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THE MATRIX DYNAMIC PROGRAMMING PROPERTY AND ITS IMPLICATIONS*

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Abstract. The dynamic programming (DP) technique rests on a very simple idea, the principle of optimality due to Bellman. This principle is instrumental in solving numerous problems of optimal control. The control law minimizes a cost functional and is determined by using the optimality principle. However, applicability of the optimality principle requires that the cost functional satisfies the property called "matrix dynamic programming (MDP) property." A simple definition of this property will be provided and functionals having it will be considered.

Key words. dynamic programming, determinants, monotonicity, target tracking

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1. Introduction. The DP technique rests on a very simple idea, the principle of optimality due to Bellman [1]. This principle simply asserts that if $\pi^* = \{\mu_0^*, \mu_1^*, \dots, \mu_n^*\}$ is an optimal control law then [1] the truncated control law $\{\mu_i^*, \mu_{i+1}^*, \dots, \mu_n^*\}$ is optimal for the ith truncated control problem.

This principle is instrumental in solving numerous problems of optimal control of a dynamic system over a finite number of stages (finite horizon). We refer to [1, 2] for a thorough and motivated presentation of the DP technique.

The control law (or the strategy [2]) must minimize a cost functional. However, to our knowledge, the authors always assume that the cost functional is additive over time.

The problem we deal with consists of optimizing the trajectory of a passive receiver. For practical purposes, we must minimize a functional depending on the state values and the control law. The functional is a functional of the Fisher information matrix (FIM) since, roughly speaking, the FIM is a general measure of the estimability problem [3, 4, 5]. A general presentation of our problem is given in [6, 7, 8].

Various choices of the FIM functional have been considered in the literature [9], even if both theoretical and practical considerations advocate for the use of the determinant [8, 10, 11]. At this point, it is necessary to stress that the determinant is not linear so the cost functional additivity no longer holds. Actually, applicability of the principle of optimality to the matrix case requires that the cost functional satisfies to the matrix dynamic programming (MDP) property. A simple definition of the MDP property will be provided, and we shall examine the functionals having it.

Then, it is shown that they are reduced to the functionals of the form $f(A) = g\,(\mathrm{tr}(AM))$ (cf. Proposition 2.2). Consequently, these functionals are "almost" linear (obviously a linear functional yields an additive cost), which may be rather restrictive.

At this point, it is worth recalling the special structure of the FIMs. Actually, if we restrict our attention to a very specific estimation problem (namely, target motion

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analysis (TMA)), which deals with the estimation of the kinematic parameters defining a source trajectory, then the FIM matrices exhibit a very special structure [12] which is very succinctly presented in the appendix. We shall then consider the applicability of the optimality principle to this class of matrices for the det functional. Some results are thus obtained, but they cannot be extended to the generic case.

Throughout the text, the following notations will be used:

- a capital letter denotes a matrix,
- a capital calligraphic letter denotes a subspace,
- the symbol (*) means transposition conjugation,
- the symbols det and tr stand, resp., for the determinant and the trace,
- $\mathcal{H}_n$ is the space of n-dimensional Hermitian matrices,
- $\mathcal{P}_n$ (resp., $\mathcal{P}_n^+$) is the subset of positive semidefinite (resp., positive definite) matrices,
- I is the identity matrix,
- $A \succeq B$ means that the matrix $A - B$ is semidefinite positive.

The paper is organized as follows. The MDP property is introduced in section 2. General results are then obtained. The validity of the optimality principle for the determinant of structured matrices is considered in section 3.

2. The MDP property and its implications. We shall say that the functional f defined from $\mathcal{H}_n$ (the vector space of n-dimensional Hermitian matrix) to $\mathbb{R}$ satisfies the MDP property if the following conditions are fulfilled.

Definition 2.1.

- *f is smooth* $\left(\mathcal{C}^2\right)$,
- *let A and B in $\mathcal{H}_n$ be two matrices and assume that $f(B) > f(A)$; then whatever the matrix C in $\mathcal{H}_n$, we have $f(B+C) > f(A+C)$.*

An interpretation of this definition in terms of dynamic programming is the following type of inequality [6, 7]:[1]

$$\min f\left\{\sum_{j\in S}[C_{i,j}(d) + F_{\pi_1^*}(k+1,j)]p_{i,j}(d)\right\} \leq f\left\{\sum_{j\in S}[C_{i,j}(d) + F_{\pi_1}(k+1,j)]p_{i,j}(d)\right\},$$

which must be valid for the strategy π_1^*, optimal up to $k+1$.

The question we deal with consists of determining the functionals f having the MDP property. An answer to this question is provided with the following result.

Proposition 2.2. *Let f satisfy the MDP property; then*

$$f(A) = g\left(\mathrm{tr}(AR)\right),$$

where g is any monotone-increasing function and R is a fixed matrix.

Proof. Since it has been assumed that f is smooth, we can consider the first-order expansion[2] of f around A

$$f\left(A + \rho\, C\right) \overset{1}{=} f(A) + \rho\, \mathrm{tr}[\nabla^* f(A)\, C] + 0(\rho) \tag{1}$$

(ρ scalar).

[1] F denotes an FIM matrix.

[2] The symbol $\overset{1}{=}$ denotes a first-order expansion.

In the above formula, $\nabla f(A)$ denotes the gradient vector of f in A. Actually, the notation $\mathrm{tr}[\nabla^* f(A)\ C]$ replaces the true expression [13, 16] of the differential of $f, Df_A(C)$ and corresponds to (see comments)

$$Df_A(C) = \mathrm{tr}[\nabla^* f(A)\ C]. \tag{2}$$

Assume now that the gradient vectors $\nabla f(A)$ are not colinear altogether. Then there exist two matrices A and B s.t. $\nabla f(A) \neq \nabla f(B)$. Denoting $F^\perp$ as the subspace orthogonal to F (for the classical scalar matrix product [14]), we thus have

$$(P_1) \quad (\nabla f(A))^\perp \neq (\nabla f(B))^\perp. \tag{3}$$

At this point, stress that the matrices A and B satisfying (P_1) may be chosen as close (for the Frobenius norm [15]) as we want.

As a consequence of (P_1) there exists a matrix C such that

$$\mathrm{tr}[\nabla^* f(A)\ C] \neq 0 \ \text{ and } \ \mathrm{tr}[\nabla^* f(B)\ C] = 0.$$

If $\mathrm{tr}[\nabla^* f(A)\ C] < 0$, then $\mathrm{tr}[\nabla^* f(A)\ (-C)] \ > 0$, so we can assume that

$$\mathrm{tr}[\nabla^* f(A)\ C] \ > 0 \text{ and } \mathrm{tr}[\nabla^* f(B)\ C] = 0. \tag{4}$$

Now consider the function $g(\rho)$

$$g(\rho) \stackrel{\Delta}{=} f\,(B + \rho\, C) - f\,(A + \rho\, C)$$

and its first-order expansion around 0, i.e.,

$$g(\rho) = f(B) - f(A) - \rho\, \mathrm{tr}[\nabla^* f(A)\ C] + 0(\rho). \tag{5}$$

Since the functional f is continuous on $\mathcal{H}_n$, we may choose (A, B) such that

$$f(B) - f(A) = \frac{\rho}{2} \mathrm{tr}[\nabla^* f(A)\ C],$$

and, consequently,

$$\begin{aligned} &f(B + \rho\, C) - f(A + \rho\, C) = -\frac{\rho}{2}\ \mathrm{tr}[\nabla^* f(A)\ C] + 0(\rho) \\ &(\mathrm{tr}[\nabla^* f(A)\ C] \ \ > 0)\,. \end{aligned} \tag{6}$$

The above equality implies that f does not satisfy the MDP property.

Therefore, if f has the MDP property then all its gradients are colinear and proportional to a unique vector. Denote this vector by $\mathbf{G}$; we thus have

$$\begin{aligned} &\nabla f(A) = \lambda(A)\mathbf{G} \quad \forall A \in \mathcal{H}_n \\ &(\lambda(A) \text{ scalar })\,. \end{aligned} \tag{7}$$

Now if we recall the intermediate value theorem and the differentiation chain rule [16]

$$\nabla g\,[h(A)] = g'\,(h(A))\, \nabla h(A)$$

$$(g : \mathbb{R} \longrightarrow \mathbb{R}, h : \mathcal{H}_n \longrightarrow \mathbb{R})\,,$$

we conclude that f is the composition of a scalar function g and a linear form h. Such a linear form h may always be written $h(A) = \mathrm{tr}(AR)$, where R is a fixed matrix.

Reciprocally, it is a trivial matter to show that $f(A) = g\,(\mathrm{tr}(AR))$ with a g monotone increasing function that satisfies the MDP property. The proof is thus complete. $\square$

Comments.

1. Consider for instance $f(A) = \log\det A$; then [14]

$$Df_A(C) = \operatorname{tr}\left(A^{-1}C\right) = \operatorname{tr}[\nabla^* f(A)\, C]$$

(A invertible),
and we see immediately that f does not have the MDP property.

2. The same remark is valid for functionals as simple as $f(A) = \operatorname{tr}\left(A^{-1}\right)$.

But actually we only need the following condition on f.

DEFINITION 2.3. *The functional f has the MDP1 property if the following conditions are satisfied.*

For all positive definite matrices A and B and positive semidefinite matrix C, the following property holds:

$$f(B) > f(A) \Rightarrow f(B+C) > f(A+C) \tag{8}$$

$\left(f : \mathcal{C}^2\right)$.

Note that MDP1 is a refinement of the MDP property and may, possibly, be less demanding than MDP. At this point, it is worth mentioning the following lower bound of $\det(A+B)$. We refer, for instance, to [17, pp. 229–230, 18] for a proof.

LEMMA 2.4. *Let A, B be Hermitian $n \times n$ matrices, and suppose that A is positive definite and that B is nonnegative definite. Then*

$$\det(A+B) \geq \det(A) + \frac{\det(A)}{n\lambda_{\max}} \operatorname{tr}(B). \tag{9}$$

Here $\lambda_{\max}$ denotes the maximum eigenvalue of A.

However, the following counterexample shows that MDP1 is not satisfied by the "det" functional.

Counterexample.

$$A_\varepsilon = \begin{pmatrix} 1+\varepsilon & 0 \\ 0 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix},$$

$$\det(B) = 5, \det\left(A_\varepsilon\right) = 5(1+\varepsilon), A_\varepsilon \text{ and } B \succ 0 \text{ (} \varepsilon \text{ suff. small)},$$

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and thus

$$\det\left(A_\varepsilon + C\right) = 11 + 6\varepsilon \text{ and } \det(B+C) = 7.$$

It is quite obvious that this (elementary) counterexample is not restricted to a rank deficient C matrix since C may be slightly perturbed without changing the sign of $\det(B+C) - \det(A+C)$. Therefore for ε sufficiently small (and negative) we have

$$\det(B) > \det\left(A_\varepsilon\right) \text{ and } \det\left(A_\varepsilon + C\right) > \det(B+C).$$

Similarly, we can show that MDP1 is not satisfied by the functional $f(A) = \operatorname{tr}\left(A^{-1}\right)$ but is trivially satisfied by any functional $f(A) = g\left(\operatorname{tr}(AR)\right)$ (g: monotone increasing). We thus consider the following problem.

What are the functionals satisfying the MDP1? We may reasonably suspect that they are reduced to those satisfying the MDP. However, the proof of Proposition 2.2 cannot be trivially extended since the subset of semidefinite matrices is not a subspace. Actually, if C is in $\mathcal{P}_n^+$ then $-C$ is not in $\mathcal{P}_n^+$ and the reasoning leading to (4) is not valid. The difficulty comes from the fact that $\mathcal{P}_n^+$ is a convex subset of $\mathcal{H}_n$ and not a subspace.

3. Structured determinants and the MDP property. Our attention will now be restricted to structured matrices. Various structures will be considered corresponding to various scenarios (see [6]) of target motion analysis. The statistical motivations of such special matrix structures are beyond the scope of this paper, but the true problems are thus conveniently described [12, 7].

Since the general MDP property is not satisfied by the determinant, we shall consider special cases associated with particular matrix structures and specific definitions of the "addition." It will then be shown that the DP property may be extended, but the validity of these extensions is limited.

3.0.1. One-leg scenario. In this case, the elementary FIM $F(A,C)$ takes the following form:

$$F(A,C) = \begin{pmatrix} A & iA \\ iA & i^2A \end{pmatrix} + \begin{pmatrix} C & jC \\ jC & j^2C \end{pmatrix} = \begin{pmatrix} A+C & iA+jC \\ iA+jC & i^2A+j^2C \end{pmatrix}$$

with

$$A, C \in \mathcal{P}_n^+,\ i,j \in \mathbb{N}.$$

Then, the following result holds.

PROPOSITION 3.1.

$$\det\left(F(A,C)\right) = (j-i)^{2n} \det A\ \det C.$$

Proof. An elementary proof relies on the following factorization:

$$F = \begin{pmatrix} A & I \\ iA & jI \end{pmatrix} \begin{pmatrix} I & iI \\ C & jC \end{pmatrix},$$

hence

$$\det F = \det \begin{pmatrix} A & I \\ iA & jI \end{pmatrix} \det \begin{pmatrix} I & iI \\ C & jC \end{pmatrix}. \tag{10}$$

Now using the classical lemma about the determinant of a partitioned matrix [15, 19] we obtain directly

$$\begin{aligned} \det \begin{pmatrix} A & I \\ iA & jI \end{pmatrix} &= \det A \det\left(jI - iAA^{-1}\right) \\ &= \det A \det\left((j-i)I\right) \end{aligned}$$

and similarly

$$\det \begin{pmatrix} I & iI \\ C & jC \end{pmatrix} = \det C\ \det\left((j-i)I\right). \qquad \square \tag{11}$$

Using the proof of Proposition 3.2 (see below), Proposition 3.1 may be extended to the case $A \in \mathcal{P}_n^+$, $C \in \mathcal{P}_n$.

A direct consequence of Proposition 3.1 is

$$\det B > \det A > 0 \Rightarrow \det F(B,C) > \det F(A,C) \quad (C \succeq 0). \tag{12}$$

Actually, as we shall see later, the simplicity of $\mathbf{Res}_1$ is a consequence of the rank deficiency of the following matrix:

$$\begin{pmatrix} A & iA \\ iA & i^2A \end{pmatrix} = \begin{pmatrix} 1 & i \\ i & i^2 \end{pmatrix} \otimes A$$

($\otimes$: Kronecker product [15]).

Thus, the MDP property is verified for this particular matrix structure. However, for practical applications, Proposition 3.1 should be extended to the following two problems.

3.0.2. Problem 1. In fact, statistical considerations may lead us to consider a slightly more general structure

$$\begin{pmatrix} A & \alpha A \\ \alpha A & \beta A \end{pmatrix}.$$

This matrix is no longer rank deficient (in general) so that previous calculations are not valid. However, the following result holds.

PROPOSITION 3.2.

$$\det\left[\begin{pmatrix} A & \alpha A \\ \alpha A & \beta A \end{pmatrix} + \begin{pmatrix} C & jC \\ jC & j^2C \end{pmatrix}\right]$$
$$= \det A \det\left[\left(\beta - \alpha^2\right) A + \left(\beta - 2j\alpha + j^2\right) C\right].$$

Proof. That A is positive definite admits a decomposition in triangular factors ($A = TT^*$) so that

$$\begin{pmatrix} A + C & \alpha A + jC \\ \alpha A + jC & \beta A + j^2C \end{pmatrix} = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} I + S & \alpha I + jS \\ \alpha I + jS & \beta I + j^2S \end{pmatrix} \begin{pmatrix} T^* & 0 \\ 0 & T^* \end{pmatrix}$$

with

$$S \triangleq T^{-1}CT^{-1*}. \tag{13}$$

Now

$$\det\begin{pmatrix} I + S & \alpha I + jS \\ \alpha I + jS & \beta I + j^2S \end{pmatrix} = \det\left[(I+S)\left(\beta I + j^2S\right) - (\alpha I + jS)^2\right]$$

(where we have used the fact that $(I+S)$ and $(\alpha I + jS)$ commute)

$$= \det\left(\left(\beta - \alpha^2\right) I + \left(\beta - 2j\alpha + j^2\right) S\right), \tag{14}$$

which ends the proof. $\square$

If $\beta = \alpha^2$, then we have

$$\det F(A,C) = \det A \det\left[(\alpha - j)^2 C\right]$$

and therefore

$$\det B > \det A \Rightarrow \det F(B,C) > \det F(A,C) \quad (C \succeq 0). \tag{15}$$

If $\beta \neq \alpha^2$, then the previous implication does not hold and the MDP property is not valid.

3.0.3. Problem 2. Another important problem arises when we try to extend the previous calculations to more than two matrices, i.e., to calculate the following structured determinant:

$$\det \begin{pmatrix} A + C_1 + C_2 & A + jC_1 + kC_2 \\ A + jC_1 + kC_2 & A + j^2 C_1 + k^2 C_2 \end{pmatrix}. \tag{16}$$

The previous simple results (Proposition 3.1 or 3.2) cannot be extended to this structure. It seems impossible to obtain a simple expression of this determinant even when using more sophisticated algebra [20]. This constitutes a major problem. A last example considers a very special addition law where the dimension of the matrices is increasing. As previously noted, this special structure may be justified by statistical considerations [6, 7, 8] associated with multileg scenarios.

3.1. Multileg scenarios. Considering this type of scenario leads to increasing the dimensionality of the state vector and thus to considering the following elementary structure of the matrix F:

$$\mathcal{F}(A, C) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} C & iC \\ iC & i^2 C \end{pmatrix},$$

$$A \in \mathcal{P}_n^+,\ C \in \mathcal{P}_n^+,\ i \in \mathbb{N}.$$

We then obtain the following result.

PROPOSITION 3.3.

$$\det \mathcal{F}(A, C) = (i)^{2n} \det A \det C.$$

Proof.

$$\begin{aligned} \det\left(\mathcal{F}(A, C)\right) &= \det\left[i^2 C - i^2 C (A + C)^{-1} C\right] \det(A + C) \\ &= i^{2n} (\det C)^2 \det\left[C^{-1}(A + C) - I\right] \\ &= i^{2n} \det C \det A. \qquad \square \end{aligned} \tag{17}$$

In view of Proposition 3.3, the following property holds:

$$\det B > \det A \Rightarrow \det \mathcal{F}(B, C) > \det \mathcal{F}(A, C). \tag{18}$$

4. Conclusion. Applicability of the principle of optimality to matrix cost functionals requires that the MDP property be satisfied. A simple definition of this property has been given, and we have determined the functionals that have it. Since the det functional does not satisfy the MDP property, various special structures have been considered.

5. Appendix. The aim of this appendix is to provide a very succinct presentation of the calculation of the FIM matrices in the TMA context. For more details, we refer to [8, 12]. First consider a rectilinear and uniform motion of the source. The source, located at the coordinates (r_{xs}, r_{ys}), moves with a constant velocity vector $\mathbf{v}\,(v_{xs}, v_{ys})$ and is thus defined to have the state vector

$$\mathbf{X}_s \triangleq \left[r_{xs}, r_{ys}, v_{xs}, v_{ys}\right]^* . \tag{19}$$

The receiver state is similarly defined as

$$\mathbf{X}_{rec} \triangleq [r_{x\ rec}, r_{y\ rec}, v_{x\ rec}, v_{y\ rec}]^*$$

so that, in terms of the relative state vector $\mathbf{X}$ defined by

$$\mathbf{X} = \mathbf{X}_s - \mathbf{X}_{rec} \triangleq [r_x, r_y, v_x, v_y]^*,$$

the discrete time equation of the system (i.e., the equation of the relative motion) takes the following form:

$$\mathbf{X}_{k+1} = F\mathbf{X}_k + \mathbf{U}_k,$$

where

$$F = \Phi(k, k+1) = \begin{pmatrix} Id & \alpha Id \\ 0 & Id \end{pmatrix}, \ Id \triangleq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\alpha \triangleq t_{k+1} - t_k = cst\ . \tag{20}$$

The measurement noise w_k is usually modelled by an independently and identically distributed (i.i.d.) zero-mean, Gaussian process.

The partial derivative matrix of the bearing vector $\mathbf{\Theta}(\mathbf{X})$ with respect to the state components is easily calculated [12] yielding

$$\frac{\partial \mathbf{\Theta}(\mathbf{X})}{\partial \mathbf{X}} = \begin{pmatrix} \frac{\cos\theta_1}{r_1} & -\frac{\sin\theta_1}{r_1} & \frac{\cos\theta_1}{r_1} & -\frac{\sin\theta_1}{r_1} \\ \vdots & & & \\ \frac{\cos\theta_n}{r_n} & -\frac{\sin\theta_n}{r_n} & \frac{n\cos\theta_1}{r_n} & -\frac{n\sin\theta_n}{r_n} \end{pmatrix},$$

where $\{\theta_i\}$ represents the source bearing (angle) at the instant i, and $\{r_i\}$ is the source-receiver distance.

Consider the case of a nonmaneuvering source (constant-velocity vector); then the calculation of the FIM is a routine exercise yielding [12]

$$\text{FIM} = \left(\frac{\partial \mathbf{\Theta}(\mathbf{X})}{\partial \mathbf{X}}\right)^* \Sigma^{-1} \left(\frac{\partial \mathbf{\Theta}(\mathbf{X})}{\partial \mathbf{X}}\right) \tag{21}$$

$$\triangleq \begin{pmatrix} \sum_{i=1}^n \Omega_i & \sum_{i=1}^n i\Omega_i \\ \sum_{i=1}^n i\Omega_i & \sum_{i=1}^n i^2\Omega_i \end{pmatrix}.$$

A realistic assumption consists of modelling the source trajectory by a sequence of elementary rectilinear uniform motions (named "legs"). The previous calculation of the FIM may be extended to this modelling, and the FIM then takes the following form [7, 6] (l legs):

$$\text{FIM} = \sum_{m=1}^{l} \sum_{k=(m-1)+j}^{mj} [\mathbf{d}_{m-1,l+1}(k)\mathbf{d}_{m-1,l+1}(k)^*] \otimes \Omega_k,$$

where $\mathbf{d}$ is a vector describing the index leg, consisting of 0 and 1, and Ω_k is a 2×2 elementary FIM.

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