E}_1, (C_t \otimes J)\mathbf{E}_1] \end{aligned} \quad (20)$$

Since $\det R_t$ is equal to 1 we deduce from expr. 16 and eqn. 20 that $\det A(t)$ itself is independent of β_t.

The last step is proved by invoking the same property of tensor products. More precisely, the following factorisations are obtained:

$$\begin{aligned} C_t \otimes J &= (C_t \otimes Id)(Id \otimes J) \\ D \otimes Id &= (C_t D) \otimes Id = (C_t \otimes Id)(D \otimes Id) \end{aligned} \quad (21)$$

from which the following equality holds:

$$\begin{aligned} \text{Expr.16} = \det(C_t \otimes Id) \det[&\mathbf{E}_1, (Id \otimes J)\mathbf{E}_1, \\ &(D \otimes Id_2)\mathbf{E}_1, (Id \otimes J)\mathbf{E}_1] \end{aligned} \quad (22)$$

Since the above reasoning holds for any expression of the type in expr. 16, property 3 is thus proved.

3 Applications

We shall consider various applications of the general results obtained in the previous Section.

3.1 Classical planar problem and some extensions

We shall first restrict our attention to the classical TMA problem: the source and the observer are moving on the same plane. The measurement is the bearing β.

Assuming a rectilinear and uniform motion of the source, $\det A(t)$ may be calculated by differentiating (with respect to time) the vector $\mathbf{M}_t$ for $\beta_t = 0$ and a reference time equal to 0, yielding:

$$\mathbf{M}_t = \left| \begin{array}{l} \cos(\beta_t) \\ -\sin(\beta_t) \\ t\cos(\beta_t) \\ -t\sin(\beta_t) \end{array} \right. \quad (23a)$$

$$\mathbf{M}_t^{(1)} = \left| \begin{array}{r} -\sin(\beta_t)\dot{\beta}_t \\ -\cos(\beta_t)\dot{\beta}_t \\ -t\sin(\beta_t)\dot{\beta}_t + \cos(\beta_t) \\ -t\cos(\beta_t)\dot{\beta}_t - \sin(\beta_t) \end{array} \right. \quad (23b)$$

Taking $t = 0$ and $\beta_t = 0$ (see the main result) in the expressions of $\{\mathbf{M}_t, \mathbf{M}_t^{(1)}, ., \mathbf{M}_t^{(3)}\}$, we thus obtain

$$\det A(t) = \det(\mathcal{M}_{t,t+3})_{(t=0;\beta_t=0)}$$

$$\det A(t) = \det\begin{pmatrix} -\dot\beta_t & -\beta_t^{(2)} & \dot\beta_t^3 - \beta_t^{(3)} \\ 1 & 0 & -3\dot\beta_t^2 \\ 0 & -2\dot\beta_t & -3\beta_t^{(2)} \end{pmatrix}$$

$$= 4\dot\beta_t^4 + 2\dot\beta_t\beta_t^{(3)} - 3(\beta_t^{(2)})^2 \quad (24)$$

whence

$$\det(\mathrm{FIM}_{t,t+k}) = c_k[4\dot\beta_t^4 + 2\dot\beta_t\beta_t^{(3)} - 3(\beta_t^{(2)})^2]^2 \quad (25)$$

Thus, nullity of det(FIM) is precisely the observability criterion of Nardone and Aidala [1]. Note that despite its apparent complexity, the associated differential equation admits a simple solution [1]. We stress also that we are generally interested in *maximising* (and *not* minimising) an estimability criterion. The Nardone and Aidala criterion is much more than an observability criterion with simply a binary response, it provides also a (local) estimability criterion. The problem is then to determine the controls so as to maximise this criterion. Practically, this is possible since the bearing-rates may be directly estimated from the data. These points will be developed below.

At this point, it is worth recalling the expressions of the various terms of the Nardone and Aidala criterion. Thus, in the absence of observer manoeuvre, elementary calculations yield:

$$\dot\beta = \frac{1}{\|\mathbf{r}\|^2}(\mathbf{r}\wedge\mathbf{v})$$

$$\beta^{(2)} = -2\dot\beta g = \frac{-2}{\|\mathbf{r}\|^4}(\mathbf{r}\wedge\mathbf{v})(\mathbf{r}.\mathbf{v})$$

$$\beta^{(3)} = 6\dot\beta g^2 - 2(\dot\beta)^3 = 2\frac{\mathbf{r}\wedge\mathbf{v}}{\|\mathbf{r}\|^4}\left[\frac{4(\mathbf{r}.\mathbf{v})^2}{\|\mathbf{r}\|^4} - \|\mathbf{v}\|^2\right] \quad (26)$$

where:

$$g = \frac{\dot r}{r} = \frac{\mathbf{r}.\mathbf{v}}{r^2}, \quad \mathbf{r}\wedge\mathbf{v} = \det(\mathbf{r},\mathbf{v})$$

Inserting these expressions of $\dot\beta$, $\beta^{(2)}$, $\beta^{(3)}$ in eqn. 25, we immediately verify that the observability criterion is null in the absence of observer manoeuvre. Moreover, these relations may be presented in a more systematic perspective. More precisely, using eqn. 26 and calculating $\dot g$, we obtain

$$\dot g = \dot\beta^2 - g^2$$

so that

$$\beta^{(3)} = -2\beta^{(2)}g - 2\dot\beta(\dot\beta^2 - g^2) = 6\dot\beta g^2 - 2\dot\beta^3$$
$$\beta^{(4)} = -24\dot\beta g^3 + 24\dot\beta^3 g, \ldots \quad (27)$$

From eqn. 27, we see that $\beta^{(n)}$ can be expressed as an homogeneous polynomial in $\dot\beta$ and g, of degree n. First, this justifies the third order expansion of $\mathbf{G}_t$ since we note (cf. eqn. 23) that the higher order derivatives (in β) of the components of $\mathbf{M}^{(2)}_{(0,0)}$ is $\beta^{(2)}$ or $\dot\beta g$, that of $\mathbf{M}^{(3)}_{(0,0)}$ is $\beta^{(3)}$ or $\lambda\dot\beta g^2 + \mu(\dot\beta)^3$, etc. ... The basic role of $\dot\beta$ and g for the estimability analysis is thus evident. Furthermore, if higher order expansions of the vectors $\mathbf{M}^{(i)}$ are considered, then the dominant term of the associated expansion is still eqn. 25. So, as we shall see below, the definition and the interest of modified polar coordinates (MPC) is not at all fortuitous.

Actually, previous studies have shown that the Cartesian coordinate extended Kalman filter exhibits unstable behaviour in the TMA context. In contrast, formulating the TMA estimation problem in MPC leads to extended Kalman filter which is both stable and asymptotically unbiased [12, 13]. It is then worth considering this problem from a nonlinear point of view [14, 15]. The system equations are

$$\mathbf{v}_t = \mathbf{v}_0 + \int_0^t \mathbf{a}_o(\tau)d\tau$$
$$\mathbf{r}_t = \mathbf{v}_t, \quad \dot{\mathbf{v}}_t = \mathbf{a}_{o,t}$$
$$h_t \triangleq \beta_t = \tan^{-1}\left(\frac{r_x(t)}{r_y(t)}\right) \quad (28)$$

Considering the state equation $\dot{\mathbf{x}} = f(t, \mathbf{x}, \mathbf{a})$ (see eqn. 28), the observation space Θ is the linear space of functions containing h and all repeated Lie derivatives [11], $L_{X_1}L_{X_2} \ldots L_{X_k}h$ where $\mathbf{X}_i$ is a vector field spanned by f.

Restricting our attention to the case where no receiver manoeuvre occurs and denoting $f_1, \ldots, f_4$ the system functions, the Lie derivative of the observation h along the vector field spanned by f is [11]

$$L_f h = \sum_{i=1}^4 f_i \frac{\partial h}{\partial x_i}$$

where, in this case, (time index is omitted):

$$f_1 = v_x, \quad f_2 = v_x, \quad f_3 = f_4 = 0 \quad (29)$$

By a direct calculation, we then prove that the Lie derivatives of β are [11] (no observer manoeuvre), equal to the simple time derivatives of β ($L_f^i\beta = \beta^{(i)}$). Using eqn. 27, the following set inclusion is thus deduced:

$$sp(DL_f^0 h, \ldots, DL_f^k h, \ldots) \subset sp(D_\beta, D_{\dot\beta}, D_g) \quad (30)$$

In eqn. 30 the symbol D denotes the differential [11].

Once again, this result proves that this TMA problem is not observable without observer manoeuvre. It also enlightens the basic role of the modified polar coordinates (β, $\dot\beta$, g and $1/r$) in TMA.

Consider now the case of a source with a constant acceleration [16]. The state vector $\mathbf{X}$ is then six-dimensional ($\mathbf{X} = (\mathbf{x}, \mathbf{v}, \mathbf{a})^*$). Consequently, the $\mathbf{M}_t$ vector is also six-dimensional and similarly to eqn. 24, $\det A(t) = \det(\mathcal{M}_{t,t+5})_{(t=0;\beta_t=0)}$, so that

$$\begin{aligned}\det(\mathrm{FIM}_{t,t+k}) = c_k[&-64\dot\beta^9 + 288\dot\beta^5(\beta^{(2)})^2 \\ &+ 540\dot\beta(\beta^{(2)})^4 + 192\dot\beta^6\beta^{(3)} \\ &- 720\dot\beta^2(\beta^{(2)})^2\beta^{(3)} + 40{\beta^{(3)}}^3 \\ &+ 240\dot\beta^3\beta^{(2)}\beta^{(4)} - 60\beta^{(2)}\beta^{(3)}\beta^{(4)} \\ &+ 15\dot\beta(\beta^{(4)})^2 - 24\dot\beta^4\beta^{(5)} \\ &+ 18(\beta^{(2)})^2\beta^{(5)} - 12\dot\beta\beta^{(3)}\beta^{(5)}]^2\end{aligned}$$

The vector $\mathbf{M}_t$ is only slightly modified, but we note that the above result is considerably more complicated than eqn. 25. Further, we note that det(FIM) is a polynomial in $\dot\beta$, g of degree homogeneously equal to 9. Once again, compare with eqn. 25 where the degree was 4. Denoting $\mathbf{a}$ the acceleration vector, relations eqn. 26 are replaced by the following:

$$\beta^{(2)} = -2\dot\beta g + \frac{1}{r^2}\mathbf{a}\wedge\mathbf{r}$$

$$\beta^{(3)} = 6\dot\beta g^2 - 2\dot\beta^3 - 2\dot\beta\frac{\mathbf{a}^*\mathbf{r}}{r^2} - 4g\frac{\mathbf{a}\wedge\mathbf{r}}{r^2} + \frac{1}{r^2}\mathbf{a}\wedge\mathbf{v}$$

$$\text{etc.} \quad (31)$$

We note that all the additional terms involve the acceleration vector. Quite analogously to the classical planar case, we define the extended modified polar coordinates $\{\beta, \dot{\beta}, g, \mathbf{a} \wedge \mathbf{r}/r^2, \mathbf{a} \wedge \mathbf{v}/r^2, 1/r\}$. The fifth coordinate is a function of the trajectory curvature $\mathbf{v} \wedge \mathbf{a}/v^3$.

Finally, practical considerations plead for including small variations of the source or observer trajectories in the motion model. Actually, this is the general case. Then, an interesting modelling consists in considering independent increments of $\dot{\beta}$, that is,

$$\dot{\beta}_{t+i} = \dot{\beta}_t + w_i, \quad w_i \text{ w.g.n } \mathcal{N}(0, \tau^2)$$

Using exterior algebra, we easily obtain

$$\det(A_t) = 4\sin(\dot{\beta} + w_1)\sin(\dot{\beta} + w_3 - w_2) - \sin(2\dot{\beta} + w_3 - w_1)\sin(2\dot{\beta} + w_2)$$

We are now in position to calculate the mean value of det(FIM) (denoted $\mathbf{E}[\det(\mathrm{FIM}_{t,t+k})]$). More precisely, the expectations of the $\det(A_t)$ are calculated by means of the characteristic functions of $\cos(w_i)$, yielding

$$\mathbb{E}[\det(\mathrm{FIM}_{t,t+3})] \approx \frac{16c_3}{r^8}\exp\left(-\frac{3}{2}\tau^2\right)(\sin(\dot{\beta}))^8 \tag{32}$$

This result may be extended to the general case, the only change being the exponential term which becomes $\exp(-m_k\tau^2)$. We thus see that $(\dot{\beta}/r)^8$ is a convenient upper bound of det(FIM). The effect of small variations is also evident ($(\exp(-3\tau^2/2))$ multiplicative term).

3.2 A local optimisation of the observer manoeuvres

Consider the previous planar problem and let us denote v the modulus of the velocity and u its heading, the equations of the relative motion are

$$\begin{aligned} \dot{r}_x &= v_x = v\sin u - v_{s,x}; \quad r_x = r\sin\beta \\ \dot{r}_y &= v_y = v\cos u - v_{s,y}; \quad r_y = r\cos\beta \end{aligned} \tag{33}$$

yielding

$$\begin{aligned} \dot{\beta} &= \frac{1}{r}[v\sin(u-\beta) + cv] \\ \dot{r} &= v\cos(u-\beta) - cv' \end{aligned}$$

with:

$$\begin{aligned} cv &= v_{s,y}\sin(\beta) - v_{s,x}\cos(\beta) \\ cv' &= -v_{s,y}\cos(\beta) - v_{s,x}\sin(\beta) \end{aligned} \tag{34}$$

In eqn. 34, cv, cv' represent 'cross velocity' terms. Since our objective is to determine the optimal heading u which maximises the cost functional $\mathcal{C} = (4(\dot{\beta})^4 + 2\dot{\beta}\beta^{(3)} - 3(\beta^{(2)})^2)^2$, we must calculate $\beta^{(2)}$, $\beta^{(3)}$. Assuming that v is constant and using the preceding results, we obtain:

$$\begin{aligned} \beta^{(2)} &= -2g\dot{\beta} + g\dot{u} - \frac{\dot{\beta}}{r}cv' - \frac{g}{r}cv \\ \beta^{(3)} &= -2\dot{g}\dot{\beta} - 2g\beta^{(2)} + \dot{g}\dot{u} + gu^{(2)} + \frac{\beta^{(2)}}{r}cv \\ &\quad - 2\frac{g\dot{\beta}}{r}cv + \frac{(\dot{\beta})^2}{r}cv' + \frac{\dot{g}}{r}cv' - \frac{g^2}{r}cv' \end{aligned} \tag{35}$$

where

$$\dot{g} = \left(\frac{\dot{r}}{r}\right)^{(1)} = (\dot{\beta}^2 - g^2) - \dot{\beta}\dot{u} + \frac{\dot{\beta}}{r}cv + \frac{g}{r}cv'$$

Inserting the above expressions of $\beta^{(2)}$ and $\beta^{(3)}$ in $\mathcal{C}$ ($\mathcal{C} \triangleq \det(\mathrm{FIM})$), $\mathcal{C}$ becomes a function of the control u and its derivatives $\dot{u}$ and $u^{(2)}$. The other terms of $\mathcal{C}$ ($1/r$ excepted) can be estimated from the observations, since they only involve β, $\dot{\beta}$, g. The problem is then to determine the heading u and its derivatives which maximise $\mathcal{C}$. Clearly, this can be treated by means of optimal control. However, the derivation of an explicit solution seems hopeless so we refer to [17] for a numerical analysis.

The problem may be considerably simplified if we assume that the receiver velocity is far superior to the source one, so that $v_{s,x}$, $v_{s,y}$ may be neglected, the estimability criterion $\mathcal{C} = [4\dot{\beta}_t^4 + 2\dot{\beta}_t\beta_t^{(3)} - 3(\beta_t^{(2)})^2]^2$ then takes the following form:

$$\mathcal{C} = \left(6\left(\frac{v}{r}\right)^2\dot{u}\dot{\beta} - \dot{u}^2\left[\left(\frac{v}{r}\right)^2 + g^2\right] + 2g\dot{\beta}\ddot{u}\right)^2$$

In all the cases, we stress that this approach is restricted to a local optimisation of the observer manoeuvres.

3.3 Multiple observers

In the absence of observer manoeuvre, the TMA problem is not observable. But, if we consider multiple observers (each one in rectilinear and uniform motion), the TMA problem becomes observable [18]. Consider, for instance, the case of two observers: the scalar observation $y(t)$ is replaced by a vectorial one $\mathbf{y}(t) = (y_1(t), y_2(t))^*$ but the statistical nature of the problem is unchanged. In particular, denote $\mathcal{M}_1$ and $\mathcal{M}_2$ the $\mathcal{M}$ matrices associated with observer 1 and 2, then property 3 still holds, so that

$$\det(\mathrm{FIM}_{t,t+k}) = c(k)\det(\mathcal{M}_1\mathcal{M}_1^* + \mathcal{M}_2\mathcal{M}_2^*) \tag{36}$$

$$= c(k)\det\left[(\mathcal{M}_1, \mathcal{M}_2)\begin{pmatrix}\mathcal{M}_1^* \\ \mathcal{M}_2^*\end{pmatrix}\right] \tag{37}$$

The matrix $\mathcal{M}_1$ and $\mathcal{M}_2$ are not detailed but have the standard form (cf. eqn. 4). Opposite to the previous case, an explicit calculation of $\det(\mathrm{FIM}_{t,t(+k)})$ is not an easy task. However, the Binet–Cauchy formula gives us the dominant terms of det(FIM). More precisely, denoting $\mathrm{col}(\mathcal{M}_i)$ the columns (vectors) of $\mathcal{M}_i$, we have

$$\det(\mathrm{FIM}_{t,t+k}) \approx \Sigma[(\mathbf{M}_1 \wedge \mathbf{M}_2) \wedge (\mathbf{N}_1 \wedge \mathbf{N}_2)]^2 \tag{38}$$

where $\mathbf{M}_i \in \mathrm{col}\ (\mathcal{M}_1)$, $\mathbf{N}_i \in \mathrm{col}\ (\mathcal{M}_2)$ and the terms of the form: $[\mathbf{M}_1 \wedge \mathbf{M}_2 \wedge \mathbf{M}_3 \wedge \mathbf{N}_1]$ are sufficiently small to be neglected.

The calculation of $\det(\mathbf{M}_1, \mathbf{M}_2, \mathbf{N}_1, \mathbf{N}_2)$ is easily achieved by means of exterior algebra. More precisely, the components (denoted resp. α_0, β_0, γ_0) of $\mathbf{M}_1 \wedge \mathbf{M}_2$, in the 'reduced' basis $\{\mathbf{E}_1 \wedge \mathbf{E}_2, \mathbf{E}_1 \wedge \mathbf{E}_3, \mathbf{E}_1 \wedge \mathbf{E}_4\}$ of $\Lambda^2(\mathbf{R}^4)$ are given below with $\mathbf{E}_1, \ldots, \mathbf{E}_4$ canonical basis of $\mathbf{R}^4$.

$$\begin{aligned} \alpha_0 &= \sin((i-j)x) \leftarrow \mathbf{E}_1 \wedge \mathbf{E}_2 \\ \beta_0 &= (j-i)\cos(ix)\cos(jx) \leftarrow \mathbf{E}_1 \wedge \mathbf{E}_3 \\ \gamma_0 &= -j\sin(jx)\cos(ix) + i\sin(ix)\cos(jx) \leftarrow \mathbf{E}_1 \wedge \mathbf{E}_4 \end{aligned}$$

where:

$$\begin{aligned} \mathbf{M}_1 &= (\cos(ix), -\sin(ix), i\cos(ix), -i\sin(ix))^* \\ \mathbf{M}_2 &= (\cos(jx), -\sin(jx), j\cos(jx), -j\sin(jx))^* \\ x &= \dot{\beta}_1 \end{aligned} \tag{39}$$

Similarly, the components (denoted α_1, β_1, γ_1) of $\mathbf{N}_1 \wedge \mathbf{N}_2$, in the 'reduced' basis $\{\mathbf{E}_3 \wedge \mathbf{E}_4, \mathbf{E}_2 \wedge \mathbf{E}_4, \mathbf{E}_2 \wedge \mathbf{E}_3\}$ stand as follows:

$$\alpha_1 = kl\sin((l-k)y) \leftarrow \mathbf{E}_3 \wedge \mathbf{E}_4$$
$$\beta_1 = (l-k)\sin(\alpha+ky)\sin(\alpha+ly) \leftarrow \mathbf{E}_2 \wedge \mathbf{E}_4$$
$$\gamma_1 = -l\cos(\alpha+ly)\sin(\alpha+ky) + k\cos(\alpha+ky)\sin(\alpha+ly) \leftarrow \mathbf{E}_2 \wedge \mathbf{E}_3$$

where:

$$\mathbf{N}_1 = (\cos(\alpha+ky), -\sin(\alpha+ky), k\cos(\alpha+ky), - k\sin(\alpha+ky))^*$$
$$\mathbf{N}_2 = (\cos(\alpha+ly), -\sin(\alpha+ly), l\cos(\alpha+ly), - l\sin(\alpha+ly))^*, y = \dot{\beta}_2 \quad (40)$$

In eqns. 39 and 40, x and y are the bearing-rates ($\dot{\beta}_1$, $\dot{\beta}_2$) on the two platforms. It is frequently assumed that they are equal. The parameter α is the angle between the two vectors joining the platforms and the source. We then have

$$\det(\mathbf{M}_1, \mathbf{M}_2, \mathbf{N}_1, \mathbf{N}_2) = \alpha_0\alpha_1 - \beta_0\beta_1 + \gamma_0\gamma_1 \quad (41)$$

The following approximations are then deduced from eqns. 38–40:

1) $-\alpha \gg x$, then: eqn.(41) $\approx (l-k)(i-j)(\sin\alpha)^2$
2) $\alpha \approx \delta x$, then: eqn.(41) $\approx (l-k)(i-j) \times (\delta+k+i)(\delta+l-i)(\delta+l-j)(\delta+k-j)x^4$ (42)

So, in the first case ($\alpha >> x$), det(FIM) is proportional to m^6 $(\sin\alpha)^4$, while in the second one it is proportional to m^{16} x^8. The interest of a large array baseline ($\alpha >> x$) is thus evident.

A similar analysis can be made if the second set of measurements (i.e. $\{y_2(t)\}$) is replaced by Doppler measurements.

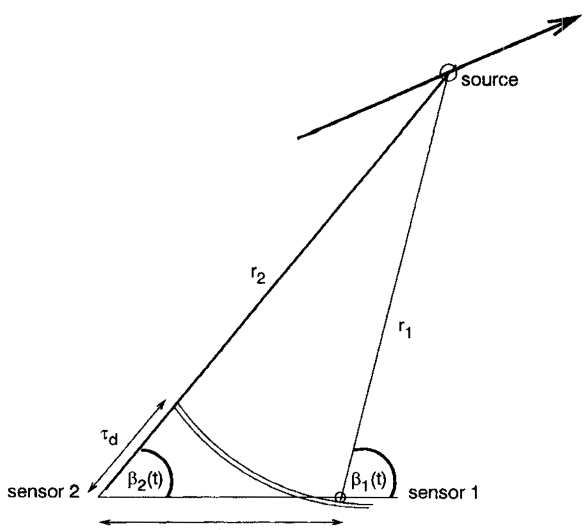

Fig. 1 *Notations for sensor, source positions and trajectory*

3.4 TDOA and TMA

We consider now a passive system where the observations are time delay of arrival (TDOA) between two or more sensors. To begin, consider the two sensor system represented in Fig. 1, with the notations of Section 2. The source trajectory is determined by its state vector **X** (see eqns. 2 and 3). Assuming the TDOA $\tau_d(\mathbf{X}, t)$, proportional to the range difference $r_d(\mathbf{X}, t)$, we have (c, wave celerity):

$$c\tau_d = r_d(\mathbf{X}, t)$$

where

$$r_d(\mathbf{X}, t) = r_1(\mathbf{X}, t) - r_2(\mathbf{X}, t) \triangleq h(\mathbf{X}, t) \quad (43)$$

Denoting $z(t)$ the observation (i.e. $z(t) = h(\mathbf{X}, t) + w_z(t)$), direct calculations [19] yield the following expression for the FIM associated with this problem:

$$\mathrm{FIM} = \frac{1}{\sigma_z^2}\sum_{k=1}^{n} \nabla_{\mathbf{X}} h(\mathbf{X}, t) \nabla_{\mathbf{X}}^* h(\mathbf{X}, t) \quad (44)$$

The components of the gradient vector $\nabla_X h(\mathbf{X}, t)$ have been calculated in [19], yielding:

$$\frac{\partial}{\partial r_x} h(t) = -2b(t)\sin\left(\frac{1}{2}(\beta_1(t)+\beta_2(t))\right)$$
$$\frac{\partial}{\partial r_y} h(t) = 2b(t)\cos\left(\frac{1}{2}(\beta_1(t)+\beta_2(t))\right)$$
$$\frac{\partial}{\partial v_x} h(t) = -2b(t)t\sin\left(\frac{1}{2}(\beta_1(t)+\beta_2(t))\right)$$
$$\frac{\partial}{\partial v_y} h(t) = 2b(t)t\cos\left(\frac{1}{2}(\beta_1(t)+\beta_2(t))\right)$$

where:

$$b(t) \triangleq \sin\left(\frac{1}{2}(\beta_1(t)+\beta_2(t))\right) \quad (45)$$

In eqn. 45, $\beta_1(t)$ and $\beta_2(t)$ denote the bearings angles at time t with respect to an axis passing through the two sensors. We are now in position to apply the fundamental results of Section 2. Note that the gradient vector $\nabla_X h(\mathbf{X}, t)$ is quite similar (see eqn. 9) to the BOT gradient vector $\mathbf{G}_t$, the differences being that $\beta(t)$ is replaced by 1/2 $(\beta_1(t) - \beta_2(t))$ in the first hand, and the term $1/r_t$ by $b(t)$ in the second one. We can now apply properties 2 and 3, yielding:

$$\det(\mathrm{FIM}) \approx (2b(t))^8 \left[4\dot{\tilde{\beta}}_t^4 + 2\dot{\tilde{\beta}}_t\tilde{\beta}_t^{(3)} - 3(\tilde{\beta}_t^{(2)})^2\right]^2 \quad (46)$$

where $\tilde{\beta}_t \triangleq \frac{1}{2}(\beta_1(t) + \beta_2(t))$.

To analyse the theoretical performance of this TMA system, it is worth considering an expansion of the various terms of det(FIM). Since the derivatives $\beta^{(n)}$ are polynomials in $\dot{\beta}$ and g, we consider the corresponding polynomial approximations of $\tilde{\beta}^{(n)}$. More precisely, we use the following approximations:

$$\dot{\tilde{\beta}} = \dot{\beta}_1 + \frac{\Delta\dot{\beta}}{2} \text{ with : } \Delta\dot{\beta} \triangleq \dot{\beta}_2 - \dot{\beta}_1 \quad (47a)$$

so that (with the assumption: $\varepsilon = \frac{\Delta\dot{\beta}}{\dot{\beta}} \ll 1$)

$$\dot{\tilde{\beta}}^4 \approx \dot{\beta}^4 + 2\dot{\beta}^{(3)}\Delta\dot{\beta} \quad (47b)$$

Quite similarly, we have

$$\beta_2^{(2)} = -2(\dot{\beta}_1 + \Delta\dot{\beta})(g_1 + \Delta g) \approx \beta^{(2)} - \dot{\beta}\Delta g - g\Delta\dot{\beta}$$

whence

$$(\tilde{\beta}^{(2)})^2 \approx (\beta^{(2)})^2 + 4\dot{\beta}g[g\Delta\dot{\beta} + \dot{\beta}\Delta g]$$
$$\tilde{\beta}^{(3)} \approx \beta^{(3)} + 3(g^2 - \dot{\beta}^2)\Delta\dot{\beta} + 6g\dot{\beta}\Delta g$$
$$\dot{\tilde{\beta}}\tilde{\beta}^{(3)} \approx \dot{\beta}\beta^{(3)} - 4\dot{\beta}^3\Delta\dot{\beta} + 6g\dot{\beta}^2\Delta g \quad (48)$$

Collecting the above results, we obtain the following approximation of det(FIM):

$$\det(\mathrm{FIM}_{t,t+k}) \approx 144 c_k k^{16} \frac{\sin\left(\frac{1}{2}(\beta_1-\beta_2)\right)^8}{\sigma_z^8} \dot{\beta}^2 g^4 (\Delta\dot{\beta})^2 \quad (49)$$

To give a more geometric interpretation of the above formula, it is worth considering the following expressions of $\dot{\beta}$ and g:

$$\dot{\beta} = \frac{1}{r^2}(\mathbf{r} \wedge \mathbf{v}) = \left(\frac{v}{r}\right) \sin\theta$$
$$g = \frac{\dot{r}}{r} = \frac{1}{r^2}(\mathbf{r}^* \mathbf{v}) = \left(\frac{v}{r}\right) \cos\theta$$

where θ is the angle formed with the vectors **r** and **v**.

Denoting d the intersensor distance and **d** the vector joining the two sensors, additional approximations (see the Appendix, Section 7.2) are as follows:

$$\sin\left(\frac{1}{2}(\beta_2-\beta_1)\right) \approx \frac{1}{r}\sin(\beta_1)(d/2)$$
$$(\Delta\dot{\beta})^2 \approx (-2g\dot{\beta} + \frac{1}{r^2}(\mathbf{d} \wedge \mathbf{v}))^2 \quad (50)$$

Finally, the following result has been obtained:

$$\det(\mathrm{FIM}_{t,t+k}) \approx c_k k^{16} \left(\frac{\sin\beta d}{\sigma_z r}\right)^8 \left(\frac{v}{r}\right)^6 (\sin^2\theta\cos^4\theta)(\Delta\dot{\beta})^2 \quad (51)$$

From eqn. 51, we note that the ratio d/r plays a major role. This means that the source must be 'sufficiently' close to the sensors. Another important factor is ($\sin^2\theta \cos^4\theta$). This factor is maximum for $\theta = \pi/3$, $2\pi/3$, and null for $\theta = 0$, $\pi/2$, π. So, the system will perform quite poorly when $\dot{\beta}$ or $g = \dot{r}/r$ are (almost) null. On the contrary, the geometry corresponding to $\theta = \pi/3$, $2\pi/3$ is quite favourable. Practically, a good sensor configuration may consist of (at least) three sensors forming a regular triangle. Data fusion step then becomes a fundamental step in the system design.

The case of Doppler measurements may be treated in the same way, even if it is *intrinsically* more intricate. The measurements are then differential Dopplers, i.e. $f_2 - f_1 = (f_0/c)(\dot{r}_1 - \dot{r}_2)$ where f_0 is the central source frequency. Using the differentiation chain rule, the FIM takes the following form:

$$\mathrm{FIM} \propto (\mathbf{M}_t^{(1)}, \ldots, \mathbf{M}_t^{(4)})(\mathbf{M}_t^{(1)}, \ldots, \mathbf{M}_t^{(4)})^* \quad (52a)$$

where

$$\mathbf{M}_t = b_t \mathbf{G}_t, \quad \mathbf{M}_t^{(1)} = b_t^1 \mathbf{G}_t + b_t \mathbf{G}_t^{(1)}, \text{etc.} \quad (52b)$$

The problem seems quite similar to the previous one. However, it is complicated by the form of $\mathbf{M}_t^{(1)}$. The following remark is then instrumental: all the terms det ($\mathbf{G}$, $\mathbf{G}^{(1)}$, $\mathbf{G}^{(2)}$, $\mathbf{G}^{(4)}$), ..., det ($\mathbf{G}^{(1)}$, $\mathbf{G}^{(2)}$, $\mathbf{G}^{(3)}$, $\mathbf{G}^{(4)}$) are small relatively to det ($\mathbf{G}$, $\mathbf{G}^{(1)}$, $\mathbf{G}^{(2)}$, $\mathbf{G}^{(3)}$). We then obtain

$$\det(\mathbf{M}_t^{(1)}, \ldots, \mathbf{M}_t^{(4)}) \approx \det(\mathbf{G}_t, \ldots, \mathbf{G}_t^{(3)}) f(\dot{\tilde{\beta}}, \frac{\dot{\tilde{r}}}{\tilde{r}}) \quad (53a)$$

where

$$f(\dot{\tilde{\beta}}, \frac{\dot{\tilde{r}}}{\tilde{r}}) = 12\dot{b}_t^2(2\dot{b}_t^2 - b_t b_t^{(2)}) + 8b_t^2 \dot{b}_t b_t^{(3)} - b_t b_t^{(4)} \quad (53b)$$

Examining the above approximation, we note that the information contained in the differential Doppler measurements on the first hand and in the time delays on the second one, may be of the same order of magnitude and are generally complementary [19]. This fact may be quite useful in the design of fused estimators.

4 Conclusion

Using basic results of multilinear algebra, a new interpretation of observability criteria has been derived. The interest of this approach is quite beyond observability analysis since it appears to be a simple, versatile and efficient tool for analysing estimability for various TMA problems.

First, general results have been obtained, allowing us to simplify considerably the calculation of the FIM determinant. Furthermore, invariance properties are thus proved. Second, these basic results are applied to a variety of practical situations, rendering evident their intrinsic unity.

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7 Appendixes

7.1 Analysis of the observability for TDOA TMA

The aim of this Appendix is to analyse observability for the passive TMA system considered in Section 3.4. Since the sensors are fixed, we consider a system with no input. This means that the Lie derivatives (eqn. 29) of the observation z are identical to the time derivatives (of z). Now, the following inclusion is easily proved:

$$sp(DL_f^0 z, \ldots, DL_f^k z, \ldots) \subset sp(D_{r_1^2}.D_{r_2^2}, D_{h_1}, D_{h_2}) \tag{54a}$$

where

$$\begin{aligned} r_1^2(t) &= (r_{x,0} + tv_x)^2 + (r_{y,0} + tv_y)^2 \\ r_2^2(t) &= (r_{x,0} + d + tv_x)^2 + (r_{y,0} + tv_y)^2 \\ h_1(t) &= (r_{x,0} + tv_x)v_x + (r_{y,0} + tv_y)v_y \\ h_2(t) &= (r_{x,0} + d + tv_x)v_x + (r_{y,0} + tv_y)v_y \end{aligned} \tag{54b}$$

We then consider the matrix M_t whose columns are $\{\nabla_X r_1^2(t), \nabla_X r_2^2(t), \nabla_X h_1(t), \nabla_X h_2(t)\}$, $(X = r_{x,0}, r_{y,0}, v_x, v_y)^*)$, and calculate its determinant, yielding

$$\det(M_t) = d^2(r_{y,0} + tv_y)^2$$

Thus, the system is (locally) unobservable on the bisector line of the segment joining the two sensors ($r_1 = r_2$), and on the line joining the two sensors ($(r_{y,0} + tv_y) = 0$). Note that the dimension of the (locally) observable space is then equal to 2 or 3. These two lines excepted, the system is observable.

7.2 TDOA approximations

In this Appendix, we deal with the approximations given in eqn. 50. The first one is a straightforward consequence of the law of sines (in a triangle). More precisely, we remark (see Fig. 1) that the angle formed with the two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ has a measure equal to $(\beta_2 - \beta_1)$, hence

$$\frac{\sin\left(\frac{1}{2}(\beta_2 - \beta_1)\right)}{d/2} = \frac{\sin\beta}{r}$$

which proves the first part. For the second one, we deal with

$$\Delta\dot{\beta} = \left(\frac{1}{r_2^2}\mathbf{r}_2 - \frac{1}{r_1^2}\mathbf{r}_1\right) \wedge \mathbf{v}$$

now: $r_2^{-2} \approx r_1^{-2}(1 + \alpha)$ with: $\alpha = -2\dfrac{\mathbf{r}_1^*\mathbf{d}}{r_1^2}$

so, that: $$\Delta\dot{\beta} \approx -2g\dot{\beta} + \frac{1}{r^2}(\mathbf{d} \wedge \mathbf{v}) \tag{55}$$