Discrete-Time Observability and Estimability Analysis for Bearings-Only Target Motion Analysis

J. P. LE CADRE, Member, IEEE
IRISA/CNRS
France

C. JAUFFRET
DCN/Ingeniérie/Sud
France

Observability in the context of bearings-only tracking (BOT) is still the subject of important literature. Different from previous approaches, where continuous-time analysis was considered, our approach relies on discrete-time analysis. It is then shown that this allows us to use directly and efficiently the simple formalisms of linear algebra. Using the direct approach, observability analysis is essentially reduced to basic considerations about subspace dimensions. Even if this approach is conceptually quite direct, it becomes more and more complex as the source-encounter scenario complexity increases. For complex scenarios, the dual approach may present some advantages essentially due to the direct use of multilinear algebra. New results about BOT observability for maneuvering sources are thus obtained. Observability analysis is then extended to unknown instants of source velocity changes.

Even if observability analysis provides thorough insights about the algebraic structure of the BOT problem, the optimization of the observer maneuvers is essentially a control problem. Basic algebraic considerations prove that a relevant cost functional for this control problem is the determinant of the Fisher information matrix (FIM). So, a large part of this work is devoted to the analysis of this cost functional. Using multilinear algebra, general approximations of this functional are given. They present the great interest to involve only directly estimable parameters, the source bearing-rates. Using these approximations, a general framework for optimizing the observer trajectory is derived which allow us to approximate the optimal sequence of controls. It is worth stressing that our approach does not require the knowledge of the source trajectory parameters and is still valid for a maneuvering source.

Manuscript received May 10, 1995; revised December 27, 1995.

IEEE Log No. T-AES/33/1/01086.

Authors' addresses: J. P. Le Cadre, IRISA/CNRS, Campus de Beaulieu, 35042, Rennes, France; C. Jauffret, ISITV, Gessy Avenue G. Pompidou, BP 56, 83 162 LaValette du Var, France.

0018-9251/97/$10.00

I. INTRODUCTION

Passive bearings-only tracking (BOT) techniques are used in a variety of applications [1–8]. In the ocean environment, two-dimensional target motion analysis (TMA) has represented an important research area [1].

In the sonar context, the BOT problem is to estimate the trajectory of an acoustic source (target). A reasonable hypothesis consists in modelling this trajectory by a collection of consecutive legs. Thus, on each leg, the source travels with a constant velocity vector. The geometric configuration is depicted in Fig. 1, where both ownship and target are presumed to lie in the same $(x-y)$ plane. This configuration is assumed throughout the rest of this work.

An important aspect of the problem is the existence of unique tracking solutions. From a system-theoretic viewpoint, this involves the notion of observability. Since observability plays a critical role in the design and subsequent performance of any tracking system, the class of observer accelerations and maneuvers which render the state observable needs to be characterized.

The first attempt at deriving criteria for BOT observability in a rigorous way appears to be the works of Lindgren and Gong (1978) [2] and Nardone and Aidala (1981) [9]. These results have been extended to three-dimensional BOT by Hammel and Aidala (1985) [10]. Observer maneuvers which ensure system observability have been characterized by Payne (1989) but always for a nonmaneuvering source [11]. An extension to an Nth-order dynamics target first appears in the works of Jauffret and Pillon (1988) [12, 13], Fogel and Gavish (1988) [14], and Becker [15].

Ordinarily, the presence of measurement nonlinearities would dictate an analysis utilizing difficult nonlinear techniques as those developed by Hermann and Krener [16]. However, as previously pointed out [9–15], the application of these relatively complicated procedures may be avoided by rewriting and embedding the BOT problem in an equivalent linear form. This simple idea is the key for solving the observability problem for BOT and has been widely utilized by previous contributors [9–15]. Moreover note that—further its simplicity—the linear approach provides us with global observability criteria [9–17].

Using pseudomeasurements [1], the observability problem is recast in a linear framework. More precisely, the measurements are modeled by a linear time-varying system. Note that now the changes may affect both the measurement and the system equations. While the observability analysis of a constant dynamic system is rather simple, the analysis of a time-varying system is much more cumbersome. Actually, the system may be modeled by a piecewise constant linear system (PWCS). Interesting insights into the observability of a PWCS can be found in the work of Goshen-Meskin and Bar-Itzhack [18].

Rather curiously, the observability analysis for BOT has been performed until now in a continuous time framework. A main objective of this work consists in demonstrating that a discrete-time analysis may be seriously simpler and, simultaneously, provides new and thorough geometric insights. Actually, the discrete-time analysis allows us to utilize all the tools of linear and multilinear algebra [18, 19]. The geometric interpretation of observability appears then both enlightening and powerful. For example, it is possible to analyze observability for the general case of target-observer encounter, say the case of a maneuvering (multilegs) source with unknown maneuver instants. Observability analysis for maneuvering sources is particularly important since interesting sources are essentially the maneuvering ones. Using the PWCS approach new results for BOT observability are demonstrated in Sections IV and V. Please note that extensions to multiple sources or observers become even quite straightforward. First, we consider a direct approach (Section IV). The analysis of the algebraic structure then leads us to consider a dual approach (Section V) for which the dual spaces and the Cayley-Hamilton Theorem are basic ingredients.

Observability analysis is then extended to the unknown instants of source velocity changes. Our approach consists of jointly using the methods developed for jump linear systems and the linear formalism previously developed (Section VI).

As noted by various authors, observability analysis may be rather frustrating (especially for statistical performance analysis) since it is an algebraic problem which has only a simple answer: yes or no. The concept of "more or less observable" is actually an estimability concept even if it occurs very frequently in the observability literature. So, a precise definition of estimability is given and its implications for the BOT problem are considered (see Section IX). In a first time, our analysis is restricted to PWCS describing maneuvering sources. For deterministic parameters (PWCS in particular), a natural tool for studying the statistical performance is the Fisher information matrix (FIM). Properties of FIM functionals are considered in Section VIII.

Under a long range hypothesis, it is shown that maximizing det(FIM) amounts to maximizing the sphericity criterion (see Section VIII). Therefore det(FIM) plays a fundamental role and is a relevant cost functional. So, a large part of this work is devoted to the study of the properties of this functional. Using multilinear algebra, new approximations are obtained (Section VII). In particular, we prove that det(FIM) can be tightly approximated by a functional involving only the consecutive source bearing-rates. Since these parameters (bearing-rates) may be directly estimated from the data, these results lead to a feasible approach for determining the optimal sequence of controls (observer maneuvers). It is proved that, under the long range hypothesis, the sequence of optimal control lies in the class of bang-bang controls.

Estimability and stochastic observability concepts have been introduced by various researchers working in the area of system theory (Markov models). We consider the applications of such concepts for the BOT problem for a Markovian modeling of the source trajectory. Estimability conditions are derived (Section VIII). Even if these conditions are almost always satisfied, it is thus possible to derive an estimability functional quite analogous to the FIM functional but, this time, for random parameters.

The paper is organized as follows. The general BOT model is briefly presented in Section II, followed by the discrete-time analysis of BOT observability in Sections III and IV. A dual approach is then considered in Section V. The analysis of observability is extended to unknown instants of source velocity changes in Section VI. Sections VII and VIII deal with the statistical performance analysis. Detailed calculations are provided in Appendix A, B, C. Finally, stochastic observability is considered in Section IX.

The following standard notations are used throughout this work.

1) A bold letter denotes a vector while a capital letter denotes a matrix.

2) A capital script letter generally denotes a subspace.

3) The symbol (*) means transposition.

4) The variables r_x and r_y represent x and y relative coordinates.

5) Relative x and y velocities are denoted by v_x and v_y.

6) The time variable is t or k.

7) The number of sensors is p.

8) A diagonal matrix is denoted by diag.

9) The symbol $\stackrel{i}{=}$ means approximation at the order i.

10) The determinant is represented by det.

11) The symbol tr stands for the trace.

12) Im and ker are the image and the kernel of a mapping (or its associated matrix).

13) $A \succ B$ means that the matrix $A - B$ is positive definite.

14) Id_n is the n-dimensional identity matrix.

15) The symbol cov stands for the covariance matrix.

16) BOT is bearings-only tracking, RUN is rectilinear uniform motion, and TMA is target motion analysis.

II. BOT MODEL

The general notations are identical to those of [1]. The physical parameters are depicted in Fig. 1. The source, located at the coordinates (r_{xs}, r_{ys}) moves with

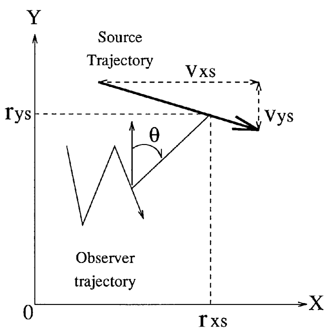

Fig. 1. Typical TMA scenarios.

a constant velocity vector $\mathbf{v}(v_{xs}, v_{ys})$ and is thus defined to have the state vector:

$$\mathbf{X}_s \stackrel{\Delta}{=} [r_{xs}, r_{ys}, v_{xs}, v_{ys}]^*. \tag{1}$$

The observer state is similarly defined as

$$\mathbf{X}_{\text{obs}} \stackrel{\Delta}{=} [r_{x\text{obs}}, r_{y\text{obs}}, v_{x\text{obs}}, v_{y\text{obs}}]^*$$

so that, in terms of the relative state vector $\mathbf{X}$ defined by

$$\mathbf{X} = \mathbf{X}_s - \mathbf{X}_{\text{obs}} \stackrel{\Delta}{=} [r_x, r_y, v_x, v_y]^*$$

the discrete-time equation of the system (i.e., the equation of the relative motion) takes the following form:

$$\mathbf{X}_{k+1} = F\mathbf{X}_k + \mathbf{U}_k$$

where

$$F = \Phi(k, k+1) = \begin{pmatrix} Id & \alpha Id \\ 0 & Id \end{pmatrix},$$

$$Id \stackrel{\Delta}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\alpha \stackrel{\Delta}{=} t_{k+1} - t_k = cst. \tag{2}$$

In the above formula t_k is the time at the kth sample while the vector $\mathbf{U}_k = (0, 0, u_x(k), u_y(k))^*$ accounts for the effects of observer accelerations (or control). Equation (2) assumes that between t_k and t_{k+1} the source motion is rectilinear and uniform (RUN). This hypothesis is used throughout the remainder of this work. Also, in this work, the vector $\mathbf{X}$ denotes the *relative* state vector.

As usual in TMA [1], the available measurements are the estimated angles $\hat{\theta}_1$ (bearings) from the observer to the source, so that the observation equation stands as follows:

$$\hat{\theta}_k = \theta_k + w_k$$

with

$$\theta_k = \tan^{-1}\left(\frac{r_x(k)}{r_y(k)}\right) \tag{3}$$

and w_k is the measurement noise.

Equivalently the measurement equation may be written as

$$0 \equiv z_k = H_k \mathbf{X}_k$$

with

$$H_k = (\cos\theta_k, -\sin\theta_k, 0, 0). \tag{4}$$

This simplistic remark is nevertheless a basic trick to investigate BOT system observability. The BOT model can thus be described by the following linear and *time-varying* state-space model:

$$\begin{cases} \mathbf{X}_{k+1} = F\mathbf{X}_k + \mathbf{U}_k \\ z_k = H_k \mathbf{X}_k. \end{cases} \tag{5}$$

Note that this system is time varying due to H_k. This induces an *apparent* difficulty for observability analysis.

III. SYSTEM OBSERVABILITY, THE SIMPLEST PROBLEM

We now briefly consider the simplest problem of observability analysis. Although this section does not provide anything but classical results, the general framework is introduced allowing us to analyze more and more complex scenarios in subsequent sections.

In this section, both source and observer motion are assumed to have RUN. Using (5) the system outputs z_k are directly calculated, yielding

$$\left| \begin{array}{l} z_0 = H_0 \mathbf{X}_0 \\ z_1 = H_1 F \mathbf{X}_0 \\ \vdots \quad \vdots \\ z_k = H_k F^k \mathbf{X}_0. \end{array} \right. \tag{6}$$

Now:

$$F^k = \begin{pmatrix} Id & k\alpha Id \\ 0 & Id \end{pmatrix}$$

so that, the observability matrix $\mathcal{O}$ is

$$\mathcal{O} = \begin{pmatrix} H_0 \\ H_1 F \\ \vdots \\ H_k F^k \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta_0 & -\sin\theta_0 & 0 & 0 \\ \cos\theta_1 & -\sin\theta_1 & \alpha\cos\theta_1 & -\alpha\sin\theta_1 \\ \vdots & \vdots & \vdots & \vdots \\ \cos\theta_k & -\sin\theta_k & k\alpha\cos\theta_k & -k\alpha\sin\theta_k \end{pmatrix}. \tag{7}$$

A basic trick consists of factorizing the matrix $\mathcal{O}$ by using the definition of sines and cosines, i.e.,

$$\mathcal{O} = \Delta_r \begin{pmatrix} r_y(0) & -r_x(0) & 0 & 0 \\ r_y(1) & -r_x(1) & \alpha r_y(1) & -\alpha r_x(1) \\ \vdots & \vdots & \vdots & \vdots \\ r_y(k) & -r_x(k) & k\alpha r_y(k) & -k\alpha r_x(k) \end{pmatrix}$$

with

$$\Delta_r \stackrel{\Delta}{=} \mathrm{diag}(r^{-1}(0), r^{-1}(1), \ldots, r^{-1}(k))$$
$$r(k) \stackrel{\Delta}{=} (r_x^2(k) + r_y^2(k))^{1/2}. \qquad (8)$$

Obviously, this factorization is valid only if all the $r(j)$ are non-zero which means that the source and observer positions are not identical. For obvious reasons, this assumption is not restrictive in our context.

Let us denote $\mathcal{O}'$ the "modified" observability matrix defined by

$$\mathcal{O}' = \begin{pmatrix} r_x(0) & r_y(0) & 0 & 0 \\ r_x(1) & r_y(1) & \alpha r_x(1) & \alpha r_y(1) \\ \vdots & \vdots & \vdots & \vdots \\ \underbrace{r_x(k)}_{\mathbf{R}_x} & \underbrace{r_y(k)}_{\mathbf{R}_y} & \underbrace{k\alpha r_x(k)}_{\mathbf{V}_x} & \underbrace{k\alpha r_y(k)}_{\mathbf{V}_y} \end{pmatrix}. \qquad (9)$$

Then Δ_r being an invertible matrix, we have

$$\mathrm{rank}\,\mathcal{O} = \mathrm{rank}\,\mathcal{O}'. \qquad (10)$$

So, in order to study system observability, it is equivalent to analyze the matrix $\mathcal{O}'$ defined by (9). This idea constitutes the cornerstone for the "direct" approach of observability and is intensively used in the next section.

Using (2), the vectors $\mathbf{R}_x, \mathbf{R}_y, \mathbf{V}_x, \mathbf{V}_y$ can be expressed as linear combinations of the three vectors $\mathbf{1}, \mathbf{Z}, \mathbf{Z}^2$, i.e.,

$$\left| \begin{aligned} \mathbf{R}_x &= r_x(0)\mathbf{1} + \alpha v_x \mathbf{Z} \\ \mathbf{V}_x &= \alpha r_x(1)\mathbf{Z} + \alpha^2 v_x \mathbf{Z}^2 \\ \mathbf{R}_y &= r_y(0)\mathbf{1} + \alpha v_y \mathbf{Z} \\ \mathbf{V}_y &= \alpha r_y(1)\mathbf{Z} + \alpha^2 v_y \mathbf{Z}^2 \end{aligned} \right.$$

with

$$\left| \begin{aligned} &\mathbf{1} \stackrel{\Delta}{=} (1, 1, \ldots, 1)^* \\ &\mathbf{Z} \stackrel{\Delta}{=} (0, 1, 2, \ldots, k)^* \\ &\mathbf{Z}^2 \stackrel{\Delta}{=} (0, 0, 2, \ldots, k(k-1))^*. \end{aligned} \right. \qquad (11)$$

It is then quite obvious from (11) that rank($\mathcal{O}'$) and thus rank($\mathcal{O}$) are bounded by 3 since the range (or image [20]) of $\mathcal{O}'$ is spanned by the three vectors $\{\mathbf{1}, \mathbf{Z}, \mathbf{Z}^2\}$, which are themselves linearly independent ($k \geq 3$). There is a rank degeneracy for $\mathcal{O}$ in the following case ($\mathrm{rank}\,\mathcal{O} = 2$):

$$r_x(0)v_y = r_y(0)v_x. \qquad (12)$$

This condition is itself equivalent to a zero bearing-rate assumption. Denote $\mathcal{N}$ the null-subspace (or the kernel [20]) of $\mathcal{O}$. Then, except for the zero bearing-rate case, this is a 1-dimensional subspace ($4 = \dim \mathrm{Im}\mathcal{O} + \dim \mathcal{N}$). Using (11) and solving the associated linear system, we obtain directly:

$$\mathcal{N} = sp(\mathbf{X}_0). \qquad (13)$$

Note that (13) expresses nothing else than the Thales Theorem [21]. Even if the above results are quite classical [9–15], a new approach has been introduced which allows us to consider much more complicated problems.

For instance, this approach may be utilized to analyze observability for multiple arrays [22]. We consider now a specific application of this formalism, which we call the direct approach.

IV. OBSERVABILITY ANALYSIS, THE DIRECT APPROACH

Using the previous formalism (see Section III), we consider various source-observer encounters and analyze the corresponding observability problems. Actually, we find that the main limitations come from the expanded expressions of $\mathcal{O}'$ which become rather cumbersome as the complexity of the scenario increases. So, a dual approach is developed in the next section.

A. Maneuvering Observer and Rectilinear Uniform Motion of the Source

Note that (14) is the "classical" case for observability analysis [9–15]. Since there is no source maneuver, the transition matrix remains unchanged throughout the scenario. Therefore, for a two-legs observer trajectory, the observability matrix is

$$\mathcal{O} = \begin{pmatrix} H_0 \\ H_1 F \\ \vdots \\ H_k F^k \\ H_{k+1} F^{k+1} \\ \vdots \\ H_{k+j} F^{k+j} \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow \text{observer maneuver.} \\ \\ \\ \\ \end{matrix} \qquad (14)$$

As previously, the modified observability matrix $\mathcal{O}'$ defined as in (9) is a $(k+j)\times 4$ matrix defined by

$$\mathcal{O}' = (\mathbf{R}_x, \mathbf{R}_y, \mathbf{V}_x, \mathbf{V}_y)$$

with

$$\begin{cases} \mathbf{R}_x = r_x(0)\begin{pmatrix}\mathbf{1}\\ \mathbf{1}'\end{pmatrix} + \alpha v_{x,1}\begin{pmatrix}\mathbf{Z}_1\\ k\mathbf{1}'\end{pmatrix} + \alpha v_{x,2}\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'_1\end{pmatrix} \\ \mathbf{R}_y = r_y(0)\begin{pmatrix}\mathbf{1}\\ \mathbf{1}'\end{pmatrix} + \alpha v_{y,1}\begin{pmatrix}\mathbf{Z}_1\\ k\mathbf{1}'\end{pmatrix} + \alpha v_{y,2}\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'_1\end{pmatrix} \\ \mathbf{V}_x = \alpha r_x(1)\begin{pmatrix}\mathbf{Z}_1\\ \mathbf{Z}'_2\end{pmatrix} + \alpha^2 v_{x,1}\begin{pmatrix}\mathbf{Z}_1^2\\ k\mathbf{Z}'_2\end{pmatrix} + \alpha^2 v_{x,2}\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}_2^2\end{pmatrix} \\ \mathbf{V}_y = \alpha r_y(1)\begin{pmatrix}\mathbf{Z}_1\\ \mathbf{Z}'_2\end{pmatrix} + \alpha^2 v_{y,1}\begin{pmatrix}\mathbf{Z}_1^2\\ k\mathbf{Z}'_2\end{pmatrix} + \alpha^2 v_{y,2}\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}_2^2\end{pmatrix} \end{cases}$$

and

$$\begin{aligned} \mathbf{Z}_1 &= (0,1,\ldots,k)^* \\ \mathbf{Z}_1^2 &= (0,2,\ldots,k(k-1))^* \\ \mathbf{Z}_2^2 &= ((k+1),(k+2)2,\ldots,(k+j)j)^* \\ \mathbf{1}' &= (1,1,\ldots,1)^* \\ \mathbf{Z}'_1 &= (1,2,\ldots,j)^* \\ \mathbf{Z}'_2 &= (k+1,k+2,\ldots,k+j)^* \\ v_{x,2} &= v_{x,1}+u_x; \qquad v_{y,2} = v_{y,1}+u_y \end{aligned} \tag{15}$$

where u_x and u_y are the observer accelerations (controls) at time $(k+1)$.

From the preceding equalities, we obtain straightforwardly:

$$\begin{aligned} \mathbf{R}_x &= \left(\begin{pmatrix}\mathbf{1}\\ \mathbf{1}'\end{pmatrix}, \begin{pmatrix}\mathbf{Z}_1\\ k\mathbf{1}'\end{pmatrix}, \begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'_1\end{pmatrix}\right)\begin{pmatrix} r_x(0)\\ v_{x,1}\\ v_{x,2}\end{pmatrix} \\ \mathbf{V}_x &= \left(\begin{pmatrix}\mathbf{Z}_1\\ \mathbf{Z}'_2\end{pmatrix}, \begin{pmatrix}\mathbf{Z}_1^2\\ k\mathbf{Z}'_2\end{pmatrix}, \begin{pmatrix}\mathbf{0}\\ \mathbf{Z}_2^2\end{pmatrix}\right)\begin{pmatrix} r_x(1)\\ v_{x,1}\\ v_{x,2}\end{pmatrix} \end{aligned} \tag{16}$$

idem for $\mathbf{R}_y$ and $\mathbf{V}_y$.

For the sake of simplicity, we shall now assume that α is equal to 1. From (16), the following equalities are deduced:

$$\begin{aligned} \mathbf{R}_x &= \left(\begin{pmatrix}\mathbf{1}\\ \mathbf{1}'\end{pmatrix}, \begin{pmatrix}\mathbf{Z}_1\\ \mathbf{Z}'_2\end{pmatrix}\right)\begin{pmatrix} r_x(0)\\ v_{x,1}\end{pmatrix} + u_x\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'_1\end{pmatrix} \\ \mathbf{V}_x &= \left(\begin{pmatrix}\mathbf{Z}_1\\ \mathbf{Z}'_2\end{pmatrix}, \begin{pmatrix}\mathbf{Z}_1\\ \mathbf{Z}'_2\end{pmatrix} + \begin{pmatrix}\mathbf{Z}_1^2\\ k\mathbf{Z}'_2\end{pmatrix} + \begin{pmatrix}\mathbf{0}\\ \mathbf{Z}_2^2\end{pmatrix}\right) \\ &\quad \times \begin{pmatrix} r_x(0)\\ v_{x,1}\end{pmatrix} + u_x\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}_2^2\end{pmatrix}. \end{aligned} \tag{17}$$

Denoting $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3\}$, the vectors defined as below:

$$\mathbf{V}_1 = \begin{pmatrix}\mathbf{1}\\ \mathbf{1}'\end{pmatrix}, \qquad \mathbf{V}_2 = \begin{pmatrix}\mathbf{Z}_1\\ \mathbf{Z}'_2\end{pmatrix}$$

$$\mathbf{V}_3 = \begin{pmatrix}\mathbf{Z}_1\\ \mathbf{Z}'_2\end{pmatrix} + \begin{pmatrix}\mathbf{Z}_1^2\\ k\mathbf{Z}'_2\end{pmatrix} + \begin{pmatrix}\mathbf{0}\\ \mathbf{Z}_2^2\end{pmatrix}$$

we deduce:

$$\begin{aligned} &(\mathbf{R}_x, \mathbf{R}_y, \mathbf{V}_x, \mathbf{V}_y) \\ &= \underbrace{(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_2, \mathbf{V}_3)\overbrace{\begin{pmatrix} r_x(0) & r_y(0) & 0 & 0 \\ v_{x,1} & v_{y,1} & 0 & 0 \\ 0 & 0 & r_x(0) & r_y(0) \\ 0 & 0 & v_{x,1} & v_{y,1}\end{pmatrix}}^{(1')}}_{(1)} \\ &\quad + \underbrace{\left(\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'_1\end{pmatrix}, \begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'_1\end{pmatrix}, \begin{pmatrix}\mathbf{0}\\ \mathbf{Z}_2^2\end{pmatrix}, \begin{pmatrix}\mathbf{0}\\ \mathbf{Z}_2^2\end{pmatrix}\right)\mathrm{diag}(u_x, u_y, u_x, u_y).}_{(2)} \end{aligned} \tag{18}$$

Let us examine the two terms of the left member of the previous vectorial equality. Under the non-zero bearing-rate hypothesis (i.e. $r_x(0)v_{y,1} - r_y(0)v_{x,1} \neq 0$), the vectorial subspace (denoted E_1) spanned by the term (1) is of rank 3. This is a direct consequence of the invertibility of the matrix $(1')$. Similarly, when the control vector $\mathbf{U}$ is non null, the rank of the vectorial subset (denoted E_2) spanned by the term (2) is equal to 2.

Considering the positions of zeros in

$$\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}_2^2\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'_1\end{pmatrix},$$

it is straightforwardly shown that the intersection of E_1 and E_2 is reduced to the null vector. Assume now that the rank of $(\mathbf{R}_x, \mathbf{R}_y, \mathbf{V}_x, \mathbf{V}_y)$ is deficient (i.e. inferior to 4), this means that there is a non-null vector $(\mathbf{L})$ in its kernel. From (18), we deduce that the intersection of E_1 and E_2 is non-null.

This is a contradiction, so the rank of $\mathcal{O}$ is equal to 4, yielding the following property.

PROPERTY 1 *Assume that the source motion is rectilinear and uniform and that the ownship trajectory has multiple legs. Then the source motion is observable iff the ownship trajectory is not comprised of only zero bearing-rate legs.*

Furthermore, we have:

$$\begin{aligned} \dim(sp(E_1 \cup E_2)) &= \dim(sp(E_1) + sp(E_2)) \\ &\quad - \dim(sp(E_1 \cap E_2)) = 5. \end{aligned} \tag{19}$$

Thus, we see that the effect of the observer maneuver is (generally) to add a 2-dimensional space to the observable space. This point will be clarified by using the dual formalism (see Sect. 5). Further, we note that rank degeneracy can occur iff:

$$\det\begin{pmatrix} r_x(0) & v_{x,1} \\ r_y(0) & v_{y,1}\end{pmatrix} = \det\begin{pmatrix} v_{x,1} & v_{x,2} \\ v_{y,2} & v_{y,2}\end{pmatrix} = 0.$$

The above condition (4.6) means that the observer (ownship) trajectory is comprised of two consecutive legs with zero bearing-rate or equivalently of two "collision" legs.

As we shall see later, the previous reasoning may be straightforwardly (but tediously) extended to the general case.

Except for the zero bearing-rate case, unobservability can be achieved only if ownship motion is not a multileg one (i.e. it is a smooth trajectory) and if the observer maneuvers at each time. A rather paradoxical result!

Indeed the rank degeneracy of $\mathcal{O}$ amounts to the existence of a (nonnull) 4-dimensional vector $\mathbf{H}$ in $\ker\mathcal{O}$, which amounts to

$$h_1\cos\theta_t - h_2\sin\theta_t + th_3\cos\theta_t - th_4\sin\theta_t = 0$$
$$\Leftrightarrow \tan\theta_t = \frac{h_1 + th_3}{h_2 + th_4} \qquad (20)$$
$$(\cos\theta_t \neq 0).$$

The scalars $h_1,\ldots,h_4$ are obtained by considering the following function:

$$f(t) = \det\begin{pmatrix} r_x(t) & h_1 + th_3 \\ r_y(t) & h_2 + th_4\end{pmatrix}$$

and the following conditions:

$$\left|\begin{array}{l} 1)\ f(0) = 0 \\ 2)\ f'(0) = 0 \quad (f(t) \text{ must be null on } [t_0, t_f]) \\ 3)\ f''(0) = 0 \end{array}\right. \qquad (21)$$

yielding

$$\tan\theta_t = \frac{r_x(0) + tv_x(0)}{r_y(0) + tv_y(0)}. \qquad (22)$$

Note that this is the criterion of Nardone and Aidala obtained by solving the differential equation (71). A condition relative to observer accelerations is directly obtained after taking (21) into consideration (see Payne [11]), i.e.,

$$\begin{vmatrix} r_x(t) \\ r_y(t)\end{vmatrix} = \alpha(t)\begin{vmatrix} r_x(0) + tv_x(0) \\ r_y(0) + tv_y(0)\end{vmatrix}. \qquad (23)$$

B. Maneuvering Source and Nonmaneuvering Observer

This case study is quite enlightening even if the system is unobservable because a general result is easily obtained allowing us to guess the conditions which will ensure observability.

Consider, first, a two-legs path of the source. Opposite to the case of a nonmaneuvering source, two transition matrices (F_1 and F_2) are now required. The measurement equations take the following form:

$$\left|\begin{array}{l} z_0 = H_0\mathbf{X}_0 \\ z_1 = H_1F_1\mathbf{X}_0 \\ \vdots \\ z_k = H_kF_1^k\mathbf{X}_0 \\ z_{k+1} = H_{k+1}F_2F_1^k\mathbf{X}_0 \leftarrow \text{source maneuver} \\ \vdots \\ z_{k+j} = H_{k+j}F_2^jF_1^k\mathbf{X}_0 \end{array}\right.$$

with

$$H_\ell = (\cos\theta_\ell, -\sin\theta_\ell, 0, 0, 0, 0) \qquad 0 \le \ell \le k + j$$
$$F_1 = \begin{pmatrix} Id & \alpha Id & 0 \\ 0 & Id & 0 \\ 0 & 0 & Id\end{pmatrix}$$
$$F_2 = \begin{pmatrix} Id & 0 & \alpha Id \\ 0 & Id & 0 \\ 0 & 0 & Id\end{pmatrix} \qquad (24)$$
$$Id \triangleq Id_2$$
$$\mathbf{X}_0 = (r_x(0), r_y(0), v_{x,1}, v_{y,1}, v_{x,2}, v_{y,2}).$$

This time, the dimension of the state vector $\mathbf{X}_0$ is equal to 6. The parameters $(r_x(0), r_y(0))$ represent the initial source position and $(v_{x,i}, v_{y,i})_{i=1,2}$ are the components of the velocity vector on source legs 1 and 2. We state without proof the following lemma.

LEMMA *For* $(\ell, m \in \mathbb{N})$,

$$F_2^\ell F_1^m = \begin{pmatrix} Id & \ell\alpha Id & m\alpha Id \\ 0 & Id & 0 \\ 0 & 0 & Id\end{pmatrix}$$

and

$$F_2^\ell F_1^m = F_1^m F_2^\ell. \qquad (25)$$

The general structure of the observability matrix $\mathcal{O}$ and therefore of $\mathcal{O}'$ (defined as in (9)) immediately follows, i.e.,

$$\mathcal{O}' = \begin{pmatrix} r_x(1) & r_y(1) & \alpha r_x(1) & \alpha r_y(1) & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ r_x(k) & r_y(k) & k\alpha r_x(k) & k\alpha r_y(k) & 0 & 0 \\ r_x(k+1) & r_y(k+1) & k\alpha r_x(k+1) & k\alpha r_y(k+1) & \alpha r_x(k+1) & \alpha r_y(k+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \underbrace{r_x(k+1)}_{\mathbf{R}_x} & \underbrace{r_y(k+1)}_{\mathbf{R}_y} & \underbrace{k\alpha r_x(k+j)}_{\mathbf{S}_x} & \underbrace{k\alpha r_y(k+j)}_{\mathbf{S}_y} & \underbrace{j\alpha r_x(k+j)}_{\mathbf{T}_x} & \underbrace{j\alpha r_y(k+j)}_{\mathbf{T}_y} \end{pmatrix}. \qquad (26)$$

Using (24) and (25), the vectors $\{\mathbf{R}_x,\ldots,\mathbf{T}_y\}$ take the following form:

$$\left|\begin{aligned} \mathbf{R}_x &= r_x(1)\begin{pmatrix}\mathbf{1}\\ \mathbf{1}'\end{pmatrix} + \alpha v_{x,1}\begin{pmatrix}\mathbf{Z}\\ k\mathbf{1}'\end{pmatrix} + \alpha v_{x,2}\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'\end{pmatrix}\\ \mathbf{R}_y &= r_y(1)\begin{pmatrix}\mathbf{1}\\ \mathbf{1}'\end{pmatrix} + \alpha v_{y,1}\begin{pmatrix}\mathbf{Z}\\ k\mathbf{1}'\end{pmatrix} + \alpha v_{y,2}\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'\end{pmatrix}\\ \mathbf{S}_x &= \alpha r_x(1)\begin{pmatrix}\mathbf{Z}\\ k\mathbf{1}'\end{pmatrix} + \alpha^2 v_{x,1}\begin{pmatrix}\mathbf{Z}^2\\ k^2\mathbf{1}'\end{pmatrix} + \alpha^2 v_{x,2}\begin{pmatrix}\mathbf{0}\\ k\mathbf{Z}'\end{pmatrix}\\ \mathbf{S}_y &= \alpha r_y(1)\begin{pmatrix}\mathbf{Z}\\ k\mathbf{1}'\end{pmatrix} + \alpha^2 v_{y,1}\begin{pmatrix}\mathbf{Z}^2\\ k^2\mathbf{1}'\end{pmatrix} + \alpha^2 v_{y,2}\begin{pmatrix}\mathbf{0}\\ k\mathbf{Z}'\end{pmatrix}\end{aligned}\right.$$

and

$$\left|\begin{aligned} \mathbf{T}_x &= \alpha r_x(1)\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'\end{pmatrix} + \alpha^2 v_{x,1}\begin{pmatrix}\mathbf{0}\\ k\mathbf{Z}'\end{pmatrix} + \alpha^2 v_{x,2}\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'^2\end{pmatrix}\\ \mathbf{T}_y &= \alpha r_y(1)\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'\end{pmatrix} + \alpha^2 v_{y,1}\begin{pmatrix}\mathbf{0}\\ k\mathbf{Z}'\end{pmatrix} + \alpha^2 v_{y,2}\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'^2\end{pmatrix}\end{aligned}\right. \qquad (27)$$

Note that in (27), the index $-'-$ stands for the second leg. The definition of the vectors $\mathbf{1},\mathbf{1}',\mathbf{Z},\mathbf{Z}',\mathbf{Z}^2,\mathbf{Z}'^2$ is that of (11) and (15).

The following subspace inclusion[1] results from (27):

$$\mathrm{Im}(\mathcal{O}') \subset sp\left\{\begin{pmatrix}\mathbf{1}\\ \mathbf{1}'\end{pmatrix},\begin{pmatrix}\mathbf{Z}\\ k\mathbf{1}'\end{pmatrix},\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'\end{pmatrix},\begin{pmatrix}\mathbf{Z}^2\\ k^2\mathbf{1}'\end{pmatrix},\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'^2\end{pmatrix}\right\}$$

and therefore

$$\mathrm{rank}(\mathcal{O}') \le 5. \qquad (28)$$

Consequently the system is unobservable which is not unexpected. However under the non zero bearing-rate hypothesis (denoted $\mathcal{H}_2$),[2] i.e.,

$$\mathcal{H}_2 : \left(\det\begin{pmatrix} r_x(1) & v_{x,1}\\ r_y(1) & v_{y,1}\end{pmatrix} \text{ and } \det\begin{pmatrix} v_{x,1} & v_{x,2}\\ v_{y,1} & v_{y,2}\end{pmatrix}\right) \neq 0$$

then

$$\mathrm{rank}\,\mathcal{O}' = 5. \qquad (29)$$

It is rather surprising that the dimension of the observable space increases with the source maneuver (note that the maneuver instant is assumed to be known, this point is discussed later). Tediously, the above property may be extended to the multileg case ($\mathcal{H}_2$ becoming $\mathcal{H}_\ell$) yielding thus the following result.

PROPERTY 2 *Consider a nonmaneuvering observer. Then, under the hypothesis $\mathcal{H}_\ell$, the dimension of the observable space is $2\ell+1$ (the dimension of $\mathbf{X}_0$ is $2\ell+2$).*

A general proof of Property 2 is given in the section devoted to the dual approach.

There are various means to recover observability. One of them may consist in using multiple arrays [22]. The more classical one consists in allowing source maneuvers. Under $\mathcal{H}_\ell$, the unobservable space is spanned by $\mathbf{X}_0$.

Let us consider now the general case. At this point, we can devise from Property 2 that an observer maneuver may be sufficient to recover observability.

C. Maneuvering Source and Observer

For the sake of simplicity, we first consider the case where the source and observer paths consist

[1] The symbol "Im" denotes the image [19] of a matrix.

[2] The hypothesis $\mathcal{H}_2$ correponds to a non-zero bearing-rate hypothesis for each of the *two* legs, $\mathcal{H}_l$ corresponds to the same hypothesis for each of the l legs.

of two legs. The observations $\{z_i\}$ stand as follows:

$$\left|\begin{array}{ll} z_1 & = H_1 F_1 \mathbf{X}_0 \\ \vdots & \\ z_k & = H_k F_1^k \mathbf{X}_0 \\ z_{k+1} & = H_{k+1} F_2 F_1^k \mathbf{X}_0 \leftarrow \text{source maneuver} \\ \vdots & \quad \vdots \\ z_{k+\ell} & = H_{k+\ell} F_2^{\ell} F_1^k \mathbf{X}_0 \\ z_{k+\ell+1} & = H_{k+\ell+1} F_2^{\ell+1} F_1^k \mathbf{X}_0 \leftarrow \text{observer maneuver} \\ \vdots & \\ z_{k+\ell+m} & = H_{k+\ell+m} F_2^{\ell+m} F_1^k \mathbf{X}_0 \end{array}\right.$$

with

$$H_j = (\cos\theta_j, -\sin\theta_j, 0, 0, 0, 0) \qquad 1 \leq j \leq k+\ell+m$$

and F_1 and F_2 defined as in (25) and

$$\dim \mathbf{X}_0 = 6. \tag{30}$$

Let $\mathbf{R}_x, \mathbf{R}_y, \mathbf{S}_x, \mathbf{S}_y, \mathbf{T}_x, \mathbf{T}_y$ be the columns of the modified observability matrix $\mathcal{O}'$, i.e.,

$$\mathcal{O}' = (\mathbf{R}_x, \mathbf{R}_y, \mathbf{S}_x, \mathbf{S}_y, \mathbf{T}_x, \mathbf{T}_y). \tag{31}$$

Using (30), the vectors $\mathbf{R}_x, \ldots, \mathbf{T}_y$ may be written as linear combinations of elementary vectors, i.e.,

$$\begin{aligned} \mathbf{R}_x &= r_x(1)\begin{pmatrix}\mathbf{1}\\ \mathbf{1}'\\ \mathbf{1}''\end{pmatrix} + \alpha v_{x,1}\begin{pmatrix}\mathbf{Z}\\ k\mathbf{1}'\\ k\mathbf{1}''\end{pmatrix} + \alpha v_{x,2}\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'\\ \ell\mathbf{1}''\end{pmatrix} + \alpha v_{x,3}\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{Z}''\end{pmatrix} \\ \mathbf{R}_y &= r_y(1)\begin{pmatrix}\mathbf{1}\\ \mathbf{1}'\\ \mathbf{1}''\end{pmatrix} + \alpha v_{y,1}\begin{pmatrix}\mathbf{Z}\\ k\mathbf{1}'\\ k\mathbf{1}''\end{pmatrix} + \alpha v_{y,2}\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'\\ \ell\mathbf{1}''\end{pmatrix} + \alpha v_{y,3}\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{Z}''\end{pmatrix} \\ \mathbf{S}_x &= r_x(1)\begin{pmatrix}\mathbf{Z}\\ k\mathbf{1}'\\ k\mathbf{1}''\end{pmatrix} + \alpha^2 v_{x,1}\begin{pmatrix}\mathbf{Z}^2\\ k\mathbf{1}'\\ k\mathbf{1}''\end{pmatrix} \\ &\quad + \alpha^2 v_{x,2}\begin{pmatrix}\mathbf{0}\\ k\mathbf{Z}'\\ k\mathbf{1}''\end{pmatrix} + k\alpha^2 v_{x,3}\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{Z}''\end{pmatrix} \\ \mathbf{S}_y &= \text{''} \\ \mathbf{T}_x &= \alpha r_x(1)\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'\\ \mathbf{Z}''\end{pmatrix} + \alpha^2 v_{x,1}\begin{pmatrix}\mathbf{0}\\ k\mathbf{Z}'\\ k\mathbf{Z}''\end{pmatrix} \\ &\quad + \alpha^2 v_{x,2}\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'^2\\ \ell\mathbf{Z}''\end{pmatrix} + \alpha^2 v_{x,3}\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{Z}''^2\end{pmatrix} \\ \mathbf{T}_y &= \text{''}. \end{aligned} \tag{32}$$

In (32) the index –′– stands for $k+1 \leq j \leq k+\ell$, the index –″– stands for $k+\ell+1 \leq j \leq k+\ell+m$. The structure of $\mathbf{S}_y$ is identical to that of $\mathbf{S}_x$, that of $\mathbf{T}_y$ is identical to that of $\mathbf{T}_x$.

Equations (16)–(18) are straightforwardly (but quite tediously) extended to this scenario if ℓ is non null, that is to say the source and the observer do not maneuver at the same instant.

Hence, the following result is obtained.

1) Assume that the source and the observer do not maneuver at the same instant.

2) Assume that there are no zero bearing-rate legs then:

$$\text{rank}(\mathcal{O}') = 6.$$

Extension to the multileg case is direct but very tedious, yielding the following.

PROPERTY 3 *Suppose that:*

1) *the source and the observer do not maneuver at the same instant, the source maneuver instants are known,*

2) *the observer path is at least constituted of two legs, and*

3) *there is no zero bearing-rate leg.*

Then the vector $\mathbf{X}_0$ *is observable.*

We stress that under the hypotheses of Property 3 only **one** maneuver of the observer is necessary whatever the number of source legs. At a first glance this result may appear quite surprising but actually if we think of Property 2, Property 3 appears quite natural.

The analysis of observability has culminated with the analysis of a maneuvering source. We have presented a general approach able to cope with more and more complicated scenarios. In [22] this method is applied to the case of multiple observers. Even if the analysis in [22] was restricted to a RUN motion of the source, the discrete-time approach may be directly extend to maneuvering sources. It is then easily shown that the multiple observer system is generally observable.

Another interesting problem is the observability analysis for multiple sources. Once again, the discrete-time approach yields a simple and efficient framework and Property 3 may thus be extended to multiple maneuvering sources. It should be stressed that, in this case, the observability analysis is purely algebraic which means that the statistical assignment problem is not considered. For the sake of brevity, the calculations corresponding to these cases are omitted.

V. THE DUAL APPROACH

The aim of this section is to present a dual approach of the observability analysis.

If V is an n-dimensional vector space (over $\mathbb{R}$), then we denote by V^* (the dual space of V) the vector space of linear mappings of V into $\mathbb{R}$.

For a (linear) subspace W of V put [24, 25]:

$$W^{\perp} = \{f \in V^* \text{ s.t. } f(x) = 0 \ \forall x \in W\}$$

where $W^{\perp}$ is the annihilator of W in V and the following relations hold [24]:

$$\begin{aligned} &r_1 : (W_1 \cap W_2)^{\perp} = W_1^{\perp} + W_2^{\perp}, (W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp} \\ &r_2 : \dim W^{\perp} = n - \dim W \\ &r_3 : V^*/W^{\perp} \simeq W^* \end{aligned} \tag{34}$$

(V/W is the factor space of V over W).

Let φ be an endomorphism of V then we denote φ^* the dual mapping of φ, φ^* itself is an endomorphism of V^* defined by $\varphi^*(f) = f \circ \varphi$ ($f \in V^*, \circ$: composition). We have the relation [24]:

$$r_4 : (\ker\varphi)^{\perp} = \mathrm{Im}(\varphi^*). \tag{35}$$

Using (r_1, r_4) the following (instrumental) relation is obtained:

$$\{\ker(\mathcal{O}')\}^{\perp} = \left\{\bigcap_i \ker(H_i F^i)\right\}^{\perp} = \Sigma_i \, \mathrm{Im}(F^{i*}H_i^*). \tag{36}$$

In view of (36), determining $\Sigma_i \mathrm{Im}(F^{i*}H_i^*)$ is sufficient to obtain $\ker\mathcal{O}'$.

Consider, first, the case of a nonmaneuvering source and denote $\mathbf{e}_0$ and $\mathbf{v}_0$ the 4-dimensional vectors defined by

$$\mathbf{e}_0 \triangleq \begin{pmatrix} r_x(0) \\ r_y(0) \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_0 \triangleq \begin{pmatrix} v_x \\ v_y \\ 0 \\ 0 \end{pmatrix}.$$

For all this section we assume (for the sake of simplicity) that $\alpha = 1$.

The following equality is easily proved:

$$\begin{aligned} &(3F_1^* - 2Id)(\mathbf{e}_0 + 3\mathbf{v}_0) - 3(2F_1^* - Id)(\mathbf{e}_0 + 2\mathbf{v}_0) \\ &\quad + 3F_1^*(\mathbf{e}_0 + \mathbf{v}_0) - \mathbf{e}_0 = \mathbf{0}. \end{aligned} \tag{37}$$

If we recall the following equalities:

$$\begin{aligned} &F_1^{*2} = 2F_1^* - Id, F_1^{*3} = 3F_1^* - 2Id, \ldots, F_1^{*k} \\ &\quad = kF_1^* - (k-1)Id \\ &H_0^* = \mathbf{e}_0, H_1^* = \mathbf{e}_0 + \mathbf{v}_0, \ldots, H_k^* = \mathbf{e}_0 + k\mathbf{v}_0 \end{aligned}$$

then (37) takes the following form:

$$F_1^{*3}H_3^* - 3F_1^{*2}H_2^* + 3F_1^*H_1^* - H_0^* = \mathbf{0} \tag{38}$$

or

$$(F_1^* - Id)^3\mathbf{e}_0 + 3F_1^*(F_1^* - Id)^2\mathbf{v}_0 = \mathbf{0}. \tag{39}$$

Note that (39) may be interpreted by considering the minimal polynomial of F_1^* which is $(x-1)^2$. Further note that the derivative of the polynomial $(x-1)^3$ (associated with the initial position vector) is the polynomial $3(x-1)^2$ (associated with the velocity).

In the general case ($k \geq 3$), the following equation is obtained by direct identification of the terms factors of the vectors $\{\mathbf{e}_0, F_1^*\mathbf{e}_0, \mathbf{v}_0, F_1^*\mathbf{v}_0\}$ which constitute a basis of $\mathbb{R}^4$

$$\begin{aligned} &F_1^{*k}H_k^* - \frac{k(k-1)}{2}F_1^{*2}H_2^* + k(k-2)F_1^*H_1^* \\ &\quad - \frac{(k-1)(k-2)}{2}H_0^* = \mathbf{0}. \end{aligned} \tag{40}$$

Following (40) we obtain

$$\sum_i \mathrm{Im}(F^{*i}H_i^*) = sp[\mathbf{e}_0, F_1^*(\mathbf{e}_0 + \mathbf{v}_0), F_1^{*2}(\mathbf{e}_0 + 2\mathbf{v}_0)]$$

and consequently (36):

$$\dim(\mathcal{N} \triangleq \ker\mathcal{O}) = \dim\left(\bigcap_i \ker(H_i F^i)\right) = 1. \tag{41}$$

The classical result (nonmaneuvering source and observer) has been retrieved by the dual formalism which does not appear, at a first glance, particularly useful. But the situation is reversed for the study of more complicated situations.

Let us consider now the case of a maneuvering observer and a nonmaneuvering source. Let the vector $\mathbf{u}$ be the new velocity vector after the time $k+1$. Then we have

$$\begin{aligned} F_1^{*(k+1)}H_{k+1}^* &= F_1^{*(k+1)}[\mathbf{e}_0 + (k+1)\mathbf{v}_0 + \mathbf{u} - \mathbf{v}_0] \\ &= F_1^{*(k+1)}(\mathbf{e}_0 + (k+1)\mathbf{v}_0) + F_1^{*(k+1)}(\mathbf{u} - \mathbf{v}_0) \end{aligned}$$

and more generally:

$$\begin{aligned} F_1^{*(k+\ell)}H_{k+\ell}^* &= F_1^{*(k+\ell)}(\mathbf{e}_0 + (k+\ell)\mathbf{v}_0) \\ &\quad + \ell F_1^{*(k+\ell)}(\mathbf{u} - \mathbf{v}_0). \end{aligned} \tag{42}$$

Now, according to (40):

$$\begin{aligned} &F_1^{*(k+\ell)}(\mathbf{e}_0 + (k+\ell)\mathbf{v}_0) \\ &\quad \in sp[\mathbf{e}_0, F_1^*(\mathbf{e}_0 + \mathbf{v}_0), F_1^{*2}(\mathbf{e}_0 + 2\mathbf{v}_0)] \end{aligned} \tag{43}$$

so that the observability analysis is reduced to consideration of the vector $F_1^{*(k+\ell)}(\mathbf{u} - \mathbf{v}_0)$.

Let us denote $\mathbf{u}'$ the *2-dimensional* vector defined by

$$u'_x \triangleq u_x - v_x, \qquad u'_y \triangleq u_y - v_y \tag{44}$$

then we must calculate the following determinants $D_{k,\ell}$:[3]

$$D_{k,\ell} \triangleq \det\left(\begin{array}{c|c|c|c} \mathbf{u}' & \mathbf{e}_0 & \mathbf{e}_0 + \mathbf{v}_0 & \mathbf{e}_0 + 2\mathbf{v}_0 \\ (k+\ell)\mathbf{u}' & \mathbf{0} & \mathbf{e}_0 + \mathbf{v}_0 & 2(\mathbf{e}_0 + 2\mathbf{v}_0) \end{array}\right). \tag{45}$$

[3]By a slight abuse of notation, $\mathbf{e}_0$ is then considered as a two-dimensional vector.

These determinants may be easily calculated (see Appendix A) yielding

$$D_{k,\ell} = (k+\ell)\det(\mathbf{e}_0 + \mathbf{v}_0, \mathbf{u}')\det(\mathbf{e}_0, \mathbf{e}_\ell)$$

with

$$\begin{aligned} \mathbf{e}_\ell &= 2\mathbf{v}_0 - \beta'_{k,\ell}(\mathbf{v}_0 - \mathbf{u}') \\ \beta'_{k,\ell} &= \frac{(k+1)(k+\ell-1)}{k+\ell}\beta'_{k,1} \end{aligned} \quad . \tag{46}$$

Since it has been assumed that the vectors $\mathbf{e}_0 + \mathbf{v}_0$ and $\mathbf{u}'$ are linearly independent, the determinants $\{D_{k,\ell}\}_{\ell-1}^{j}$ are null iff the determinants $\{\det(\mathbf{e}_0, \mathbf{e}_\ell)\}_{\ell=1}^{j}$ are null altogether. Now the nullity of all these determinants implies ($j \geq 3$):

$$\mathbf{v}_0 - \mathbf{u}', \mathbf{v}_0 \text{ and } \mathbf{e}_0 \text{ are colinear}$$

or equivalently:

$$\mathbf{e}_0, \mathbf{v}_0 \text{ and } \mathbf{u}' \text{ are colinear.}$$

Property 1 is thus proved by using the dual formalism.

Although the dual formalism appears more complicated at first, more complex scenarios can be analyzed in this way. Extensions to complex scenarios are straightforward since the dual analysis of observability enlightens the thorough algebraic structure of the problem.

Consider now the case of a maneuvering source (with a velocity change at time k) and an observer with a constant velocity. For a two-legs path of the source, the dimension of the state $\mathbf{X}_0$ and of the transition matrices F_1 and F_2 (4) is equal to 6. Using (36) observability analysis is reduced to consider the following sequence of 6-dimensional vectors:

$$\begin{aligned} &H_0^*, F_1^* H_1^*, F_1^{*2} H_2^*, \ldots, F_1^{*k} H_k^*, F_2^*(F_1^{*k} H_{k+1}^*) \\ &F_2^{*2}(F_1^{*k} H_{k+2}^*), \ldots, F_2^{*j}(F_1^{*k} H_{k+j}^*). \end{aligned} \tag{47}$$

Let us denote $\mathbf{e}'_0$ the $(k+1)$st vector of the sequence, i.e.,

$$\mathbf{e}'_0 \triangleq F_1^{*k} H_k^*$$

then we have

$$F_1^{*k} H_{k+1}^* = \mathbf{e}'_0 + \mathbf{v}'_2, \ldots, F_1^{*k} H_{k+\ell}^* = \mathbf{e}'_0 + \ell \mathbf{v}'_2$$

where:[4]

$$\mathbf{v}'_2 \triangleq F_1^{*k} \mathbf{v}_2$$

($\mathbf{v}_1$ and $\mathbf{v}_2$ are the two consecutive velocity vectors).

Quite similarly to (37) the following equality is obtained by direct identification:

$$\begin{aligned} &(3F_2^* - 2Id)(\mathbf{e}'_0 + 3\mathbf{v}'_2) - 3(2F_2^* - Id)(\mathbf{e}'_0 + 2\mathbf{v}'_2) \\ &\quad + 3F_2^*(\mathbf{e}'_0 + \mathbf{v}'_2) = \mathbf{e}'_0 \end{aligned} \tag{48}$$

[4]Actually $\mathbf{v}'_2 = \mathbf{v}_2$ thanks to the structure of F_1.

and more generally:

$$\begin{aligned} &F_2^{*\ell} F_1^{*k} H_{k+\ell}^* - \frac{\ell(\ell-1)}{2} F_2^{*2}(F_1^{*k} H_{k+2}^*) \\ &\quad - \ell(\ell-2) F_2^*(F_1^{*k} H_{k+1}^*) = \frac{(\ell-1)(\ell-2)}{2}\mathbf{e}'_0. \end{aligned} \tag{49}$$

Therefore, the following inclusion holds:

$$\begin{aligned} &sp\{H_0^*, \ldots, F_1^{*k} H_k^*, F_2^*(F_1^{*k} H_{k+1}^*), \ldots, F_2^{*j}(F_1^{*k} H_{k+j})\} \\ &\quad \subset sp\{\mathbf{e}_0, F_1^*(\mathbf{e}_0 + \mathbf{v}_1), F_1^{*2}(\mathbf{e}_0 + 2\mathbf{v}_1), \\ &\qquad F_2^*(\mathbf{e}'_0 + \mathbf{v}'_2), F_2^{*2}(\mathbf{e}'_0 + 2\mathbf{v}'_2)\} \end{aligned}$$

and thus

$$\dim(\Sigma\, \mathrm{Im}(F^i H_i^*)) \leq 5. \tag{50}$$

Now the following observations, due to the structure of F_1 and F_2, are instrumental.

1) O_1: The general form of the vectors $\{\mathbf{e}_0, F_1^*(\mathbf{e}_0 + \mathbf{v}_1), F_1^{*2}(\mathbf{e}_0 + 2\mathbf{v}_1)\}$ is a vector

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \quad \text{where} \quad \mathbf{x} \in \mathbb{R}^4, \quad \mathbf{0} \in \mathbb{R}^2.$$

2) O_2: The two last components of the vectors $F_2^*(\mathbf{e}'_0 + \mathbf{v}'_2)$ and $F_2^{*2}(\mathbf{e}_0 + 2\mathbf{v}'_2)$ are non-zero (under the non-zero bearing-rate hypothesis for leg 2).

Using the above observations (O_1 and O_2), we have proved that if the non-zero bearing-rate assumption is valid on each leg ($\mathcal{H}_\ell$ hypothesis), then we have

$$\dim \mathcal{N} = 1$$

which is nothing but Property 2.

The above reasoning can be directly extended to any number of legs thus yielding a general proof of Property 2 (under $\mathcal{H}_\ell$, $\dim\mathcal{N} = 1$).

The case of maneuvering source **and** observer can be treated by the same way. In this case, the sequence of vectors (49) is extended by including the vectors $F_2^{*(\ell+j)} F_1^{*k} H_{k+\ell+j}^*$ appearing in (42).

Now

$$\begin{aligned} &F_2^{*(\ell+j)} F_1^{*k} H_{k+\ell+j}^* \\ &\quad = F_2^{*(\ell+j)} F_1^{*k}[\mathbf{e}_0 + k\mathbf{v}_1 + (\ell+j)\mathbf{v}_2 - \ell\mathbf{u}] \end{aligned} \tag{51}$$

which leads us to consider (as in (44)) the following vectors:

$$F_2^{*(\ell+j)} F_1^{*k}(\mathbf{v}_2 - \mathbf{u}) \tag{52}$$

and the associated determinants (see (45)).

Using (51) it is then a trivial matter to prove Property 3.

VI. UNKNOWN INSTANTS OF SOURCE VELOCITY CHANGES

Up to now (Sections IV and V), the instants of source maneuvers were assumed known. Next supposing, as previously, a leg-by-leg source trajectory we extend the previous observability analysis to unknown instants of source velocity changes.

We consider the following jump linear system [26] (cf. (5), (14), (24), (30))

$$\begin{aligned} \mathbf{X}_{k+1} &= F(r_k)\mathbf{X}_k + \mathbf{U}_{r'_k} \\ z_k &= H_k \mathbf{X}_k. \end{aligned} \tag{53}$$

In (53) the index r_k is the source leg index. The instants of observer maneuvers are denoted by $r'_k(\mathbf{U}_{r'_k} = (0,\ldots,0,u_x(r'_k),u_y(r'_k))^*)$. Note that both r_k and r'_k are assumed discrete and deterministic. The matrix $F(r_k)$ and the state vector have the appropriate dimension which is $2(\ell+1)$ where ℓ is the number of source legs. Consider the jump linear system (53), then we define observability as follows [26].

DEFINITION Let T_{0a} be the minimum time such that equivalent outputs $z_k(\mathbf{X}_0 = \mathbf{X}_{\#1}) = z_k(\mathbf{X}_0 = X_{\#2})$ and known inputs $\mathbf{U}$ in the interval $0 \le k \le T_{0a}$ imply that $\mathbf{X}_{\#1} = \mathbf{X}_{\#2}$. We say that the jump linear system (53) is absolutely observable if T_{0a} is finite.

Equivalently, the system (53) is observable if the following conditions are fulfilled:

$$C_1 : T \ge T_{0a} \Rightarrow \operatorname{rank}(\mathcal{O}_T) = 2(l+2)$$

$$C_2 : (T \ge T_{0a})$$

for any given output sequence $\{z_0,z_1,\ldots,z_T\}$ and corresponding input sequence $\{u_0,u_1,\ldots,u_T\}$, the initial state vector $\mathbf{X}_0$ can be uniquely determined (without the knowledge of the values of $r_1,r_2,\ldots,r_{T-1}$).

We now investigate the implications of the above definition for a maneuvering source. For the sake of simplicity, the observability analysis is restricted to a two-legs source path. Consider for instance the following scenarios:

Scenario 1: Observer maneuver at time $k+1(r'_k = \delta(k+1))$, source maneuver at time $k+j+1$, state vector $\mathbf{X}_{\#1}$.

Scenario 2: Observer maneuver at time $k+1(r'_k = \delta(k+1))$ source maneuver at time $k+j'+1(j' > j)$, state vector $\mathbf{X}_{\#2}$.

We now consider the conditions ensuring the unicity of the state vectors $\mathbf{X}_0$. Practically, we show that under non-zero bearing-rate assumptions and given the input and output sequences, $\mathbf{X}_{\#1}$ and $\mathbf{X}_{\#2}$ are equal. As a by-product, we see that the source maneuvering instants are identical for the two scenarios $(j = j')$.

For that purpose, let us denote $\mathcal{O}_1$ the observability matrix associated with the scenario 1 and $\mathcal{O}_2$ the observability matrix associated with the scenario 2, then (see (53)) $\mathcal{O}_1$ and $\mathcal{O}_2$ take the following form:[5]

$$\begin{aligned} \mathcal{O}_1^* = (&H_0^*, F_1^*H_1^*, \ldots, F_1^*H_k^*, \ldots, \\ &F_1^{*k+j}H_{k+j}^* F_1^{*k+j}F_2^*H_{k+j+1}, \ldots, \\ &F_1^{*k+j}F_2^{*j'-j+1}H_{k+j'+1}, \ldots, \\ &F_1^{*k+j}F_1^{*j'-j+r}H_{k+j'+r}^*) \end{aligned}$$

and

$$\begin{aligned} \mathcal{O}_2^* = (&H_0^*, F_1^*H_1^*, \ldots, F_1^{*k}H_k^*, \ldots, \\ &F_1^{*k+j}H_{k+j}^*, F_1^{*k+j+1}H_{k+j+1}^*, \ldots, \\ &F_1^{*k+j'}F_2^*H_{k+j'+1}^*, \ldots, \\ &F_1^{*k+j'}F_2^{*r}H_{k+j'+r}^*). \end{aligned} \tag{54}$$

The system equations (5) result in the following equality:

$$\mathcal{O}\mathbf{X}_0 = \mathbf{z} \tag{55}$$

where $\mathbf{z}$ is the output vector associated with $\mathbf{X}_0$.

The output vectors $\mathbf{z}_i$ (scenario i) are directly deduced from (53), yielding

$$\begin{aligned} \mathbf{z}_1 = (&0,\ldots,0,H_{k+1}\mathbf{U},\ldots,H_{k+j}F_1^j\mathbf{U},H_{k+j+1}F_2F_1^j\mathbf{U},\ldots, \\ &H_{k+j'+1}F_2^{j'-j+1}F_1^j\mathbf{U},\ldots,H_{k+j'+r}F_2^{j'-j+r}F_1^j\mathbf{U})^* \\ \mathbf{z}_2 = (&0,\ldots,0,H_{k+1}\mathbf{U},\ldots,H_{k+j}F_1^j\mathbf{U},H_{k+j+1}F_1^{j+1}\mathbf{U},\ldots, \\ &H_{k+j'+1}F_1^{j'}F_2\mathbf{U},\ldots,H_{k+j'+r}F_1^{j'}F_2^r\mathbf{U})^*. \end{aligned} \tag{56}$$

Assuming $\operatorname{rank}\mathcal{O}_1 = \operatorname{rank}\mathcal{O}_2 = 6$, let us denote $\mathbf{X}_{\#1}$ and $\mathbf{X}_{\#2}$ the associated state vectors. Then:

$$\mathcal{O}_i\mathbf{X}_{\#i} = \mathbf{z}_i \qquad \text{for} \quad i = 1,2. \tag{57}$$

At this point it is worth considering the following decomposition of the 6-dimensional state vector $\mathbf{X}_{\#i}$:

$$\mathbf{X}_{\#i} = \begin{pmatrix} \mathbf{Y}_i \\ \mathbf{y}_i \end{pmatrix}$$

where

$$\mathbf{Y}_i \in \mathbb{R}^4, \qquad \mathbf{y}_i \in \mathbb{R}^2. \tag{58}$$

Let us denote $\mathbf{y}_2$ and $\mathbf{y}'_2$ the respective velocity vectors on the second source leg for the two scenarios[6] i.e.,

$$\mathbf{y}_2 \stackrel{\Delta}{=} (v_{x,2}, v_{y,2})$$

$$\mathbf{y}'_2 \stackrel{\Delta}{=} (v'_{x,2}, v'_{y,2}).$$

Then the consideration of the first $(k+j)$ rows of (57) implies

$$\mathbf{Y}_2 = \mathbf{Y}_1 \stackrel{\Delta}{=} (r_x, r_y, v_x, v_y)^*. \tag{59}$$

[5] F_1 and F_2 defined as in (24) and $H_k = (\cos\theta_k, -\sin\theta_k, 0,0,0,0)$.

[6] (The index 2 stands for the 2nd source leg) (cf. (57)).

Similarly, considering the $k+j+s$ rows of (57) where s runs from 0 to $j'-j$ we obtain the following set of equalities:

$$\begin{aligned} &H_{k+j+s}F_1^{k+j}F_2^s\mathbf{X}_{\#1} - H_{k+j+s}F_1^{k+j+s}\mathbf{X}_{\#2} \\ &\quad = sH_{k+j+s}F_{12}\mathbf{U}H_{k+j+s}(F_1^jF_2^s - F_1^{j+s})\mathbf{U} \\ &\quad = sH_{k+j+s}F_{12}\mathbf{U} \end{aligned}$$

with

$$F_{12} \triangleq \begin{pmatrix} 0 & -Id & Id \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad s = 0,\ldots,j'-j. \tag{60}$$

Note that F_{12} may be equivalently defined by the following matrix equality:

$$F_1^{k+j}F_2^s = F_1^{k+j+s} + sF_{12}. \tag{61}$$

Using (61), (60) simply results in

$$\begin{aligned} &s[\cos(\theta_{k+j+s})(v_{x,2} - v_x - u_x) \\ &\quad - \sin(\theta_{k+j+s})(v_{y,2} - v_y - u_y)] = 0 \\ &\quad s = 0 \text{ to } j'-j. \end{aligned}$$

Similarly, considering the last r rows of (57) the following set of equalities holds:

$$\begin{aligned} &s[\cos\theta_{k+j'+s}(v'_{x,2} - v_{x,2}) - \sin\theta_{k+j'+s}(v'_{y,2} - v_{y,2})] \\ &\quad + (j'-j)[\cos\theta_{k+j'+s}(v_{x,2} - u_x) \\ &\quad - \sin\theta_{k+j'+s}(v_{y,2} - u_y)] = 0 \\ &\quad s = 0 \text{ to } r. \end{aligned} \tag{62}$$

Thus under the non-zero bearing-rate hypothesis, we see that (61) and (62) imply that $j = j'$ (see (62)) and $v'_{x,2} = v_{x,2}$ (see (63)). Therefore:

$$\mathbf{X}_{\#1} = \mathbf{X}_{\#2} \quad \text{and} \quad j = j'. \tag{63}$$

The following property has thus been obtained.

PROPERTY 4 *Assume that the observer maneuvers occur at times different from all the source maneuver instants and that there are no zero bearing-rate legs. Then the jump linear system is observable.*

VII. SOME PROPERTIES OF THE FIM DETERMINANT AND OPTIMIZATION OF THE OBSERVER TRAJECTORY

Up to now, we have been only concerned with observability analysis which is a binary yes/no parameter. However a practical fundamental question remains: if the system is observable what is the accuracy of the state estimate? A classic approach consists then in considering the FIM and more precisely its determinant. The choice of the determinant functional is reasonable. This is a common cost functional in the estimation literature [1, 28, 29, 34, 35]. It is the inverse of the square of the volume of the uncertainty ellipsoid. Furthermore, we show (Section VIII) that, under hypotheses reasonable in the BOT context, the maximum of det(FIM) is attained when the sphericity criterion is maximum. However, as we see later, the det functional does not own the monotonicty property (see Section VIII) so it is not evident that adding an optimal control for the time $t+1$ to a control sequence optimal up to time t will yield a control sequence up to time $t+1$.

This explains, for a large part, the relative complexity of this section. We show that using elementary multilinear algebra accurate approximations of det(FIM) may be obtained. More specifically, we prove that det(FIM) may be approximated by a functional involving only the successive source bearing-rates yielding thus the general form of the optimal controls (observer maneuvers). In particular it is shown that, under the long-range and bounded controls hypotheses, the sequence of optimal controls lies in the general class of bang-bang controls. These results demonstrate the interest of maneuver diversity. More generally, they provide a general framework for optimizing the observer trajectory by means of feedback control.

First, approximations of det(FIM) are derived for a constant source bearing-rate (Section VIIA). Using the same approach, these results are extended to the case of time-varying (PWCS) source bearing-rates. A geometric interpretation of these results is presented in Section VIIB.

A. Some Approximations of the FIM Determinant and Their Consequences

Consider the case of a nonmaneuvering source (constant velocity vector), then the calculation of the FIM is a routine exercice yielding, under the Gaussian assumption [1]:

$$\text{FIM} = \left(\frac{\partial\Theta(\mathbf{X})}{\partial\mathbf{X}}\right)^* \Sigma^{-1} \left(\frac{\partial\Theta(\mathbf{X})}{\partial\mathbf{X}}\right) \tag{64}$$

where $\Theta(\mathbf{X})$ is the measurement vector generated by the state vector $\mathbf{X}$ and Σ is the diagonal matrix whose diagonal terms are the inverses of the variances of the measured bearings. The partial derivative matrix of the bearing vector $\Theta(\mathbf{X})$ with respect to the state vector is directly calculated [1, 2] yielding

$$\frac{\partial\boldsymbol{\Theta}(\mathbf{X})}{\partial\mathbf{X}} = \begin{pmatrix} \frac{\cos\theta_1}{r_1} & -\frac{\sin\theta_1}{r_1} & \frac{\cos\theta_1}{r_1} & -\frac{\sin\theta_1}{r_1} \\ \vdots & & & \\ \frac{\cos\theta_n}{r_n} & -\frac{\sin\theta_n}{r_n} & \frac{n\cos\theta_1}{r_n} & -\frac{n\sin\theta_n}{r_n} \end{pmatrix} \tag{65}$$

where $\{\theta_i\}_{i=1}^n$ represent the source bearing at the instant i and $\{r_i\}$ the source-observer distance. In (65) the reference time is the instant 0. Obviously, another reference time may be chosen but it is quite remarkable that the determinant of the FIM does not depend on the reference time.

The distance is assumed to be constant (at first). Further, we assume that the diagonal noise matrix Σ is proportional to the identity (i.e., $\Sigma = \sigma^2 Id$).

We denote $F_{k,4}$ the FIM corresponding to a reference time k and 4 consecutive measurements, $\theta_k,\ldots,\theta_{k+3}$. Then the FIM $F_{k,4}$ takes the following form (4 measurements):

$$F_{k,4} = (\sigma r)^{-2}\mathcal{G}_{k,4}\mathcal{G}_{k,4}^*$$

where

$$\mathcal{G}_{k,4} = (\mathbf{G}_k, \mathbf{G}_{k+1}, \mathbf{G}_{k+2}, \mathbf{G}_{k+3})$$

and $\mathbf{G}_k$ is the gradient vector of θ_k with respect to $\mathbf{X}_0$, i.e.,

$$\mathbf{G}_k = (\cos\theta_k, -\sin\theta_k, k\cos\theta_k, -k\sin\theta_k)^*. \tag{66}$$

Assuming $\mathcal{G}_{k,4}$ invertible, we have

$$\det(F_{k,4}) = (\sigma r)^{-8}(\det\mathcal{G}_{k,4})^2.$$

It is thus sufficient to calculate $\det\mathcal{G}_{k,4}$. For that purpose, we consider a second-order expansion of $\cos(\theta_{k+i})$ and $\sin(\theta_{k+i})$ $(i = 1,2,3)$, i.e.,

$$\begin{aligned} \cos(\theta_{k+i}) &\overset{2}{=} \cos\theta_k - i(\sin\theta_k)\dot\theta \\ &\quad + \frac{i^2}{2}(-\dot\theta^2\cos\theta_k - \ddot\theta\sin\theta_k) \\ \sin(\theta_{k+i}) &\overset{2}{=} \sin\theta_k + i(\cos\theta_k)\dot\theta \\ &\quad + \frac{i^2}{2}(-\dot\theta^2\sin\theta_k + \ddot\theta\cos\theta_k) \\ \dot\theta &\triangleq \left(\frac{\partial\theta(t)}{\partial t}\right)_{t=k}. \end{aligned} \tag{67}$$

The calculation of $\det(\mathcal{G}_{k,4})$ is detailed in Appendix B. Denoting $\{\alpha_i\}_{i=1}^3$ and $\{\beta_i\}_{i=1}^3$ constants deduced from (67), we obtain (see Appendix B)

$$\det(\mathcal{G}_{k,4}) = \alpha_1(2\alpha_3\beta_2' - \alpha_2\beta_3')$$

where

$$\begin{aligned} \alpha_i &= \frac{i^2}{2}\ddot\theta + i\dot\theta, \qquad \beta_i = -\frac{i^2}{2}\dot\theta^2 \\ \beta_2' &= 2(1+\beta_2) - \frac{\alpha_2}{\alpha_1}(1+\beta_1) \\ \beta_3' &= 3(1+\beta_3) - \frac{\alpha_3}{\alpha_1}(1+\beta_1). \end{aligned} \tag{68}$$

Notice that the formula (68) is valid whatever the order of the approximation since it involves only the two vectors $\mathbf{g}_k$ and $\mathbf{v}_k$.

It is now quite enlightening to calculate explicitly the second-order approximation of $\det(F_{k,4})$ given by (68).

PROPERTY 5 *Approximating the source-observer distance as constant, then the second-order approximation of* $\det F_{k,4}$ *is given by*

$$\det F_{k,4} \overset{2}{=} (\sigma r)^{-8}(6\dot\theta^4 - 3\ddot\theta^2)^2. \tag{69}$$

This approximation is null under the following condition:

$$6\dot\theta^4 - 3\ddot\theta^2 = 0$$

which is equivalent to

$$4\dot\theta^4 - 6\ddot\theta^2 + 2\dot\theta\dddot\theta = 0. \tag{70}$$

At this point, it is worth recalling the observability criterion given by Nardone and Aidala [9], i.e., with their notations:

$$\det[A(t)] = 4\dot\theta^4 - 3\ddot\theta^2 + 2\dot\theta\dddot\theta \neq 0. \tag{71}$$

We can remark that (70) and (71) only differ on the scalar coefficients affecting $\ddot\theta$ which are respectively -6 and -3.

If a third-order approximation of $(\cos(\theta_{k+i}), \sin(\theta_{k+i}))$ is considered then (68) is still valid, only the $\{\alpha_i\}$ and $\{\beta_i\}$ are slightly modified:

$$\begin{aligned} \alpha_i &= i\dot\theta + \frac{i^2}{2}\ddot\theta + \frac{i^3}{6}(\dddot\theta - \dot\theta^3) \\ \beta_i &= -\frac{i^2}{2}\dot\theta^2 - \frac{i^3}{6}\dot\theta\ddot\theta. \end{aligned} \tag{72}$$

The following approximation of the determinant of the third-order approximation of $F_{k,4}$ is then

$$\det F_{k,4} \simeq (\sigma r)^{-8}\cdot(4\dot\theta^4 - 3\ddot\theta^2 + 2\dot\theta\dddot\theta)^2 \tag{73}$$

which leads to the observability criterion of Nardone and Aidala.

Obviously, our attention is not limited to four measurements per legs. So, the previous calculations are now extended to any number of measurements. Let ℓ be the number of measurements and consider now the (4×4) FIM $F_{k,\ell}$ $(l \geq 4)$ defined as in (66) by

$$F_{k,\ell} = (\sigma r)^{-2}\mathcal{G}_{k,\ell}\mathcal{G}_{k,\ell}^*$$

where

$$\mathcal{G}_{k,\ell} = (\mathbf{G}_k, \mathbf{G}_{k+1}, \dots, \mathbf{G}_{k+\ell}) \qquad \ell \geq 0. \tag{74}$$

Note that in (74) the source-observer distance is again assumed to be constant. Using classical properties of multilinear algebra, namely the Cauchy–Binet formula [19, 24], $\det(F_{k,\ell})$ is given by the following formula:

$$\det(F_{k,\ell}) = (\sigma r)^{-8} \sum_E \det(\mathcal{G}_E)^2$$

where

$$E = \{i_1, i_2, i_3, i_4\} \qquad \text{s.t.} \quad 1 \leq i_1 < i_2 < i_3 < i_4 \leq \ell$$

and

$$\mathcal{G}_E = (\mathbf{C}_{i_1}, \mathbf{C}_{i_2}, \mathbf{C}_{i_3}, \mathbf{C}_{i_4}). \tag{75}$$

In (75) $\mathbf{C}_{i_j}$ stands for the i_jth the column of the matrix $\mathcal{G}$. Considering for instance, a first-order expansion of the bearings θ_{k+i} (i.e., $\theta_{k+i} \overset{1}{=} \theta_k + i\dot{\theta}$), the calculation of $\det(F_{k,\ell})$ is reduced to the calculation of the determinants $\det(\mathcal{G}_E)$. Now each of these determinants is the determinant of a 4×4 matrix and may be calculated by using the general calculation given in Appendix B, yielding the following.

RESULT 1

$$\det F_{k,4} \overset{1}{=} \left(\frac{\sin\dot{\theta}}{\sigma r}\right)^8 16$$

$$\det F_{k,5} \overset{1}{=} \left(\frac{\sin\dot{\theta}}{\sigma r}\right)^8 32[18 + 16\cos(2\dot{\theta}) + \cos(4\dot{\theta})]$$

$$\det F_{k,6} \overset{1}{=} \left(\frac{\sin\dot{\theta}}{\sigma r}\right)^8 32 \times [313 + 416\cos(2\dot{\theta}) + 136\cos(4\dot{\theta}) + 16\cos(6\dot{\theta}) + \cos(8\dot{\theta})] \tag{76}$$

$$\det F_{k,8} \overset{1}{=} \left(\frac{\sin\dot{\theta}}{\sigma r}\right)^8 32 \times [26691 + 44912\cos(2\dot{\theta}) + 27608\cos(4\dot{\theta}) + 12368\cos(6\dot{\theta}) + 3867\cos(8\dot{\theta}) + 816\cos(10\dot{\theta}) + 136\cos(12\dot{\theta}) + 16\cos(14\dot{\theta}) + \cos(16\dot{\theta})] \quad \text{etc.}$$

The general form of this approximation of $\det(F_{k,\ell})$ is thus:

$$\det F_{k,\ell} \overset{1}{=} \left(\frac{\sin\dot{\theta}}{\sigma r}\right)^8 32 P_\ell[\cos(2\dot{\theta}), \dots, \cos(4(\ell-4)\dot{\theta})]. \tag{77}$$

This expression of $\det F_{k,\ell}$ is directly deduced from the general expression of $\det \mathcal{G}_E$ obtained in Appendix B. This leads to the following result.

PROPERTY 6 *The following approximation of* $\det \mathcal{G}_E$ *holds:*

$$\det \mathcal{G}_E = (c-b)(d-b)\sin(c_1 - d_1)\sin b_1 + (c-b)(c-a)\sin(b_1 - d_1)\sin c_1 + (b-d)(b-a)\sin(b_1 - c_1)\sin d_1$$

where

$$a = i_1, \qquad b = i_2, \qquad c = i_3, \qquad d = i_4$$
$$b_1 \overset{\Delta}{=} (i_2 - i_1)\dot{\theta} \qquad c_1 \overset{\Delta}{=} (i_3 - i_1)\dot{\theta} \qquad d_1 \overset{\Delta}{=} (i_4 - i_1)\dot{\theta}. \tag{78}$$

Note that the only approximation lies in the first-order approximation of θ_{k+i} since the other steps are exact calculations. Using this formalism, an extension of (76)–(77) to higher order expansions of θ_{k+i} is quite straightforward but not truly enlightening. Instead the effect of observer maneuvers is now considered.

Consider that the temporal evolutions of the source bearings on two successive legs are described by the two following linear models:

$$\theta_{k+i} \overset{1}{=} \theta_k + i\dot{\theta}_1 \quad \text{on the 1st leg}$$
$$\theta_{k'+j} \overset{1}{=} \theta_{k'} + j\dot{\theta}_2 \quad \text{on the 2nd leg.} \tag{79}$$

Then the following property holds (cf. Appendix B) and extends the previous result (78).

PROPERTY 7

$$\det(\mathcal{G}_E) = (c-b)(a-b)\sin(b_1 + c_1)\sin d_1 + (b-d)(a-d)\sin(b_1 + d_1)\sin c_1 + (c-d)(d-b)\sin(c_1 - d_1)\sin b_1 \tag{80}$$

where b_1, c_1, d_1 *have, this time, the following meanings:*

$$b_1 = (i_2 - i_1)\dot{\theta}_1 \qquad c_1 = (i_3 - i_2)\dot{\theta}_2 \qquad d_1 = (i_4 - i_2)\dot{\theta}_2$$
$$a = i_1, \qquad b = i_2 \qquad c = i_3 \qquad d = i_4$$
$$(i_1, i_2) \in \text{1st leg,} \qquad (i_3, i_4) \in \text{2nd leg.}$$

Since the parameters $\dot{\theta}_1$ and $\dot{\theta}_2$ are usually small, we examine an expansion of $\det(\mathcal{G}_E)$ wrt $\dot{\theta}_1$ and $\dot{\theta}_2$ around the point (0,0). Then, we obtain the following types[7] of fourth-order expansions (in $\dot{\theta}_1$ and $\dot{\theta}_2$) of $\det(\mathcal{G}_E)$ given by (80):

$$(\det \mathcal{G}_E)^2 \simeq K(x^2y^2 - 2xy^3 + y^4)$$
$$\text{or:} \quad K(x^2y^2 - 2x^3y + y^4) \tag{81}$$
$$\text{or:} \quad K(x^2y^2 - 2x^3y + x^4)$$

with

$$K \overset{\Delta}{=} (b-a)^2(c-b)^2(c-d)^2(d-b)^2$$
$$x \overset{\Delta}{=} \dot{\theta}_1, \qquad y \overset{\Delta}{=} \dot{\theta}_2. \tag{82}$$

[7] The type of the expansion only depends on the relative values of i_1, i_2, i_3, i_4.

Also, it is worth recalling the following expression of $\dot{\theta}$, i.e., [27]:

$$\dot{\theta} = \frac{v_x r_y - r_x v_y}{r} = \frac{\det\begin{pmatrix} v_x & r_x \\ v_y & r_y \end{pmatrix}}{r}. \tag{83}$$

Since the velocity modulus v (i.e., $v = (v_x^2 + v_y^2)^{1/2}$) is physically limited (i.e., $v \leq v_m$), it follows from (81) that

$$0 \leq |\dot{\theta}| \leq \dot{\theta}_{\max}$$

with

$$\dot{\theta}_{\max} = \frac{1}{r}\det\begin{pmatrix} v_{xm} & r_x \\ v_{ym} & r_y \end{pmatrix}$$

and

$$v_{xm} = v_m\left(\frac{r_y}{r}\right), \qquad v_{ym} = v_m\left(\frac{-r_x}{r}\right). \tag{84}$$

In other words, $\dot{\theta}$ is maximized when the velocity vector (components v_x, v_y) is orthogonal to the position vector (components r_x, r_y). It is therefore quite legitimate to consider the optimization of the FIM determinant on the following 2D domain:

$$\begin{aligned} -\dot{\theta}_{\max} \leq \dot{\theta}_1 \leq \dot{\theta}_{\max} \\ -\dot{\theta}_{\max} \leq \dot{\theta}_2 \leq \dot{\theta}_{\max}. \end{aligned} \tag{85}$$

Consider a two-legs path for the observer and a nonmaneuvering source and denote $F_{\ell,m}$ the associated FIM. The index k is omitted since $\det(F_{k,\ell,m})$ is independent of k. Then, as previously, we have:

$$\det F_{\ell,m} = (\sigma r)^{-8}\sum_{E}\det^2(\mathcal{G}_E) \tag{86}$$

$\det(\mathcal{G}_E)$ given by (80)[8]

$$(\{\theta_1,\ldots,\theta_\ell\} \leftarrow \text{leg } 1, \{\theta_{\ell+1},\ldots,\theta_{\ell+m}\} \leftarrow \text{leg } 2).$$

Considering $\det F_{\ell,m}$ as a function of $\dot{\theta}_1$ and $\dot{\theta}_2$, we see directly that $\det F_{\ell,m}$ is maximum when each of its elementary terms (i.e., $\det^2(\mathcal{G}_E)$) is maximum. Now it follows directly from (81) that each $\det^2(\mathcal{G}_E)$ is maximized when

$$\begin{aligned} |\dot{\theta}_1| = |\dot{\theta}_2| = \dot{\theta}_{\max} \\ \dot{\theta}_2 = -\dot{\theta}_1. \end{aligned} \tag{87}$$

For these values of the parameters $\dot{\theta}_1$ and $\dot{\theta}_2$, the maximum of $\det F_{\ell,m}$ has the following expression:

$$\max_{\dot{\theta}_1,\dot{\theta}_2}(\det F_{\ell,m}) \simeq (\sigma r)^{-8} K_{\ell,m}\dot{\theta}_{\max}^4. \tag{88}$$

The positive scalar $K_{\ell,m}$ depends only on ℓ and m. The minimum value (for a given value of $\dot{\theta}$) of $\det F_{\ell,m}$ is attained for $\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}$ giving

$$\min_{\dot{\theta}_1,\dot{\theta}_2}\det F_{\ell,m} \simeq \left(\frac{\dot{\theta}}{\sigma r}\right)^8 k_{\ell+m}. \tag{89}$$

[8]The symbol $\leftarrow$ means that $\{\theta_1,\ldots,\theta_l\}$ are bearings corresponding to the leg 1.

Actually (89) constitutes a minoration of $\det F_{\ell,m}$ which has been rather intensively used in the literature [28, 29] for deriving an integral criterion.[9] However, when we compare (88) and (89) this minoration appears quite pessimistic since it roughly corresponds to a nonmaneuvering observer. On the other hand, (88) shows us that optimized observer maneuvers may improve the values of the FIM criterion dramatically. Note that the optimal observer maneuvers then appear as a bang-bang control sequence. This general result is confirmed by numerical results [29, 30]. A difficult problem consists then in determining the switching instants.

Let us now illustrate (80) with some examples. By using (81) and (85) we obtain

$$\begin{aligned} (\sigma r)^8 \det F_{4,3} &\simeq 200x^4 - 800x^3 y \\ &\quad + 5120x^2y^2 - 8640xy^3 + 4120y^4 \\ (\sigma r)^8 \det F_{6,7} &\simeq 694820x^4 - 2075640x^3 y \\ &\quad + 3599540x^2y^2 - 3751440xy^3 \\ &\quad + 1532720y^4. \end{aligned} \tag{90}$$

where

$$x = \dot{\theta}_1, \qquad y = \dot{\theta}_2.$$

The values of $\det F_{6,7}(x,y)$ are plotted in Fig. 2. Considering unsymmetric legs, we obtain for instance,

$$\begin{aligned} (\sigma r)^8 \det F_{2,7} &\simeq 56448y^4 - 112896xy^3 + 56448x^2y^2 \\ (\sigma r)^8 \det F_{3,6} &\simeq 94864y^4 - 193648xy^3 \\ &\quad - 5488xy^3 + 103488x^2y^2. \end{aligned} \tag{91}$$

The above calculations may be extended to a three-leg path, yielding

$$\begin{aligned} \arg_{\dot{\theta}_1,\dot{\theta}_2,\dot{\theta}_3}\max\det F_{\ell,m,n} = \varepsilon(\dot{\theta}_{\max}, -\dot{\theta}_{\max}, \dot{\theta}_{\max}) \\ \varepsilon = \pm 1. \end{aligned} \tag{92}$$

Up to now, the effects of range variations have not been considered. However, the analysis is greatly simplified if we remark that the effects of range and bearing-rate variations are uncoupled. Consider for instance, a first-order expansion of the source-observer distance (i.e., $r_{k+i} \stackrel{1}{=} r_k + i\dot{r}$). Then, as previously, the calculation of $\det F_{k,\ell}$ is reduced to the calculation of the determinants $\det(\mathcal{G}_E)$ yielding for instance,

$$\det F_{k,6} \simeq \frac{32(\sin\dot{\theta})^8}{\sigma^8(r_0+\dot{r})^2\cdots(r_0+6\dot{r})^2}P_\ell(\dot{\theta},\dot{r})$$

where

$$P_\ell(\dot{\theta},\dot{r}) = P_\ell(\dot{\theta})\cdot Q_\ell(\dot{r})$$

[9]Even if the analysis was restricted to localization.

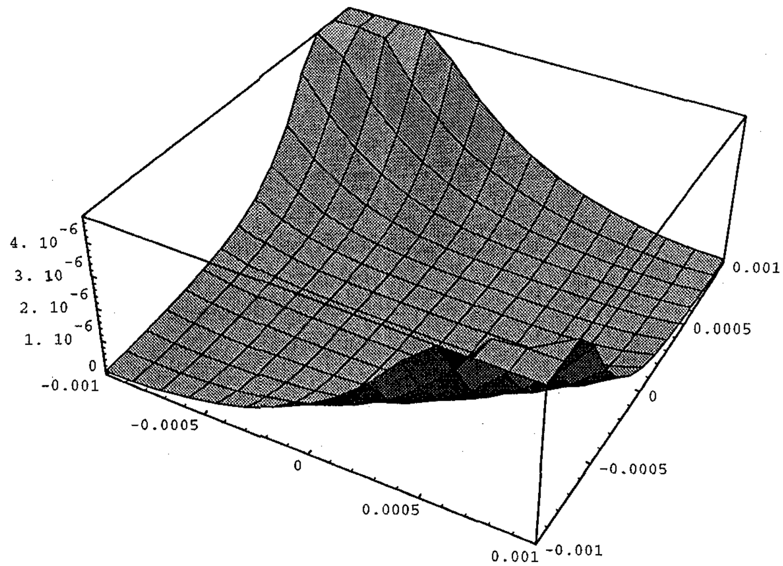

Fig. 2. Values of approximation (see (90)) of $\det(F_{6,7})$; $-10^{-3} \le \dot\theta_1 \le 10^{-3}$ and $-10^{-3} \le \dot\theta_2 \le 10^{-3}$.

with (see (77))

$$\left|\begin{array}{l} P_\ell(\dot\theta) = 313 + 416\cos(2\dot\theta) + 136\cos(4\dot\theta) \\ \qquad + 16\cos(6\dot\theta) + \cos(8\dot\theta) \\ Q_\ell(\dot r) = r_0^4 + 14 r_0^3 \dot r + 73 \dot r_0^2 \dot r^2 + 168 r_0 \dot r^3 + 144 \dot r^4 \\ r_0 \triangleq r_k. \end{array}\right. \tag{93}$$

From (93) we see that the polynomial $P_\ell(\dot\theta, \dot r)$ is actually the product of the two polynomials $P_\ell(\dot\theta)$ and $Q_\ell(\dot r)$. This factorization is quite general and is simply due to the basic properties of the determinant. Therefore, the effects of range variations are easily taken into account. More precisely, it is sufficient to replace the matrix R_1 by the matrix $(1 + \dot r/r)^{-1} R_1$ in the previous analysis. The coefficients $\alpha_l, \ldots, \gamma_l$ (100) are then subsequently modified.

The practical interest of the preceding results is evident since explicit forms of the FIM determinant have been obtained. We stress that these explicit forms involve only directly observable [27, 30] parameters. More precisely $\dot\theta$ may be directly estimated (i.e., without any prior knowledge about the source trajectory) and even $\dot r/r$ may be estimated (since $\dot r/r = -\ddot\theta/2\dot\theta$) from the spatio-temporal data received on the sensor array. Hence, the above results allow us to optimize the observer trajectory without any prior knowledge about the source trajectory [31, 32].

B. Geometric Interpretations of the Properties of the FIM Determinant

The preceding results advocate for a more systematic and geometric interpretation. Thus, we consider the determinant $\det \mathcal{G}_E$ (see (74)) where as previously (see (75)), $E = \{i_1, i_2, i_3, i_4\}$ and $i_1 < i_2 < i_3 < i_4$

$$\begin{aligned} \det \mathcal{G}_E &= \det(\mathbf{G}_{i_1}, \ldots, \mathbf{G}_{i_4}) \\ &= \det(R_1^{i_1}\mathbf{G}_k, R_1^{i_2}\mathbf{G}_k, R_1^{i_3}\mathbf{G}_k, R_1^{i_4}\mathbf{G}_k) \end{aligned}$$

where

$$R_1 \triangleq \begin{pmatrix} R_0 & 0 \\ R_0 & R_0 \end{pmatrix} \qquad \text{and} \qquad R_0 \triangleq \begin{pmatrix} \cos\dot\theta & \sin\dot\theta \\ -\sin\dot\theta & \cos\dot\theta \end{pmatrix}. \tag{94}$$

In the same spirit, the vector $\mathbf{G}_k$ may be written as

$$\mathbf{G}_k = S_1^k \mathbf{E}$$

where

$$S_1 \triangleq \begin{pmatrix} S_0 & 0 \\ S_0 & S_0 \end{pmatrix} \qquad \text{and} \qquad S_0 \triangleq \begin{pmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{95}$$

$$\theta \triangleq \theta_k/k, \qquad \mathbf{E} = (1,0,0,0)^*.$$

Now the following property is instrumental: the matrices R_0 and S_0 commute. The matrices R_1 and S_1 then also commute and using this property $\det \mathcal{G}_E$ then becomes

$$\det \mathcal{G}_E = \det(R_1^{i_1}\mathbf{E}, R_1^{i_2}\mathbf{E}, R_1^{i_3}\mathbf{E}, R_1^{i_4}\mathbf{E}). \tag{96}$$

The following property has thus been proved: $\det \mathcal{G}_E$ is independent of k and θ_k. This remarkable property is due to the basic property of the determinant ($\det AB = \det A \det B$). A further step yields

$$\begin{aligned} \det \mathcal{G}_E &= \det(\mathbf{E}, R_1^{i_2 - i_1}\mathbf{E}, R_1^{i_3 - i_1}\mathbf{E}, R_1^{i_4 - i_1}\mathbf{E}) \\ &= \det \left(\begin{array}{c|c|c|c} \mathbf{e} & R_0^{i'_2}\mathbf{e} & R_0^{i'_3}\mathbf{e} & R_0^{i'_4}\mathbf{e} \\ \mathbf{0} & i'_2 R_0^{i'_2}\mathbf{e} & i'_3 R_0^{i'_3}\mathbf{e} & i'_4 R_0^{i'_4}\mathbf{e} \end{array}\right) \end{aligned} \tag{97}$$

where

$$\mathbf{e} \stackrel{\Delta}{=} (1,0)^*, \qquad i'_k \stackrel{\Delta}{=} i_k - i_1 \qquad k = 2,3,4. \tag{98}$$

The calculation of $\det \mathcal{G}_E$ may then be achieved by recalling the expression of the minimal polynomials of R_0:

$$R_0^2 = 2\cos\dot{\theta} R_0 - Id_2 \tag{99}$$

so that the minimal polynomial of R_1 is $[(x-\lambda) \cdot (x-\overline{\lambda})]^2$, $\lambda = \exp(i\dot{\theta})$.

Thanks to the Cayley–Hamilton Theorem, the following equality holds ($l \geq 4$):

$$R_1^l = \alpha_l Id + \beta_l R_1 + \gamma_l R_1^2 + \delta_l R_1^3.$$

The coefficients $\alpha, \beta, \gamma, \delta$ are determined by the following recursion:

$$\begin{aligned} \delta_{l+1} &= \gamma_l + 4\cos\dot{\theta}\delta_l \\ \gamma_{l+1} &= \beta_l - 2(1+2\cos^2\dot{\theta})\delta_l \\ \beta_{l+1} &= \alpha_l + 4\cos\dot{\theta}\delta_l, \qquad \alpha_{l+1} = \delta_l. \end{aligned} \tag{100}$$

The determinant $\det \mathcal{G}_E$ can thus be calculated for any subset E, yielding the general form (see (77)) of $\det F_{k,l}$. Further note that the vector sequence $\{\mathbf{E}, R_1^{i_2-i_1}\mathbf{E}, R_1^{i_3-i_1}\mathbf{E}, R_1^{i_4-i_1}\mathbf{E}\}$ is a part of a Krylov sequence [33].

Another way to calculate $(\det \mathcal{G}_E)^2$ consists in considering the determinant of the gramian matrix of the Krylov sequence, more precisely:

$$\begin{aligned} (\det \mathcal{G}_E)^2 &= \det(\mathcal{G}_E^* \mathcal{G}_E) \\ &= \det[\mathrm{Gram}(\mathbf{E}, R_1^i\mathbf{E}, R_1^j\mathbf{E}, R_1^k\mathbf{E})]. \end{aligned} \tag{101}$$

Now, the calculation of the elements of the gramian matrix is direct since we have

$$\begin{aligned} \mathbf{E}^* R_1^{*i} R_1^j \mathbf{E} &= (1+ij)\mathbf{e}^* R_0^{*i} R_0^j \mathbf{e} \\ &= (1+ij)\cos(i-j)\dot{\theta}. \end{aligned} \tag{102}$$

Using (102) an explicit form of $(\det \mathcal{G}_E)^2$ is straightforwardly obtained:

$$(\det \mathcal{G}_E)^2 = (\det(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}'_1, \mathbf{V}'_2))^2$$

where[10]

$$\begin{aligned} \mathbf{V}_1^* &= (1, \cos i\dot{\theta}, \cos j\dot{\theta}, cos k\dot{\theta}) \\ \mathbf{V}_2^* &= (1, \sin i\dot{\theta}, \sin j\dot{\theta}, \sin k\dot{\theta}) \\ \mathbf{V}_1'^{,*} &= (0, i\cos i\dot{\theta}, j\cos j\dot{\theta}, k\cos k\dot{\theta}) \\ \mathbf{V}_2'^{,*} &= (0, i\sin i\dot{\theta}, j\sin j\dot{\theta}, k\sin k\dot{\theta}) \end{aligned} \tag{103}$$

yielding finally (80).

The previous calculations provide interesting insights about the optimization of the observer maneuvers. For instance, using (97), (100) and the determinant properties the following expression is obtained:

$$\det(F_{k,5}) = \det(F_{k,4})(2 + \alpha_4{}^2 + \beta_4{}^2 + \gamma_4{}^2). \tag{104}$$

A more general expression of $\det(F_{k,l})$ can be easily obtained by this way, since the calculation is reduced to a simple enumeration (see Appendix C). Furthermore, interesting insights about the observer maneuvers may be thus obtained. Consider for instance the following determinant:

$$f(y) = \det(\mathbf{E}, R_{1,x}\mathbf{E}, R_{1,x}^2\mathbf{E}, R_{1,x}^2 R_{1,y}\mathbf{E}) \tag{105}$$

where $x = \dot{\theta}_1$, $y = \dot{\theta}_2$.

Let us now calculate the partial derivative $\partial f/\partial y(x)$. We obtain

$$\frac{\partial f}{\partial y}(x) = \det(\mathbf{E}, R_{1,x}\mathbf{E}, R_{1,x}^2\mathbf{E}, R_{1,x}^2 S_{1,x}\mathbf{E}) \tag{106}$$

where $S_{1,x} = ((\partial/\partial y)R_{1,y})_{(y=x)}$, or, explicitly:

$$S_{1,x} = \begin{pmatrix} S_{0,x} & 0 \\ S_{0,x} & S_{0,x} \end{pmatrix} \tag{107}$$

with

$$S_{0,x} = \begin{pmatrix} -\sin x & -\cos x \\ \cos x & -\sin x \end{pmatrix}.$$

Using (94) and (107) the following property is then easily proved:

$$S_{1,x} R_{1,x}^2 = R_{1,x}^2 S_{1,x} = S_{1,3x}$$

so that

$$\begin{aligned} \frac{\partial f}{\partial y}(x) &= \det(\mathbf{E}, R_{1,x}\mathbf{E}, R_{1,x}^2\mathbf{E}, S_{1,3x}\mathbf{E}) \\ &= 2\sin(2x) - \frac{\sin(4x)}{2} \end{aligned} \tag{108}$$

and, therefore

$$\begin{aligned} f(y) &\approx f(x) + (y-x)\frac{\partial f}{\partial y}(x) \\ &\approx cx^4 + (y-x)(2x) \\ &\approx 2x(y-x). \end{aligned} \tag{109}$$

At this point, it is worth noting that the vector $S_{1,3x}\mathbf{E} = (-\sin 3x, \cos 3x, -3\sin 3x, 3\cos 3x)^*$ (see (108)) is approximately orthogonal to the vectors $\{\mathbf{E}, R_{1,x}\mathbf{E}, R_{1,x}^2\mathbf{E}\}$. This fact is typical of a four-dimensional state vector and corresponds to a diversity in maneuvers. The effect of an observer maneuver corresponds to a change from $x(\dot{\theta}_1)$ to $y(\dot{\theta}_2)$. From (109) it is clear that the increase of $\det(F_{x,y})$ is maximized when the term $(2x(x-y))^2$ is maximized. Since $\dot{\theta}$ is bounded, an optimal sequence of controls is necessarily a bang-bang one or, more precisely, a sequence of the form $\{\dot{\theta}_{\max}, -\dot{\theta}_{\max}, \dot{\theta}_{\max}, \ldots\}$.

It remains to determine the optimal number of control commutations (from $\dot{\theta}_{\max}$ to $-\dot{\theta}_{\max}$) as well

[10] $E = \{0, i, j, k\}$.

as their locations. Using the previous results, the problem may be formulated as follows. Consider a multilegs observer trajectory, then the problem consists in maximizing $\det(F_{l_1,l_2,l_3,\ldots})$ (l_i is length of the ith leg) given below (see (93))

$$(\sigma r)^8 \det(F_{l_1,l_2,l_3}) \approx P(l_1,\dot\theta_{\max})\theta^8 + P(l_1,l_2,\dot\theta_{\max})(2\dot\theta_{\max})^4 + P(l_1,l_3,\dot\theta_{\max})(2\dot\theta_{\max})^4 + \cdots. \quad (110)$$

The instants of commutation may then be determined by maximizing the previous expression. In (110), the polynomials $P(l_i,l_j,\dot\theta)$ may be obtained by means of the Cayley-Hamilton Theorem (see (99)).

VIII. SOME CONSIDERATIONS ABOUT THE FIM FUNCTIONALS

The previous section has been devoted to the study of the determinant of the FIM. As it has been shown this functional inherits quite interesting properties from multilinear algebra. Furthermore, it has been widely considered in the TMA literature [34–36]. We now see how elementary considerations advocate for the use of this functional.

Consider the inequality of the arithmetic-geometric means [19], it yields

$$(\det F)^{1/4} \le \tfrac{1}{4}\mathrm{tr}(F). \quad (111)$$

Now if—as in the previous section—the source receiver distance is assumed to be constant. The trace (denoted by tr) is directly calculated, providing (n is the number of estimated bearings):

$$\mathrm{tr}F = \frac{1}{r^2\sigma^2}\frac{n\cdot n+3}{2}. \quad (112)$$

Equality in (111) can occur only if F is proportional to the identity matrix. Due to the particular structure of F (66) this is generally impossible. The scalar $\det F$ may be interpreted as the inverse of the volume of the uncertainty ellipsoid. Another interesting criterion may be the condition number $\kappa(F)(\kappa(F) = \lambda_{\max}/\lambda_{\min})$. However there exists no general relation between the eigenvalues of F and the variances of $\{\hat r_x, \hat r_y, \hat v_x, \hat v_y\}$. The determinant thus appears as a good criterion involving all the eigenvalues. Furthermore, as it is shown in (112), maximizing $\det F(F \succ 0)$ amounts to optimize the sphericity test (111). The smallest eigenvalue of F thus cannot tend towards zero.

Indeed, since, under the long range hypothesis, $\mathrm{tr}(F)$ (cf. (112)) does not depend on the parameters $\dot\theta_i$ it is thus equivalent to maximize $\det F$ and the sphericity criterion [37] $s(F)$ defined by

$$s(F) = \frac{p\,\mathrm{tr}(F)}{(\det F)^p} \quad (113)$$

where p is the dimension of F (here $p = 4$).

Thanks to the basic inequality of the arithmetic-geometric means, the following property holds:

$$0 \le s(F) \le 1 \quad (F \succeq 0)$$

with

$$s(F) = 1 \Longleftrightarrow F = \lambda Id \quad (\lambda > 0). \quad (114)$$

However, as stressed in [1], the FIM inherits a very special structure from its definition (65), (66). More precisely with the notations of [1], the general structure of F is given below:[11]

$$F = \frac{1}{\sigma^2 r^2}\sum_{i=1}^{n}\begin{pmatrix}\Omega_i & i\Omega_i\\ i\Omega_i & i^2\Omega_i\end{pmatrix}$$

where

$$\Omega_i = \begin{pmatrix}\cos^2\theta_i & -\frac{1}{2}\sin 2\theta_i\\ -\frac{1}{2}\sin 2\theta_i & \sin^2\theta_i\end{pmatrix}. \quad (115)$$

Very interesting insights about the eigenvalues of F have been given in [1]. More precisely, using the Weyl's Theorem [19] and the Cauchy's inclusion principle [38, 39], it has been shown that, under reasonable hypotheses, the two largest eigenvalues of F are within a distance n of the eigenvalues of the submatrix $\Omega_{22}(\Omega_{22} = \sum_i i^2\Omega_i)$. The two smallest eigenvalues, which usually correspond to the position estimate are always less than or equal to eigenvalues of $\Omega_{11}(\Omega_{11} = \sum_i \Omega_i)$. Therefore, the matrix F cannot be proportional to the identity matrix.

Remarkable bounds for the ratios of eigenvalues using only $\mathrm{tr}(F)$ and $\mathrm{tr}(F^2)$ have been obtained in [40]. More precisely, let p be the dimension of F and assume $1 \le k < \ell \le p$, then the ratio $\gamma_{k,\ell}(\gamma_{k,\ell} \stackrel{\Delta}{=} \lambda_k/\lambda_\ell)$ admits the following bound:

$$\gamma_{k,\ell} \le \frac{c+k+\left\{\frac{p-\ell+1}{k}(c+k)(p-\ell+1-c)\right\}^{1/2}}{c+k-\left\{\frac{k}{p-\ell+1}(c+k)(p-\ell+1-c)\right\}^{1/2}}$$

where

$$c = \frac{(\mathrm{tr}F)^2}{\mathrm{tr}(F^2)} - (\ell - 1). \quad (116)$$

However, the special structure of F is not considered in this bound, but, actually, maximizing the determinant (and thus the sphericity criterion) will "almost" optimize the condition number.

More generally, if we consider a functional f (of F), it would be worth that the following monotonicity property be satisfied by the functional f.

DEFINITION

1) Simple Monotonicity Property:

$$F' \succ F \succ 0 \Rightarrow f(F') > f(F) > 0. \quad (117)$$

[11] 0: reference time.

2) Additive Monotonicity Property (AMP):

$$F, F' \succ 0 \qquad \text{and} \qquad G \succeq 0$$
$$f(F') > f(F) \Rightarrow f(F' + G) > f(F + G). \tag{118}$$

Actually, the second property should be highly desirable for optimizing the observer motion since it justifies the use of dynamic programming [41, 42]. Unfortunately, it is not satisfied by the determinant ($f(F) = \det F$) and is trivially satisfied by the trace. Indeed, this property is very strong and it can be proved that it may only be satisfied by functionals derived from the trace [43]. More precisely, the only functionals satisfying the AMP are of the form $f(F) = g(\mathrm{tr}(AF))$, g increasing function and A fixed matrix. The nonvalidity of the AMP for the determinant largely explains the relative complexity of the previous section.

IX. STOCHASTIC OBSERVABILITY AND ESTIMABILITY

We now consider a Markovian sequence of state vectors:

$$\mathbf{X}_{k+1} = F\mathbf{X}_k + \mathbf{U}_k + \mathbf{W}_k$$

with

$$\mathbf{W}_k \quad \text{IID sequence} \quad (\mathrm{cov}(\mathbf{W}) = Q). \tag{119}$$

where IID is independent identically distributed. The measurement equation is unchanged (4) since, as previously, we use the pseudomeasurement equation. According to the definition of Boguslavskij [43], we say that the system is stochastically observable if, in estimating its states from its outputs, the posterior error variances of all the state components are strictly smaller than the priors.

Let $\hat{\mathbf{X}}_k$ be the linear least-mean-square estimate of $\mathbf{X}_k$ given the measurements $\{y_k, \ldots, y_0\}$ and define the matrices Π_k and P_k ($\Pi_k = \mathrm{cov}(\mathbf{X}_k), P_k = \mathrm{cov}(\mathbf{X}_k - \hat{\mathbf{X}}_k)$) then we consider the following definition of observability.

DEFINITION The system (4), (120) is said to be observable[12] iff:

$$\mathbf{e}_i^* P_k \mathbf{e}_i < \mathbf{e}_i^* \Pi_k \mathbf{e}_i \qquad 1 \le i \le n. \tag{120}$$

It can easily shown that the general form of P_k is

$$P_k = \Pi_k - L_k \Sigma_k^{-1} L_k^*. \tag{121}$$

The rectangular $(n \times k) L_k$ is defined later. Since the matrix Σ_k (the covariance matrix of the noise measurements) is positive definite, the matrix $\Pi_k - P_k$ is positive semidefinite. So, the inequality $\mathbf{e}_i^* P_k \mathbf{e}_i \le \mathbf{e}_i^* \Pi_k \mathbf{e}_i$ always holds, whatever i. The values of i ensuring a strict inequality are the observable state components. In fact, the above definition is rather arbitrary and does not take into account the possible coupling between the estimates of the state components. A convenient definition may then be the estimability condition of Baram and Kailath [44].

[12] $\{\mathbf{e}_i\}_{i=1}^n$ usual orthogonal basis of $\mathbb{R}^n$ ($n = \dim \mathbf{X}$).

DEFINITION The system (4), (120) is said to be estimable iff: $\Pi_k - P_k$ is positive definite.

Denote $\theta(L_k)$ the number of rows of the matrix L_k with non-zero elements, then a direct consequence of (121) is that the system is stochastically observable iff $\theta(L_k)$ is equal to n. Another direct consequence of (121) is that the system is estimable iff the rank of L_k is n. Direct calculations [43,44] yield the following expression of L_k:

$$L_k = \{\Phi_{k,0} N_0, \Phi_{k,1} N_1, \ldots, \Phi_{k,k} N_k\}$$

where

$$\Phi_{k,j} = F^{k-j} \qquad (j \le k)$$
$$N_j = \Pi_j H_j^*$$

and Π_j satisfies the following Lyapunov state equation:

$$\Pi_{j+1} = F \Pi_j F^* + Q. \tag{122}$$

From (122) and quite direct but lengthly calculations it is easily seen that the BOT system is stochastically observable except for very pathological cases, i.e., the sequences $\{\cos\theta_0, \ldots, \cos\theta_k\}$ or $\{\sin\theta_0, \ldots, \sin\theta_k\}$ are identically null. As pointed in [43] stochastic observability is thus less demanding than deterministic observability which requires a maneuver of the observer.

We now consider estimability for BOT. In view of the following factorization of the matrix L_k [43, 44]:

$$L_k = F^k (\Pi_0 H_0^*, \Pi_1' H_1^*, \ldots, \Pi_k' H_k^*) \qquad (\Pi_i' = F^{-i} \Pi_i) \tag{123}$$

it is easily shown ($F^k = kF - (k-1)Id$) that the matrices Π_i' are spanned by the three matrices Π_0, Π_1', Π_2', i.e.,

$$\Pi_3' = 4\Pi_0 - 6\Pi_1' + 4\Pi_2',$$
$$\Pi_4' = 13\Pi_0 - 20\Pi_1' + 10\Pi_2', \ldots. \tag{124}$$

From (124) we deduce that the rank of L_k is generally equal to 3 for a zero-bearing-rate scenario. The system is then not estimable. Excepting this case, the BOT system is generally estimable since the rank of L_k is equal to 4.

A convenient "measure" of the information contained in the measurements may be $\det(L_k \Sigma_k^{-1} L_k^*)$. Direct calculations yield the following results rather similar to (88):

$$(\sigma r)^8 \det(L_4 \Sigma_4^{-1} L_4^*) = 16[43 - 47\cos 2\dot{\theta}]^2 (\sin\dot{\theta})^4$$
$$(\sigma r)^8 \det(L_k \Sigma_k^{-1} L_k^*) = Q_l[\cos 2\dot{\theta}, \ldots, \cos(4(k-4)\dot{\theta})](\sin\dot{\theta})^4. \tag{125}$$

In order to optimize the observer maneuvers, a reasonable criterion may consist in maximizing $\det(L_k \Sigma_k^{-1} L_k^*)$ for whom an explicit form is given by (122), (125). Note, however, that this formula is valid only if $\dot{\theta}$ is approximatively constant and that the true problem is a difficult stochastic control problem.

X. CONCLUSION

A discrete-time BOT observability analysis has been developed allowing us to immerse it in the simple formalism of linear algebra. Using the direct approach, observability analysis is essentially reduced to basic considerations about subspace dimensions. Even if this approach is conceptually straightforward, it becomes more and more cumbersome as the source-encounter scenario complexity increases. For complex scenarios, the dual approach may present certain advantages due to the direct use of multilinear algebra. New results for BOT observability of a maneuvering source were obtained by this way. Thus, observability analysis has been extended to piecewise linear systems which allows us to analyze observability for general source and observer trajectories. Using the same formalism, the observability analysis was then extended to unknown instants of source velocity changes.

Even if observability analysis provides thorough insights about the algebraic structure of the BOT problem, the optimization of the observer maneuvers is essentially a control problem. It was shown that a relevant cost functional to this problem is the FIM determinant. So, a large part of this paper was devoted to the analysis of this functional. Using multilinear algebra, general approximations of this determinant were obtained. Although if this problem has been widely investigated in the literature, new insights were given. More specifically, it has been shown that the FIM determinant can be approximated by a functional involving only the consecutive source bearing-rates. The structure of this functional has been carefully detailed. It was then possible to determine the general form of the optimal control sequence (observer maneuvers). In particular it was proved that, under the long range and bounded controls hypotheses, the sequence of optimal controls lies in the general class of bang-bang controls. This result illustrates the interest of maneuver diversity. More generally, these results provide a general framework for optimizing the observer trajectory by means of feedback control.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the reviewers and the editor (Sonar & Undersea Syst.) for their many helpful comments.

XI. APPENDIX A

This Appendix is devoted to the calculation of the determinants $D_{k,\ell}$ defined in (45).

If we assume that the vectors $(\mathbf{e}_0 + \mathbf{v}_0)$ and $\mathbf{u}'$ are linearly independent, they constitute a basis of $\mathbb{R}^2$ and there exists two scalars α and $\beta_{k,\ell}$ such that:

$$2(\mathbf{e}_0 + 2\mathbf{v}_0) = \alpha(\mathbf{e}_0 + \mathbf{v}_0) + \beta_{k,\ell}(k+\ell)\mathbf{u}'.$$

Notice that we have necessarily:

$$\beta_{k,\ell} = \left(\frac{k+1}{k+\ell}\right)\beta_{k,1}.$$

Let us denote $\mathbf{e}_\ell$ the 2-dimensional vector defined by

$$\mathbf{e}_\ell \stackrel{\Delta}{=} \mathbf{e}_0 + 2\mathbf{v}_0 - \alpha(\mathbf{e}_0 + \mathbf{v}_0) - \beta_{k,\ell}\mathbf{u}'$$

then

$$-\mathbf{e}_\ell = \mathbf{e}_0 + 2\mathbf{v}_0 - \beta_{k,\ell}(k+\ell-1)\mathbf{u}'.$$

Let us denote $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \mathbf{C}_4$ the columns of $D_{k,\ell}$ and perform the following algebraic manipulation of the columns:

$$\mathbf{C}_4' = \mathbf{C}_4 - \alpha\mathbf{C}_3 - \beta_{k,\ell}\mathbf{C}_1 = \begin{pmatrix} \mathbf{e}_\ell \\ \mathbf{0} \end{pmatrix}$$

then, we obtain (determinant of block matrices [20])

$$D_{k,\ell} = \det\left(\begin{array}{c|c|c|c} \mathbf{e}_0 & \mathbf{e}_\ell & \mathbf{e}_0 + \mathbf{v}_0 & \mathbf{u}' \\ \mathbf{0} & \mathbf{0} & \mathbf{e}_0 + \mathbf{v}_0 & (k+\ell)\mathbf{u}' \end{array}\right) = (k+\ell)\det(\mathbf{e}_0, \mathbf{e}_\ell)\det(\mathbf{e}_0 + \mathbf{v}_0, \mathbf{u}').$$

XII. APPENDIX B

This Appendix deals with the calculation of $\det(\mathcal{G}_k)$ defined by (66) and (67).

Equation (66) may be written in vectorial form, yielding

$$\underbrace{\begin{pmatrix} \cos\theta_{k+i} \\ -\sin\theta_{k+i} \end{pmatrix}}_{\mathbf{g}_{k+i}} \stackrel{2}{=} \underbrace{\begin{pmatrix} \cos\theta_k \\ -\sin\theta_k \end{pmatrix}}_{\mathbf{g}_k} + \underbrace{\left(\frac{i^2}{2}\ddot{\theta} + i\dot{\theta}\right)}_{\alpha_i}\underbrace{\begin{pmatrix} -\sin\theta_k \\ -\cos\theta_k \end{pmatrix}}_{\mathbf{v}_k} + \underbrace{\left(-\frac{i^2}{2}\dot{\theta}^2\right)}_{\beta_i}\underbrace{\begin{pmatrix} \cos\theta_k \\ -\sin\theta_k \end{pmatrix}}_{\mathbf{g}_k}$$

or simply

$$\mathbf{g}_{k+i} = \mathbf{g}_k + \alpha_i\mathbf{v}_k + \beta_i\mathbf{g}_k$$

where

$$\alpha_i = \frac{i^2}{2}\ddot{\theta} + i\dot{\theta} \qquad \beta_i = -\frac{i^2}{2}\dot{\theta}^2$$

$$\|\mathbf{g}_k\| = \|\mathbf{v}_k\| = 1 \qquad \text{and} \qquad \mathbf{g}_k \perp \mathbf{v}_k.$$

Consequently, we have

$$\mathbf{G}_{k+i} \overset{2}{=} \mathbf{G}_k + \mathbf{V}_k^i$$

with

$$\mathbf{V}_k^i = \left| \begin{array}{l} \alpha_i \mathbf{v}_k + \beta_i \mathbf{g}_k \\ i\mathbf{g}_k + \alpha_i (k+i)\mathbf{v}_k + \beta_i (k+i)\mathbf{g}_k. \end{array} \right.$$

The determinant being multilinear and alternate, we have

$$\det(\mathcal{G}_k) = \det(\mathbf{G}_k, \mathbf{V}_k^1, \mathbf{V}_k^2, \mathbf{V}_k^3) \quad (C_1).$$

Assuming the 2-dimensional vectors $\mathbf{g}_k$ and $\beta_1 \mathbf{g}_k + \alpha_1 \mathbf{v}_k$ linearly independent, we can find 4 scalars $(\lambda_1, \lambda_2, \mu_1, \mu_2)$ such that:

$$\begin{cases} \alpha_2 \mathbf{v}_k + \beta_2 \mathbf{g}_k = \lambda_1(\mathbf{g}_k) + \lambda_2(\alpha_1 \mathbf{v}_k + \beta_1 \mathbf{g}_k) \\ \alpha_3 \mathbf{v}_k + \beta_3 \mathbf{g}_k = \mu_1(\mathbf{g}_k) + \mu_2(\alpha_1 \mathbf{v}_k + \beta_1 \mathbf{g}_k) \end{cases}$$

and direct calculations yield

$$\begin{cases} \lambda_1 = \dfrac{\alpha_1\beta_2 - \alpha_2\beta_1}{\alpha_1}, & \lambda_2 = \dfrac{\alpha_2}{\alpha_1} \\ \mu_1 = \dfrac{\alpha_1\beta_3 - \alpha_3\beta_1}{\alpha_1}, & \mu_2 = \dfrac{\alpha_3}{\alpha_1}. \end{cases}$$

The following operations are performed on the columns of the 4×4 matrix $(\mathbf{G}_k, \mathbf{V}_k^1, \mathbf{V}_k^2, \mathbf{V}_k^3)$:

$$\mathbf{V}_k^3 - \lambda_1 \mathbf{G}_k - \lambda_2 \mathbf{V}_k^1 = \begin{pmatrix} \mathbf{0} \\ \mathbf{d}_3 \end{pmatrix}$$

$$\mathbf{V}_k^4 - \mu_1 \mathbf{G}_k - \mu_2 \mathbf{V}_k^1 = \begin{pmatrix} \mathbf{0} \\ \mathbf{d}_4 \end{pmatrix}$$

so that

$$\begin{aligned} \det(\mathcal{G}_k) &= \alpha_1 \det(\mathbf{g}_k, \mathbf{v}_k)\det(\mathbf{d}_3, \mathbf{d}_4) \\ &= \alpha_1(2\alpha_3\beta_2' - \alpha_2\beta_3') \end{aligned}$$

with

$$\beta_2' = 2(1+\beta_2) - \frac{\alpha_2}{\alpha_1}(1+\beta_1)$$

$$\beta_3' = 3(1+\beta_3) - \frac{\alpha_3}{\alpha_1}(1+\beta_1).$$

Consider now a linear modeling of θ_{k+i} in $\dot{\theta}$, i.e.,

$$\theta_{k+i} \overset{1}{=} \theta_k + i\dot{\theta}$$

then we have

$$\begin{cases} \mathbf{g}_{k+i} = \cos(i\dot{\theta})\mathbf{g}_k + \sin(i\dot{\theta})\mathbf{v}_k \\ (k+i)\mathbf{g}_{k+i} = (k+i)\cos(i\dot{\theta})\mathbf{g}_k + (k+i)\sin(i\dot{\theta})\mathbf{v}_k. \end{cases}$$

Consider first the calculation of $\det(\mathcal{G}_{k,4})$, (C_1) yields

$$\det(\mathcal{G}_{k,4}) = \det(\mathbf{G}_k, \mathbf{V}_k^1, \mathbf{V}_k^2, \mathbf{V}_k^3)$$

where

$$\mathbf{V}_k^i = \left| \begin{array}{l} \sin(i\dot{\theta})\mathbf{v}_k \\ i\cos(i\dot{\theta})\mathbf{g}_k + (k+i)\sin(i\dot{\theta})\mathbf{v}_k \end{array} \right.$$

for $1 \le i \le 3$. It is then sufficient to consider the following algebraic manipulations of the $\mathcal{G}_{k,4}$ columns:

$$\mathbf{V}_k^i - \frac{\sin(i\dot{\theta})}{\sin(\dot{\theta})}\mathbf{V}_k^1 = \begin{pmatrix} \mathbf{0} \\ \mathbf{d}_i \end{pmatrix}$$

where

$$\begin{aligned} \mathbf{d}_i &= (i-1)\sin(i\dot{\theta})\mathbf{v}_k \\ &\quad + \left[(i-1)\cos(i\dot{\theta}) - \frac{\sin[(i-1)\dot{\theta}]}{\sin\dot{\theta}} \right] \mathbf{g}_k \end{aligned}$$

from which it follows that

$$\begin{aligned} \det(\mathcal{G}_{k,4}) &= 4(\sin\dot{\theta})^4 (\det(\mathbf{g}_k, \mathbf{v}_k))^2 \\ &= 4(\sin\dot{\theta})^4 \end{aligned}$$

which is the first result of (76).

More generally, we consider the matrix $\mathcal{G}_E$ (74):

$$\mathcal{G}_E = (\mathbf{G}_{k+i_1}, \mathbf{G}_{k+i_2}, \mathbf{G}_{k+i_3}, \mathbf{G}_{k+i_4}) \qquad \text{where}$$
$$k \le i_1 < \cdots < i_4 \le k + \ell.$$

Then using the previous reasoning, we obtain

$$\det \mathcal{G}_E = \det(\mathbf{g}_{k+i_1}, \mathbf{g}_{k+i_2}) \det(\mathbf{d}_{i_3}, \mathbf{d}_{i_4})$$

with

$$\begin{aligned} \mathbf{d}_{i_3} &= (i_3 - i_2)\sin c_1 \mathbf{v}_{k+i_1} \\ &\quad + \left[(i_2 - i_1)\frac{\sin(b_1 - c_1)}{\sin b_1} + (i_3 - i_2)\cos c_1 \right] \mathbf{g}_{k+i_1} \end{aligned}$$

idem for $\mathbf{d}_{i_4}$ (i_4 replacing i_3, d_1 replacing c_1) where

$$b_1 \overset{\Delta}{=} (i_2 - i_1)\dot{\theta}$$

$$c_1 \overset{\Delta}{=} (i_3 - i_1)\dot{\theta}$$

$$d_1 = (i_4 - i_1)\dot{\theta}.$$

The following simple expression of $\det \mathcal{G}_E$ is thus deduced:

$$\begin{aligned} \det \mathcal{G}_E &= (c-b)(d-b)\sin(c_1 - d_1)\sin b_1 \\ &\quad + (c-b)(b-a)\sin(b_1 - d_1)\sin c_1 \\ &\quad + (b-d)(b-a)\sin(b_1 - c_1)\sin d_1 \end{aligned}$$

where

$$a = i_1, \quad b = i_2, \quad c = i_3, \quad d = i_4 \quad (C_2).$$

Using the above relation, the determinant of $F_{k,5}$ is easily calculated since we have

$$E_1 = \{1,2,3,4\} \to \det \mathcal{G}_{E_1} = 4(\sin\dot{\theta})^4$$

$$E_2 = \{1,2,3,5\} \to \det \mathcal{G}_{E_2} = 16(\cos\dot{\theta})(\sin\dot{\theta})^4$$

$$E_3 = \{1,2,4,5\} \to \det \mathcal{G}_{E_3} = 8(2+\cos(2\dot{\theta}))(\sin\dot{\theta})^4$$

$$E_4 = \{1,3,4,5\} \to \det \mathcal{G}_{E_4} = 16(\cos\dot{\theta})(\sin\dot{\theta})^4$$

$$E_5 = \{2,3,4,5\} \to \det \mathcal{G}_{E_5} = 4(\sin\dot{\theta})^4$$

so that finally:

$$\det F_{k,5} = (\sigma r)^{-8} \sum_{i=1}^{5} (\det \mathcal{G}_{E_i})^2$$

$$= \left(\frac{\sin\dot{\theta}}{\sigma r}\right)^8 32[18 + 16\cos(2\dot{\theta}) + \cos(4\dot{\theta})].$$

This calculation may be extended to $\det F_{k,6}$ i.e.,

$$\det F_{k,6} = (\sigma r)^{-8} \sum_{i=1}^{15} (\det \mathcal{G}_{E_i})^2$$

with

$$E_1 = \{1,2,3,4\}, \ldots, E_{15} = \{3,4,5,6\}$$

and $\det \mathcal{G}_{E_i}$ given by (C_2).

The case of a maneuvering observer may be treated with the same approach. Assume, for instance, that the source maneuver occurs at time i_2 and consider thus the following determinant:

$$\det(\mathbf{G}_{k+i_1}, \mathbf{G}_{k+i_2}, \mathbf{G}_{k+i_3}, \mathbf{G}_{k+i_4}) \stackrel{\Delta}{=} \det(\mathcal{G}_E).$$

Assume furthermore that the two following approximations are valid on each observer leg:

$$\theta_{k+i} \stackrel{1}{=} \theta_k + i\dot{\theta}_1 \quad \text{on the 1st leg}$$

$$\theta_{k'+j} \stackrel{1}{=} \theta_{k'} + j\dot{\theta}_2 \quad \text{on the 2nd leg}$$

then

$$\begin{cases} \mathbf{g}_{k+i_2} = \cos(b_1)\mathbf{g}_{k+i_1} + \sin(b_1)\mathbf{v}_{k+i_1}, & (i_1, i_2 \in \text{1st leg}) \\ \mathbf{g}_{k+i_3} = \cos(b_1+c_1)\mathbf{g}_{k+i_1} & \\ \qquad + \sin(b_1+c_1)\mathbf{v}_{k+i_1}, & (i_3, i_4 \in \text{2nd leg}) \end{cases}$$

so that

$$\det \mathcal{G}_E = \det(\mathbf{g}_{k+i_1}, \mathbf{g}_{k+i_2}) \det(\mathbf{d}_{i_3}, \mathbf{d}_{i_4})$$

where

$$\mathbf{d}_{i_3} = (i_3 - i_2)\sin(b_1+c_1)\mathbf{v}_{k+i_1} + \left[(i_3-i_2)\cos(b_1+c_1) + (i_1-i_2)\frac{\sin c_1}{\sin b_1}\right]\mathbf{g}_{k+i_1}$$

idem for $\mathbf{d}_{i_4}$ (i_4 replacing i_3, d_1 replacing c_1), which finally yields

$$\det(\mathbf{G}_E) = (c-b)(a-b)\sin(b_1+c_1)\sin(d_1) + (b-d)(a-d)\sin(b_1+d_1)\sin(c_1) + (c-d)(d-b)\sin(c_1-d_1)\sin(b_1).$$

As an illustration, consider the following example:

$$a = k\theta_0, \qquad b = k\theta_0 + \dot{\theta}_1,$$
$$c = k\theta_0 + 4\dot{\theta}_1 + \dot{\theta}_2 \qquad d = k\theta_0 + 4\dot{\theta}_1 + 2\dot{\theta}_2$$

then applying (C_2) we obtain

$$\det \mathcal{G}_E = \tfrac{1}{2}[24\cos(\dot{\theta}_1 - \dot{\theta}_2) - 25\cos(\dot{\theta}_1 + \dot{\theta}_2) + \cos(7\dot{\theta}_1 + 3\dot{\theta}_2)].$$

The above calculations may be extended to a three leg path and we obtain, for instance,

$$(\sigma r)^8 \det F_{3,3,3} \simeq 784\dot{\theta}_1^4 + 5264\dot{\theta}_1^3\dot{\theta}_3 - 224\dot{\theta}_1^3\dot{\theta}_2 + 147114\dot{\theta}_1^2\dot{\theta}_3^2 - 56928\dot{\theta}_1^2\dot{\theta}_2\dot{\theta}_3 + 13302\dot{\theta}_1^2\dot{\theta}_2^2.$$

XIII. APPENDIX C

We consider here the use of the geometric interpretation of the FIM given in Section VIIB. First, denote $\mathbf{V}_i$ the vector $R_1^i\mathbf{E}$. Using the Cauchy–Binet formula (85), we have

$$(\sigma r)^8 \det(F_{k,5}) = \det{}^2(\mathbf{V}_0, \mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3) + \det{}^2(\mathbf{V}_0, \mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_4) + \det{}^2(\mathbf{V}_0, \mathbf{V}_1, \mathbf{V}_3, \mathbf{V}_4) + \det{}^2(\mathbf{V}_0, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4) + \det{}^2(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4).$$

Let us denote c^2 the term $\det^2(\mathbf{V}_0, \mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3)$. Using the Cayley-Hamilton Theorem we have

$$\mathbf{V}_4 = \alpha_4\mathbf{V}_3 + \beta_4\mathbf{V}_2 + \gamma_4\mathbf{V}_1 + \delta_4\mathbf{V}_0.$$

The scalars $\{\alpha_i, \ldots, \delta_i\}$ are defined by (99). Then, the following equalities are directly deduced from the alternating and multilinear properties of the determinant:

$$\det{}^2(\mathbf{V}_0, \mathbf{V}_1, \mathbf{v}_2, \mathbf{V}_4) = \alpha_4^2 c^2$$
$$\det{}^2(\mathbf{V}_0, \mathbf{V}_1, \mathbf{V}_3, \mathbf{V}_4) = \beta_4^2 c^2, \ldots$$

so that

$$(\sigma r)^8 \det(F_{k,5}) = c^2(2 + \alpha_4^2 + \beta_4^2 + \gamma_4^2).$$

The formula (76) has thus been retrieved by this way. Extension to the calculation of $\det(F_{k,l})$ are reduced to a simple enumeration of the elementary subsets E

(see (75)) and to the calculation of the scalar factors $\alpha_l, \ldots, \gamma_l$.

REFERENCES

[1] Nardone, S., Lindgren, A., and Gong, K. (1984)
Fundamental properties and performance of conventional bearings-only target motion analysis.
IEEE Transactions on Automatic Control, **AC-29**, 9 (Sept. 1984), 775–787.

[2] Lindgren, A., and Gong, K. (1978)
Position and velocity estimation via bearing observation.
IEEE Transactions on Aerospace and Electronic Systems, **AES-14** (July 1978), 564–577.

[3] Kolb, R. C., and Hollister, F. H. (1967)
Bearings-only target motion estimation.
In *Proceedings of the 1st Asilomar Conference on Circuits Systems*, 1967.

[4] Pham, D. T. (1993)
Some quick and efficient methods for bearings-only target motion analysis.
IEEE Transactions on Signal Processing, **41**, 9 (Sept. 1993), 2737–2751.

[5] Aidala, V. J., and Hammel, S. E. (1983)
Utilization of modified polar coordinates for bearings-only tracking.
IEEE Transactions on Automatic Control, **AC-28** (Mar. 1983), 283–294.

[6] Chan, Y. T., and Rudnicki, S. W. (1992)
Bearings-only and Doppler-bearing tracking using instrumental variables.
IEEE Transactions on Aerospace and Electronic Systems, **28**, 24 (Oct. 1992), 1076–1083.

[7] Passerieux, J. M., Pillon, D., Blanc-Benon, P., and Jauffret, C. (1989)
Target motion analysis with bearings and frequencies measurements via instrumental variable estimator.
In *Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing*, Glasgow, Scotland, UK, May 1989, 2645–2648.

[8] de Vlieger, J. H., and Gmelig Meyling, R. H. J. (1992)
Maximum likelihood estimation for long-range target tracking using passive sonar measurements.
IEEE Transactions on Signal Processing, **40**, 5 (May 1992), 1216–1225.

[9] Nardone, S. C., and Aidala, V. J. (1981)
Observability criteria for bearings-only target motion analysis.
IEEE Transactions on Aerospace and Electronic Systems, **AES-17**, 2 (Mar. 1981), 162–166.

[10] Hammel, S. E., and Aidala, V. J. (1985)
Observability requirements for three-dimensional tracking via angle measurements.
IEEE Transactions on Aerospace and Electronic Systems, **AES-21**, 2 (Mar. 1985), 200–207.

[11] Payne, A. N. (1989)
Observability problem for bearings-only tracking.
International Journal of Control, **49**, 3 (1989), 761–768.

[12] Jauffret, C., and Pillon, D. (1988)
New observability criterion in target motion analysis.
Nato Asi, Kingston, Canada, July 1988, 479–484.

[13] Jauffret, C. (1993)
Trajectographie passive, observabilité et prise en compte de fausses alarmes.
Ph.D. dissertation, Université de Toulon et du Var, Feb. 18, 1993.

[14] Fogel, E., and Gavish, M. (1988)
*N*th-order dynamics target observability from angle measurements.
IEEE Transactions on Aerospace and Electronic Systems, **24** (May 1988), 305–308.

[15] Becker, K. (1993)
Simple linear theory approach to TMA observability.
IEEE Transactions on Aerospace and Electronic Systems, **29**, 2 (Apr. 1993), 575–578.

[16] Hermann, R., and Krener, A. J. (1977)
Nonlinear controllability and observability.
IEEE Transactions on Automatic Control, **AC-22**, 5 (Oct. 1977), 728–739.

[17] Lévine, J., and Marino, R. (1992)
Constant-speed target tracking via bearings-only measurements.
IEEE Transactions on Aerospace and Electronic Systems, **28**, 1 (Jan. 1992), 175–182.

[18] Goshen-Meskin, D., and Bar-Itzhack, I. Y. (1992)
Observability analysis of piece-wise constant system—Part I: Theory.
IEEE Transactions on Aerospace and Electronic Systems, **28**, 4 (Oct. 1992), 1056–1067.

[19] Horn, R. A., and Johnson, C. R. (1987)
Matrix Analysis.
New York: Cambridge University Press, 1987.

[20] Rugh, W. J. (1993)
Linear System Theory.
Englewood Cliffs, NJ: Prentice-Hall, Information and System Science Series, 1993.

[21] Neuman, P. M., Stoy, G. A., and Thompson, E. C. (1994)
Groups and Geometry.
New York: Oxford, 1994.

[22] Trémois, O., and Le Cadre, J. P. (1996)
Target motion analysis with multiple arrays: Performance analysis.
IEEE Transactions on Aerospace and Electronic Systems, **32**, 3 (July 1996), 1030–1046.

[23] Lang, S. (1993)
Algebra (3rd edition).
Reading, MA: Addison-Wesley, 1993.

[24] Satake, I. (1975)
Linear Algebra.
New York: Marcel Dekker, 1975.

[25] Wonham, W. M. (1985)
Linear Multivariable Control: A Geometrical Approach (3rd ed.).
New York: Springer-Verlag, 1985.

[26] Ji, Y., and Chizek, H. J. (1988)
Controllability, observability and discrete-time Markovian jump linear quadratic control.
International Journal of J. Control, **48**, 2 (1988), 481–498.

[27] Le Cadre, J. P., and Trémois, O. (1996)
Properties and performance of extended target motion analysis.
IEEE Transactions on Aerospace and Electronic Systems, **32**, 1 (Jan. 1996), 66–83.

[28] Liu, P. T. (1988)
An optimum approach in target tracking with bearing measurements.
Journal of Optimization Theory and Applications, **56**, 2 (Feb. 1988), 205–214.

[29] Hammel, S. E., Liu, P. T., Hilliard, E. J., and Gong, K. F. (1989)
Optimal observer motion for localization with bearings measurements.
Computer and Mathematics with Applications, **18**, 1–3 (1989), 171–186.

[30] Passerieux, J. M., and Van Cappel, D. (1996)
Optimal observer maneuver for bearings-only tracking.
IEEE Transactions on Aerospace and Electronic Systems, to be published.

[31] McCabe, B. J. (1985)
Accuracy and tactical implications of bearings-only ranging algorithms.
Operations Research, **33**, 1 (Jan.–Feb. 1985), 94–108.

[32] Fawcett, J. A. (1988)
Effect of course maneuvers on bearings-only range estimation.
IEEE Transactions on Acoustics, Speech and Signal Processing, **36**, 8 (Aug. 1988), 1193–1199.

[33] Chatelin, F. (1988)
Valeurs Propres de Matrices.
Paris: Masson, 1988.

[34] Helferty, J. P., and Mudgett, D. R. (1993)
Optimal observer trajectories for bearings-only tracking by minimizing the trace of the Cramer-Rao bound.
In *Proceedings of the 32nd Conference on Decision and Control*, San-Antonio, TX, Dec. 1993, 936–939.

[35] Ucinski, D., Korbicz, J., and Zaremba, M. (1993)
On optimization of sensor motions in parameter identification of two-dimensional distributed systems.
In *Proceedings of the ECC'93*, Grenoble, France, 1993, 1359–1364.

[36] Jauffret, C. G., and Musso, C. J. (1991)
New results in target motion analysis.
In *Proceedings of the Undersea Defense Technology (UDT) Conference*, Paris, 1991, 239–244.

[37] Giri, N. C. (1977)
Multivariate Statistical Inference.
New York: Academic Press, 1977.

[38] Marshall, A. W., and Olkin, I. (1979)
Inequalities: Theory of majorization and its applications.
In *Mathematics in Science and Engineering*, vol. 143.
New York: Academic Press, 1979.

[39] Carlson, D. (1983)
Minimax and interlacing theorems for matrices.
Linear Algebra and its Applications, **54** (1983), 153–172.

[40] Merikoski, J. K., Styan, G. P. H., and Wolkowicz, H. (1983)
Bounds for ratios of eigenvalues using traces.
Linear Algebra and Its Applications, **55** (1983), 105–124.

[41] Le Cadre, J. P., and Trémois, O. (1995)
Optimization of the observer motion using dynamic programming.
In *Proceedings of the 1995 International Conference on Acoustics, Speech, and Signal Processing*, Detroit, May 9–12, 1995.

[42] Bertsekas, D. P. (1987)
Dynamic Programming: Deterministic and Stochastic Models.
Englewood Cliffs, NJ: Prentice-Hall, 1987.

[43] Boguslavskij, I. A. (1988)
Filtering and Control.
New York: Optimization Software (Publication Division), 1988.

[44] Baram, Y., and Kailath, T. (1988)
Estimability and regulability of linear systems.
IEEE Transactions on Automatic Control, **33**, 12 (Dec. 1988), 1116–1121.

J. P. Le Cadre (M'93) received the M.S. degree in mathematics in 1977, the Doctorat de 3^{-eme} cycle in 1982 and the Doctorat d'Etat in 1987, both from INPG, Grenoble, France.

From 1980 to 1989, he worked at the GERDSM (Groupe d'Etudes et de Recherche en Detection Sous-Marines), a laboratory of the DCN (Direction des Constructions Navales), mainly on array processing. Since 1989, he has been with IRISA/CNRS, where he is a director of research. His interests have now moved toward topics like system analysis, detection, data association and operations research.

He received (with O. Zugmeyer) the Eurasip *Signal Processing* best paper award in 1993. He is a member of various societies of IEEE and of SIAM.

Claude Jauffret was born on March 29, 1957, in Ollioules, France. He received the Diplôme d'Etudes Approfondies in applied mathematics from the Saint Charles University, Marseille, France in 1981 and the Diplôme D'ingénieur from the Ecole Nationale Supérieure d'Informatique et de Mathématiques Appliquées de Grenoble in 1983, the title of "docteur de l'université de Toulon" in 1993 and the "habilitation à diriger des recherches" in 1996 from the University of Toulon.

From Nov. 1983 to Sept. 1984, he worked on image processing used in passive sonar systems at the GERDSM, Six-Fours Les Plages, France. From 1984 to 1988, he worked on target motion analysis (TMA) problems at the GERDSM. A sabbatical year at the University of Connecticut, Storrs, allowed him to work on tracking problems in a cluttered environment in 1989. His research interests are mainly estimation and detection theory.

He has published papers about observability and estimation in nonlinear systems and TMA. Since September 1996, he is with the University of Toulon (France) where he is a Professor.