Target Motion Analysis With Multiple Arrays: Performance Analysis

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The performance analysis of source trajectory estimation using measurements provided by multiple platforms (or arrays) is studied. In numerous practical situations, the maneuvering ability of the receiver (e.g., a ship towing linear arrays) is limited, leading to the assumption that the observer motion is rectilinear and uniform. Even if this hypothesis appears quite restrictive, practical and tactical considerations fully justify its interest. This leads to consider multiple (platform) target motion analysis (denoted MTMA) and to analyze the performance of such trajectory estimation methods.

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I. INTRODUCTION

Conceptually, the basic problem in target motion analysis (TMA) is to estimate the trajectory of an object (i.e., position and velocity) from noise-corrupted sensor data [8]. These data are typically estimated bearings that are obtained via a standard beamforming algorithm [2]. These estimated bearings represent the basic data or observations for the passive sonar context within a direct path environment.

The performance of any TMA algorithm is conditioned by the statistical quality of the data (estimated bearings), which in turn depends on physical parameters such as the array length, the integration time, and so on. In particular, the array length is a critical parameter for the performance of the tracking and data association steps. This advocates for the use of large towed arrays. However, the maneuvering ability of the towing ship is itself limited by the array length leading thus to consider the following special case: the observer's motion is rectilinear and uniform (constant velocity vector). Even if this hypothesis seems oversimplified, practical and tactical considerations fully justify its interest. It constitutes the basic hypothesis of this work.

If the observations can be represented by a "mono-dimensional" time series $\{\hat{\theta}_1,\ldots,\hat{\theta}_n\}$, where $\hat{\theta}_i$ denotes the source bearing estimate at time $i\Delta T$, obtained by using a single array, then a classical result asserts that the TMA problem is actually nonobservable without an observer maneuver [7, 9]. This means that the state vector defining the source trajectory (constant velocity vector) cannot be determined from the exact time series $\{\theta_1,\ldots,\theta_n\}$. Assume for now that at least two estimated bearings $\hat{\theta}_{i,1}$ and $\hat{\theta}_{i,2}$ are available at each time then the problem becomes generally observable. This leads to consider multiple platform or array target motion analysis (denoted MTMA).

However, the observability concept is purely algebraic. So the main problem consists in calculating the MTMA statistical performance. Analytic formulations of the variance of the source state vector components are approximated in terms of physical parameters such as source distance, source velocity, and inter-array distance. The respective effects of these parameters on the MTMA performance will appear clearly. It is worth noting that a similar problem has been previously studied [11]. The main difference is that in this case, the data are constituted of time delay and time-delay rate, the so-called Doppler time compression. When the source signal is moving relative to the receiving arrays, the various signal components are not only time delayed but also Doppler time compressed relative to each other [11]. Therefore, according to [11], measurement of these differential Dopplers provides important additional information about the source trajectory. The paper of E. Weinstein

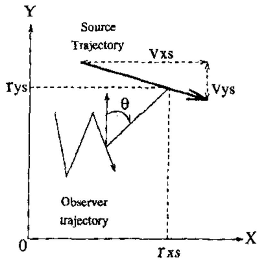

Fig. 1. Typical TMA scenario.

[1] deals with optimal source localization and tracking using differential delay and Doppler observations. However, the analysis is restricted to a short time analysis since the data consist of the differential delays and differential Dopplers at **a given time**.

Our approach is different since the data as well as the performance analysis methods are those of classical TMA [8]. Surprisingly, the results appear quite similar to E. Weinstein's results for the short integration time analysis case. Furthermore, the performance analysis is extended to long integration time MTMA. This represents the original contribution of this work.

Standard notations are used throughout this paper.

A bold capital letter denotes a vector while a capital letter denotes a matrix or a subspace.

The symbol * means transposition [4].

r_x and r_y represent x and y position.

v_x and v_y represent x and y velocity.

t is the time variable.

m is the number of arrays.

p is the number of sensors per array.

n_s is the number of snapshots (integration time) used in the array processing [2].

n is the number of estimated bearings in the TMA process.

II. MTMA MODEL

The physical parameters are depicted in Fig. 1. The source, located at the coordinates (r_{xs}, r_{ys}), moves with a constant velocity vector $v(v_{xs}, v_{ys})$ and is defined to have the state vector:

$$\mathbf{X}_s \stackrel{\Delta}{=} [r_{xs}, r_{ys}, v_{xs}, v_{ys}]^*. \tag{1}$$

The observer state is similarly defined as:

$$\mathbf{X}_{\text{obs}} \stackrel{\Delta}{=} [r_{x\text{obs}}, r_{y\text{obs}}, v_{x\text{obs}}, v_{y\text{obs}}]^*$$

so that, in terms of the relative state vector $\mathbf{X}$, defined by

$$\mathbf{X} = \mathbf{X}_s - \mathbf{X}_{\text{obs}} \stackrel{\Delta}{=} [r_x, r_y, v_x, v_y]^*$$

the discrete time equation of the relative motion takes the following form:

$$\mathbf{X}(t_k) = \Phi(t_k, t_{k-1})\mathbf{X}(t_{k-1}) + U(t_k) \tag{2}$$

where

$$\Phi(t_k, t_{k-1}) = \begin{pmatrix} Id & (t_k - t_{k-1})Id \\ O & Id \end{pmatrix}, \qquad Id \stackrel{\Delta}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In the above formula, t_k is the time at the kth sample, while the vector

$$U(t_k) = (0, 0, u_x(t_k), u_y(t_k))^*$$

accounts for the effects of the observer accelerations. *In this paper, the observer accelerations are zero* ($\mathbf{U} \equiv 0$) which means that the observer's motion is rectilinear and uniform. Throughout the document, the source state means the relative source state (eq. (2)).

As usually in TMA [8], the available measurements are the estimated bearings $\hat{\theta}_t$ from the observation platforms to the source, so that the observation equation stands as follows:

$$\hat{\theta}_{t,j} = \theta_{t,j} + w_{t,j} \tag{3}$$

($j = 1, \ldots, m$, m is number of arrays), with

$$\theta_{t,j} = \tan^{-1}\left(\frac{r_{x,j}(t)}{r_{y,j}(t)}\right) \tag{4}$$

$r_{x,j}$ and $r_{y,j}$ are the relative Cartesian coordinates of the source with respect to (w.r.t.) the center of the jth platform.

In (3), $w_{t,j}$ represents the estimation noise on the jth platform, modeled as zero mean, Gaussian, with variance given by the following approximate formula (Woodward's formula): (narrowband analysis)

$$\sigma_{w_{t,j}} = \frac{1}{\sqrt{n_s}}\theta_3 \frac{\sqrt{1 + p\rho}}{p\rho} \tag{5}$$

with

$$2\theta_3 = \frac{\sqrt{6}}{\pi}\frac{\lambda}{L}\sqrt{\frac{p-1}{p+1}}\frac{1}{\cos\theta_{t,j}}. \tag{6}$$

In (5) and (6), d represents the elementary distance between sensors (equispaced linear array), L the array length ($L = (p-1)d$), λ the wavelength, and ρ the signal-to-noise ratio.

The various noise components $w_{t,j}$ are assumed to be statistically uncorrelated from receiver to receiver. This assumption is not very restrictive since the receivers cannot be spaced very closely. Finally, the source signals at the various receivers are different from one another only by a geometrically determined factor. Temporal correlation is not really a problem as soon as each bearing estimation has nonoverlapping temporal intervals.

The four-dimensional state equation (2) and the nonlinear measurement equation (3) define the bearings-only motion analysis process (MTMA).

The classical TMA algorithms [8] can be directly extended to the multiple measurements. The single change consists in replacing the scalar observation

$\{\hat{\theta}_t\}$ by a vector of observations i.e., $\{\hat{\theta}_{1,t},\ldots,\hat{\theta}_{m,t}\}^* = \hat{\mathbf{\Theta}}_t$ $(1 \le t \le n)$, where n is the total number of estimated azimuths on each platform and * denotes the transposition. Only real data are processed here.

Given the history of measured bearings $(\hat{\mathbf{\Theta}}_1,\ldots,\hat{\mathbf{\Theta}}_n)$ the likelihood function is [8]

$$P(\hat{\mathbf{\Theta}}_1,\ldots,\hat{\mathbf{\Theta}}_n \mid \mathbf{x}) = \text{cstexp}\left[-\frac{1}{2}\sum_{t=1}^{n}\|\hat{\mathbf{\Theta}}_t - \mathbf{\Theta}_t(\mathbf{X})\|_{\Sigma}^2\right]$$

$$\left|\begin{array}{l}\mathbf{\Theta}_t(\mathbf{X}) \text{ defined by (1) and (2)}\\ \text{and}\\ \|\hat{\mathbf{\Theta}}_t - \mathbf{\Theta}_t(\mathbf{X})\|_{\Sigma}^2 \stackrel{\Delta}{=} (\hat{\mathbf{\Theta}}_t - \mathbf{\Theta}_t(\mathbf{X}))^*\Sigma^{-1}(\hat{\mathbf{\Theta}}_t - \mathbf{\Theta}_t(\mathbf{X}))\\ \Sigma = \text{diag}(\sigma_{w_{t,j}}^2)\ (\sigma_{w_{t,j}}^2 \text{ given by (5)}.\end{array}\right. \tag{7}$$

The maximum likelihood estimate (MLE) is the solution to the likelihood equation:

$$\frac{\partial}{\partial \mathbf{X}}\log P(\hat{\mathbf{\Theta}}_1,\ldots,\hat{\mathbf{\Theta}}_n \mid \mathbf{X}) = 0. \tag{8}$$

The above equation has no explicit closed form solution. However, a Newton [8] algorithm for the maximization of the likelihood functional is easily obtained. Denoting $\hat{\mathcal{C}} = (\hat{\mathbf{\Theta}}_1^*,\ldots,\hat{\mathbf{\Theta}}_n^*)^*$, the vector of concatenated measurements it takes the following form [8]:

$$\mathbf{X}_{\ell+1} = \mathbf{X}_\ell - s_\ell\left[\left(\frac{\partial\mathcal{C}}{\partial\mathbf{X}}\right)^*\Sigma^{-1}\frac{\partial\mathcal{C}}{\partial\mathbf{X}}\right]^{-1}\left(\frac{\partial\mathcal{C}}{\partial\mathbf{X}}\right)^*\Sigma^{-1}(\hat{\mathcal{C}} - \mathcal{C}) \tag{9}$$

where ℓ is the iteration index, s_ℓ = step size of the algorithm and $\mathcal{C} = \mathcal{C}(\mathbf{X}_\ell)$.

The calculation of the gradient vector is obtained from (3)

$$\tan(\theta_t) = \frac{r_x(t)}{r_y(t)}$$

$$= \frac{r_x(0) + tv_x}{r_y(0) + tv_y} \quad \text{(the index } j \text{ is omitted)},$$

and consequently:

$$\frac{\partial}{\partial r_x(0)}\tan(\theta_t) = \frac{1}{\cos^2(\theta_t)}\frac{\partial\theta_t}{\partial r_x(0)} = \frac{1}{r_y(t)}$$

so that

$$\frac{\partial\theta_t}{\partial r_x(0)} = \frac{\cos^2\theta_t}{r_y(t)} = \frac{r_y(t)}{r_t}\frac{\cos\theta_t}{r_y(t)} = \frac{\cos\theta_t}{r_t} \tag{10}$$

with $r_t = \sqrt{r_x^2(t) + r_y^2(t)}$. The other components of the partial derivative matrix $\partial\mathcal{C}/\partial\mathbf{X}$ are obtained by the same way, giving

$$\left|\begin{array}{l}\dfrac{\partial\theta_t}{\partial r_y(0)} = -\dfrac{\sin\theta_t}{r_t}\\ \dfrac{\partial\theta_t}{\partial v_x} = \dfrac{t}{r_y(t)}\cos^2\theta_t = \dfrac{t}{r_t}\cos\theta_t\\ \dfrac{\partial\theta_t}{\partial v_y} = -\dfrac{t}{r_t}\sin\theta_t.\end{array}\right. \tag{11}$$

Using the notations in [8], the matrix $\partial\mathcal{C}/\partial\mathbf{X}$ takes the following form:

$$\frac{\partial\mathcal{C}}{\partial\mathbf{X}} = \left(\left(\frac{\partial\Theta_1}{\partial\mathbf{X}}\right)^*,\ldots,\left(\frac{\partial\Theta_m}{\partial\mathbf{X}}\right)^*\right)^* \qquad \frac{\partial\mathcal{C}}{\partial\mathbf{X}} \text{ is } nm\times 4 \tag{12}$$

with

$$\frac{\partial\Theta_i}{\partial\mathbf{X}} = \begin{pmatrix}\dfrac{\cos\theta_{1,i}}{r_{1,i}} & -\dfrac{\sin\theta_{1,i}}{r_{1,i}} & (t_1 - t_m)\dfrac{\cos\theta_{1,i}}{r_{1,i}} & -(t_1 - t_m)\dfrac{\sin\theta_{1,i}}{r_{1,i}}\\ \vdots & \vdots & \vdots & \vdots\\ \dfrac{\cos\theta_{n,i}}{r_{n,i}} & -\dfrac{\sin\theta_{n,i}}{r_{n,i}} & (t_n - t_m)\dfrac{\cos\theta_{n,i}}{r_{n,i}} & -(t_n - t_m)\dfrac{\cos\theta_{n,i}}{r_{n,i}}\end{pmatrix} \tag{13}$$

where $\theta_{j,i}$ represents the source bearing, and $r_{j,i}$ the relative distance from the ith array center, corresponding to the source trajectory parameter vector $\mathbf{X}$. The time t_m is the reference time of the state estimate.

III. MEASUREMENTS

In passive radar or sonar, the location of a source can be estimated by observation of its signal at two or more spatially separated receivers. Most systems depend on the relative (differential) time delay of the signal wavefront between the various receivers.

When the source signal is moving relative to the array, the various signal components are not only time delayed but also Doppler compressed relative to each other. Measurements of these differential Dopplers provides important additional information about source speed and heading.

More precisely the time-varying delay between receivers of the ith pair is characterized by

$$\Delta\tau_i(t) = \alpha_i + \beta_i t \tag{14}$$

α_i and β_i are, respectively, the time delay and time-delay rate (Doppler time compression).

The observation $\mathbf{\Gamma}$ is then constituted of $(m-1)$ delays and $(m-1)$ differential Dopplers, i.e.,

$$\mathbf{\Gamma} = (\alpha_1,\alpha_2,\ldots,\alpha_{m-1},\beta_1,\beta_2,\ldots,\beta_{m-1})^*. \tag{15}$$

The relations between the components of $\mathbf{\Gamma}$ and the components of the state vector $\mathbf{X}$ in polar coordinates (i.e., $\mathbf{X} = (\theta, r, v_\theta, v_r)^*$, v_θ, and v_r are, respectively, the tangential and radial components of the source velocity vector) and are given below [11]:

$$\begin{cases} \alpha_i = d_i \sin\theta/c - d_i^2 \cos^2\theta/2rc \\ \beta_i = -d_i v_\theta \cos\theta/rc + d_i^2 (v_r \cos\theta - 2v_\theta \sin\theta)\cos\theta/2r^2 c \end{cases} \tag{16}$$

(d_i is the spacing between receiver i and the reference, and c is the sound speed).

Since the components of $\mathbf{X}$ enter into α_i and β_i in a nonlinear way, it is wothwhile rewriting (16) in the following form [11]:

$$\begin{cases} c\alpha_i = d_i x_1 - d_i^2 x_2 \\ c\beta_i = -d_i x_3 + d_i^2 x_4 \end{cases}$$

with:

$$\begin{cases} x_1 = \sin\theta \\ x_2 = \cos^2\theta/2r \\ x_3 = v_\theta \cos\theta/r \\ x_4 = (v_r \cos\theta - 2v_\theta \sin\theta)\cos\theta/2r^2. \end{cases} \tag{17}$$

Using this formalism and these measurements, E. Weinstein has calculated bounds for the variance of the estimated components of the $\mathbf{X}$ vector [11]. Conceptually, his approach uses the time delay and time-delay rate as measurements. The analysis is then essentially nonlinear (eqs. (16) or (17)). On the other hand, we assume in this work that the measurements are the bearings estimated on the m various subarrays. Performance analysis of MTMA using these measurements will now be performed and the results compared with those of E. Weinstein. Surprisingly, they appear quite similar for short-time analysis.

IV. SYSTEM OBSERVABILITY IN NONMANEUVERING SOURCE CASE

In order to study the system observability we reformulate the system equations (2) and (3) into the following form [8]:

$$\begin{cases} \mathbf{X}_{k+1} = F\mathbf{X}_k + U \\ 0 \equiv z_k = H_k \mathbf{X}_k \end{cases}$$

with

$$F = \Phi(t_{k+1}, t_k) = \begin{pmatrix} Id & \alpha Id \\ O & Id \end{pmatrix} \quad (\alpha \stackrel{\Delta}{=} t_{k+1} - t_k)$$

$$\mathbf{X}_k \stackrel{\Delta}{=} \mathbf{X}_{t_k} \tag{18}$$

$$H_k = (\cos\theta_k, -\sin\theta_k, 0, 0).$$

Note that $U \equiv 0$ from the nonmaneuvering assumption.

Thus the nonlinear bearing measurement [9] can be manipulated to provide a pseudomeasurement that is "linearly" related to the target state [8]. This constitutes the basis of the pseudo-linear estimate (PLE) [8]. It also provides a simpler way to investigate the system observability [1, 3, 6, 10]. From (18), it comes directly in the single platform case:

$$\begin{cases} z_0 = H_0 \mathbf{X}_0 \\ z_1 = H_1 F \mathbf{X}_0 \\ \vdots \\ z_k = H_k F^k \mathbf{X}_0. \end{cases} \tag{19}$$

Now

$$F^k = \begin{pmatrix} Id & k\alpha Id \\ O & Id \end{pmatrix}$$

so that, the observability matrix $\mathcal{O}$ [10] is defined as follows:

$$\mathcal{O} = \begin{pmatrix} H_0 \\ H_1 F \\ \vdots \\ H_k F^k \end{pmatrix} = \begin{pmatrix} \cos\theta_0 & -\sin\theta_0 & 0 & 0 \\ \cos\theta_1 & -\sin\theta_1 & \alpha\cos\theta_1 & -\alpha\sin\theta_1 \\ \vdots & & & \\ \cos\theta_k & -\sin\theta_k & k\alpha\cos\theta_k & -k\alpha\sin\theta_k \end{pmatrix}. \tag{20}$$

Note the analogy of (20) with (13). Now, it is quite enlightening to factorize $\mathcal{O}$:

$$\mathcal{O} = \Delta_r \begin{pmatrix} r_y(0) & -r_x(0) & 0 & 0 \\ r_y(1) & -r_x(1) & \alpha r_y(1) & -\alpha r_x(1) \\ \vdots & \vdots & \vdots & \vdots \\ r_y(k) & -r_x(k) & k\alpha r_y(k) & -k\alpha r_x(k) \end{pmatrix}$$

with

$$\Delta_r \stackrel{\Delta}{=} \mathrm{diag}(r^{-1}(0), r^{-1}(1), \ldots, r^{-1}(k)). \tag{21}$$

This factorization is only valid if all the $r(k)$ terms are non-zero. Fortunately, this assumption is not restrictive.

Denote now $\mathcal{O}'$ the "dual" observability matrix defined by

$$\mathcal{O}' = \begin{pmatrix} r_x(0) & r_y(0) & 0 & 0 \\ r_x(1) & r_y(1) & \alpha r_x(1) & \alpha r_y(1) \\ \vdots & \vdots & \vdots & \vdots \\ \underbrace{r_x(k)}_{\mathbf{T}_x} & \underbrace{r_y(k)}_{\mathbf{T}_y} & \underbrace{k\alpha r_x(k)}_{\mathbf{V}_x} & \underbrace{k\alpha r_y(k)}_{\mathbf{V}_y} \end{pmatrix}.$$

Then, except in the special case where the source and the observer positions are identical at a given time:

$$\mathrm{rank}\,\mathcal{O} = \mathrm{rank}\,\mathcal{O}'. \tag{22}$$

So, in order to study the observability, it is sufficient to consider the matrix $\mathcal{O}'$ defined by (22). This simple identity constitutes the cornerstone for our approach of observability.

Now, using (2) or (equivalently) (18), the vectors $\mathbf{T}_x$, $\mathbf{T}_y$, $\mathbf{V}_x$, and $\mathbf{V}_y$ can be expressed as linear combinations of the three vectors $\mathbf{1}$, $\mathbf{Z}$ and $\mathbf{Z}^2$, i.e.,

$$\begin{aligned}\mathbf{T}_x &= r_x(0)\mathbf{1} + \alpha v_x \mathbf{Z}\\ \mathbf{V}_x &= \alpha r_x(1)\mathbf{Z} + \alpha^2 v_x \mathbf{Z}^2\\ \mathbf{T}_y &= r_y(0)\mathbf{1} + \alpha v_y \mathbf{Z}\\ \mathbf{V}_y &= \alpha r_y(1)\mathbf{Z} + \alpha^2 v_y \mathbf{Z}^2\end{aligned}$$

with

$$\begin{aligned}\mathbf{1} &\stackrel{\Delta}{=} (1,1,\ldots,1)^*\\ \mathbf{Z} &\stackrel{\Delta}{=} (0,1,2,\ldots,k)^*\\ \mathbf{Z}^2 &\stackrel{\Delta}{=} (0,0,2,\ldots,k(k-1))^*.\end{aligned} \tag{23}$$

It is then clear from (23), that rank ($\mathcal{O}'$) and thus rank $\mathcal{O}$ are bounded by 3 since the range of $\mathcal{O}'$ is spanned by the three vectors $\{\mathbf{1},\mathbf{Z},\mathbf{Z}^2\}$. There is a rank degeneracy in the following case (rank $\mathcal{O} = 2$):

$$r_x(0)v_y = r_y(0)v_x. \tag{24}$$

This condition is itself equivalent to a zero bearing-rate assumption. Even if the above calculations do not provide new result, it presents a direct approach to investigate the observability of the discrete-time system.

Consider now a system constituted of two linear subarrays. The physical supports of these arrays are situated on the same line as depicted in Fig. 2. Then, using the previous notation, the observability matrix $\mathcal{O}'$ takes the following (concatenated) form:

$$\mathcal{O}' = \begin{pmatrix}\mathcal{O}'_1\\ \mathcal{O}'_2\end{pmatrix} \tag{25}$$

with

$$\left|\begin{aligned}\mathcal{O}'_1 &= (\mathbf{T}_x,\mathbf{T}_y,\mathbf{V}_x,\mathbf{V}_y)\\ \mathcal{O}'_2 &= \Delta(\mathbf{T}'_x,\mathbf{T}'_y,\mathbf{V}'_x,\mathbf{V}'_y).\end{aligned}\right. \tag{26}$$

The vectors $(\mathbf{T}_x,\mathbf{T}_y,\mathbf{V}_x,\mathbf{V}_y)$ are defined as in (23), while the vectors defining $\mathcal{O}'_2$ stand as follows:

$$\begin{aligned}\mathbf{T}'_x &= \mathbf{T}_x + (r'_x(0) - r_x(0))\mathbf{1}\\ \mathbf{V}'_x &= \alpha(r'_x(0) - r_x(0))\mathbf{Z} + \mathbf{V}_x\\ \mathbf{T}'_y &= \mathbf{T}_y\\ \mathbf{V}'_y &= \mathbf{V}_y\\ \Delta &= \mathrm{diag}\left(\frac{\cos\theta'_0}{\cos\theta_0},\ldots,\frac{\cos\theta'_k}{\cos\theta_k}\right)\\ &= \mathrm{diag}\left(\frac{r'_0}{r_0},\ldots,\frac{r'_k}{r_k}\right)\end{aligned}$$

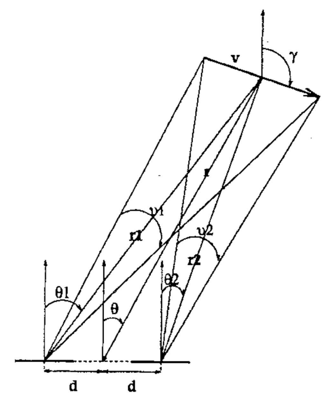

Fig. 2. Typical simulation for 2 TMA.

so that

$$\mathcal{O}' = \begin{pmatrix}\mathcal{O}'_1\\ \Delta\mathcal{O}'_1\end{pmatrix} + \beta\begin{pmatrix}\mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}\\ \Delta\mathbf{1} & \mathbf{O} & \alpha\Delta\mathbf{Z} & \mathbf{O}\end{pmatrix}$$
$$\beta \stackrel{\Delta}{=} r'_x(0) - r_x(0) \qquad (\beta \neq 0). \tag{27}$$

Denote $\ker(\mathcal{O}'_1)$ [4] the null subspace of $\mathcal{O}'_1$ and $\mathcal{R}$ its complementary subspace [4] in $\mathbb{R}^4$, i.e.,

$$\mathbb{R}^4 = \ker\mathcal{O}'_1 \oplus \mathcal{R} \tag{28}$$

(the symbol $\oplus$ meaning direct sum of two subspaces [4]).

Let $\mathbf{X}$ be a vector of $\mathbb{R}^4$. Then $\mathbf{X}$ can be uniquely decomposed as a sum of two vectors belonging respectively to $\ker\mathcal{O}'_1$ and $\mathcal{R}$:

$$\mathbf{X} = \mathbf{K} + \mathbf{Y}, \qquad \mathbf{Y} \in \mathcal{R},$$
$$\mathbf{K} \in \ker\mathcal{O}'_1 \qquad \text{and} \qquad X \stackrel{\Delta}{=} (x_1,x_2,x_3,x_4)^*.$$

The following results can be directly deduced from (27).

1) If $\mathbf{Y} \neq \mathbf{0}$, then $\mathcal{O}'\mathbf{X} \neq \mathbf{0}$ (i.e., $\mathbf{X} \notin \ker\mathcal{O}'$) since $\mathcal{O}'_1\mathbf{X} \neq \mathbf{0}$.

2) If $\mathbf{Y} = \mathbf{0}$, then $\mathcal{O}'\mathbf{X} = \beta\Delta(x_1\mathbf{1} + \alpha x_3\mathbf{Z})$.

Consider now the second case (i.e., $\mathbf{Y} = \mathbf{0}$):

$$\mathcal{O}'\mathbf{X} = \mathbf{0} \quad \text{implies}$$
$$x_1 = x_3 = 0 \quad (\mathbf{1} \text{ and } \mathbf{Z} \text{ are independent}).$$

Since $\mathbf{X}$ belongs to $\ker\mathcal{O}'_1$:

$$x_2\mathbf{T}_y + x_4\mathbf{V}_y = \mathbf{0}$$

which results (except for $\mathbf{T}_y = \mathbf{V}_y = \mathbf{0}$) in

$$x_2 = x_4 = 0.$$

Consequently, except for the special case $\mathbf{T}_y = \mathbf{V}_y = \mathbf{0}$, $\ker\mathcal{O}'$ is reduced to the null vector in the multiplatform case.

Finally, the following result has been demonstrated [5].

PROPOSITION 1 (Multiarray Case $m \geq 2$) *If the source does not move on the array axis, then the system is observable.*

Note that this result has been previously proved by using a slightly different approach [5, 6] involving a continuous-time representation of the system. Actually, the main interest of this approach is its versatility. Consider, for example, the first case (single platform with constant velocity vector) with additional measurements (Doppler shifts). Using the previous formalism, we show that the system becomes observable, that is, the vector $\mathbf{X}_0$ can be uniquely determined from the measurement sequence [5, 6].

The effect of relative motion on the source frequency is termed Doppler effect and modeled by the following equation:

$$f(t) = f_0\left(1 - \frac{v_x}{c}\sin\theta_t - \frac{v_y}{c}\cos\theta_t\right) \tag{29}$$

where f_0 is the reference frequency and c the sound speed.

The measurement now includes the instantaneous time-varying frequency $f(t)$ so that the matrix H_k becomes [5]:

$$H_k = \begin{pmatrix} \cos\theta_k & -\sin\theta_k & 0 & 0 \\ 0 & 0 & -\sin\theta_k & -\cos\theta_k \end{pmatrix}. \tag{30}$$

The observability matrix $\mathcal{O}$ (eq. (13)) becomes

$$\mathcal{O} = \begin{pmatrix} H_0 \\ H_1F \\ \vdots \\ H_kF^k \end{pmatrix} = \left(\frac{\mathcal{O}_\theta}{\mathcal{O}_D}\right) \tag{31}$$

where the $\mathcal{O}_\theta$ matrix is given by (20) and the $\mathcal{O}_D$ matrix is the observability matrix associated with Doppler measurements:

$$\mathcal{O}_D = \begin{pmatrix} 0 & 0 & -\sin\theta_0 & -\cos\theta_0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -\sin\theta_k & -\cos\theta_k \end{pmatrix}. \tag{32}$$

The dual observability matrix $\mathcal{O}'$ takes then the following form:

$$\mathcal{O}' = \left(\frac{\mathcal{O}'_\theta}{\mathbf{0},\mathbf{0},\mathbf{T}_y,\mathbf{T}_x}\right). \tag{33}$$

Let us examine the null subspace of $\mathcal{O}'$. Let $\mathbf{Y}$ be a generic vector of $\mathbb{R}^4$. Clearly if $\mathbf{Y}$ does not belong to $\ker\mathcal{O}'_\theta$ then $\mathbf{Y}$ is not in $\ker\mathcal{O}'$.

Assume now that $\mathbf{Y}$ is in $\ker\mathcal{O}'_\theta$. If it is also in $\ker\mathcal{O}'$ then:

$$y_3\mathbf{T}_y + y_4\mathbf{T}_x = 0.$$

This corresponds to the zero-bearing rate condition (24), since otherwise $\mathbf{T}_x$ and $\mathbf{T}_y$ would be independent, or:

$$r_x(0)v_y = r_y(0)v_x.$$

The following result has thus been demonstrated.

PROPOSITION 2 (Single Platform Case) *If the source does not move at a zero-bearing rate, then the system using bearings and Dopplers measurements is observable.*

Once again, this result is not new but the effect of measurements is clearly demonstrated by this approach. At first, the previous analysis calls for the inclusion of Dopplers in the measurements but, in a multifrequency context, they do not appear especially informative. More specifically, it is shown that the performance corresponding to time delay and Doppler measurements and multifrequency estimated bearings are nearly equivalent.

V. MANEUVERING SOURCE CASE

Proposition 2 can be directly extended to a multiarray system. However, it is more interesting and enlightening to consider the case of maneuvering source. The practical interest of a multiplatform system becomes more evident. The tools and general notations are identical to those used previously. The source trajectory consists of two legs with an instantaneous velocity change. In this case, the state vector of the source ($\mathbf{X}_0$) is of dimension 6 and the observation equation (19) takes the following form (single array):

$$\left|\begin{array}{l} z_0 = H_0\mathbf{X}_0 \\ z_1 = H_1F_1\mathbf{X}_0 \\ \vdots \\ z_k = H_kF_1^k\mathbf{X}_0 \\ z_{k+1} = H_{k+1}F_2F_1^k\mathbf{X}_0 \leftarrow \text{source maneuver} \\ \vdots \\ z_{k+i} = H_{k+i}F_2^iF_1^k\mathbf{X}_0 \end{array}\right. \tag{34}$$

with

$$H_\ell = (\cos\theta_\ell, -\sin\theta_\ell, 0, 0, 0, 0)$$

$$\mathbf{X}_0 = (r_x(0), r_y(0), v_{x,1}, v_{y,1}, v_{x,2}, v_{y,2})^*$$

$$F_1 = \begin{pmatrix} Id_2 & \alpha Id_2 & 0 \\ 0 & Id_2 & 0 \\ 0 & 0 & Id_2 \end{pmatrix},$$

$$F_2 = \begin{pmatrix} Id_2 & 0 & \alpha Id_2 \\ 0 & Id_2 & 0 \\ 0 & 0 & Id_2 \end{pmatrix}.$$

The first two components $r_x(0)$ and $r_y(0)$ of the state vector $\mathbf{X}_0$ represent the initial source position while $(v_{x,1}, v_{y,1})$ and $(v_{x,2}, v_{y,2})$ represent the source velocity vectors on the two consecutive legs.

The transition matrices F_1 and F_2 may be expressed as Kronecker product [4]:

$$F_1 = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes Id_2, \qquad F_2 = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes Id_2$$

and the composition of F_1 and F_2 satisfies to the following property:

$$F_2^\ell F_1^m = F_1^m F_2^\ell = \begin{pmatrix} Id & \ell\alpha Id & m\alpha Id \\ 0 & Id & 0 \\ 0 & 0 & Id \end{pmatrix}. \qquad (35)$$

The matrix $\mathcal{O}'$ (22) then takes on the following form:

$$\mathcal{O}' = (\mathbf{R}_x \mid \mathbf{R}_y \mid \mathbf{S}_x \mid \mathbf{S}_y \mid \mathbf{T}_x \mid \mathbf{T}_y)$$

with

$$\left| \begin{array}{l} \mathbf{R}_x = r_x(0)\begin{pmatrix}\mathbf{1}\\ \mathbf{1}'\end{pmatrix} + \alpha v_{x,1}\begin{pmatrix}\mathbf{Z}\\ k\mathbf{1}'\end{pmatrix} + \alpha v_{x,2}\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'\end{pmatrix} \\ \mathbf{R}_y = r_y(0)\begin{pmatrix}\mathbf{1}\\ \mathbf{1}'\end{pmatrix} + \alpha v_{y,1}\begin{pmatrix}\mathbf{Z}\\ k\mathbf{1}'\end{pmatrix} + \alpha v_{y,2}\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'\end{pmatrix} \\ \mathbf{S}_x = \alpha r_x(1)\begin{pmatrix}\mathbf{Z}\\ k\mathbf{1}'\end{pmatrix} + \alpha^2 v_{x,1}\begin{pmatrix}\mathbf{Z}^2\\ k\mathbf{1}'\end{pmatrix} + \alpha^2 v_{x,2}\begin{pmatrix}\mathbf{0}\\ k\mathbf{Z}'\end{pmatrix} \\ \mathbf{T}_x = \alpha r_x(1)\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'\end{pmatrix} + \alpha^2 v_{x,1}\begin{pmatrix}\mathbf{0}\\ k\mathbf{Z}'\end{pmatrix} + \alpha^2 v_{x,2}\begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'^2\end{pmatrix} \end{array} \right. \qquad (36)$$

($\mathbf{S}_y$ and $\mathbf{T}_y$ have the form of $\mathbf{S}_x$ and $\mathbf{T}_x$, $(r_y(1), v_{y,1}, v_{y,2})$ replacing $(r_x(1), v_{x,1}, v_{x,2})$).

The following space inclusion[1,2,3] is directly deduced from (36):

$$\mathrm{Im}\,\mathcal{O}' \subset \mathrm{sp}\left\{\begin{pmatrix}\mathbf{1}\\ \mathbf{1}'\end{pmatrix}, \begin{pmatrix}\mathbf{Z}\\ \mathbf{1}'\end{pmatrix}, \begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'\end{pmatrix}, \begin{pmatrix}\mathbf{Z}^2\\ \mathbf{1}'\end{pmatrix}, \begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'^2\end{pmatrix}\right\}.$$

Therefore,

$$\mathrm{rank}(\mathcal{O}') \leq 5.$$

From (36), the following hypotheses (denoted H_1 and H_2):

$$\det\begin{pmatrix} r_x(1) & v_{x,1} \\ r_y(1) & v_{y,1} \end{pmatrix} \quad \text{and} \quad \det\begin{pmatrix} v_{x,1} & v_{x,2} \\ v_{y,1} & v_{y,2} \end{pmatrix} \neq 0$$

imply

$$\mathrm{rank}(\mathcal{O}') = 5.$$

The dimension of the observable space is thus increased by two. The unobservable space is of dimension 1 and spanned by the vector $\mathbf{X}_0$.

Next, consider a system made of two linear subarrays situated on the same line (Fig. 2). Then, the observability matrix $\mathcal{O}'$ of the general system takes the following form:

$$\mathcal{O}' = \begin{pmatrix} \mathcal{O}_1' \\ \mathcal{O}_2' \end{pmatrix}.$$

Hence,

$$\mathcal{O}' = \begin{pmatrix} \mathcal{O}_1' \\ \Delta\mathcal{O}_1' \end{pmatrix} + \beta\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \Delta\mathbf{1} & \mathbf{0} & \alpha\Delta\mathbf{S}_2 & \mathbf{0} & \alpha\Delta\mathbf{T}_2 & \mathbf{0} \end{pmatrix} \qquad (37)$$

where:

$$\beta \triangleq r_x'(0) - r_x(0), \qquad \Delta = \mathrm{diag}\left(\frac{r_0'}{r_0}, \ldots, \frac{r_{k+i}'}{r_{k+i}}\right)$$

and (with the notations of (35)):

$$\mathbf{S}_2 \triangleq \begin{pmatrix}\mathbf{Z}\\ k\mathbf{1}'\end{pmatrix}, \qquad \mathbf{T}_2 \triangleq \begin{pmatrix}\mathbf{0}\\ \mathbf{Z}'\end{pmatrix}$$

(the index $\langle\langle ' \rangle\rangle$ corresponds for $\mathbf{S}_2$ and $\mathbf{T}_2$ to the second leg).

$\ker(\mathcal{O}_1')$ denotes the null subspace of $\mathcal{O}_1'$ and $\mathcal{R}$ its complementary subspace in $\mathbb{R}^6$:

$$\mathbb{R}^6 = \ker\mathcal{O}_1' \oplus \mathcal{R}. \qquad (38)$$

In addition, let $\mathbf{X}$ be a vector of $\mathbb{R}^6$. Then $\mathbf{X}$ can be uniquely decomposed as

$$\mathbf{X} = \mathbf{K} + \mathbf{Y}, \qquad \mathbf{Y} \in \mathcal{R} \quad \text{and} \quad \mathbf{K} \in \ker\mathcal{O}_1'$$

and if $\mathbf{Y} = \mathbf{0}$, then it comes from (38):

$$\mathcal{O}'\mathbf{X} = \beta\Delta\left(x_1\begin{pmatrix}\mathbf{0}\\ \mathbf{1}\end{pmatrix} + \alpha x_3\begin{pmatrix}\mathbf{0}\\ \mathbf{S}_2\end{pmatrix} + \alpha x_5\begin{pmatrix}\mathbf{0}\\ \mathbf{T}_2\end{pmatrix}\right). \qquad (39)$$

Since the vectors ($\mathbf{1}$), ($\mathbf{S}_2$), and ($\mathbf{T}_2$) are linearly independent, the following property holds:

$$\mathcal{O}'\mathbf{X} = \mathbf{0} \quad \text{and} \quad \mathbf{X} \in \ker\mathcal{O}_1' \Leftrightarrow x_1 = x_3 = x_5 = 0. \qquad (40)$$

Now it has been previously shown (under the two hypotheses H_1 and H_2) that $\dim\ker\mathcal{O}_1' = 1$ and more precisely $\ker\mathcal{O}_1' = \mathrm{sp}(\mathbf{X}_0)$. Therefore:

$$\begin{array}{c} \mathcal{O}'\mathbf{X} = \mathbf{0} \quad \text{and} \\ \mathbf{X} \in \ker\mathcal{O}_1' \Leftrightarrow r_x(0) = v_{x,1} = v_{x,2} = 0 \end{array} \qquad (41)$$

but this contradicts the hypotheses H_1 and H_2 since the two determinants are null if (41) is valid.

Further, note that H_1 (or H_2) is equivalent to a zero bearing-rate assumption. This reasoning can be directly but tediously extended to the general case of ℓ legs, yielding the following result.

PROPOSITION 3 *Consider a system comprising multiple nonmaneuvering arrays. If on each leg of the source trajectory the source bearing rate is non-zero, then the system is observable.*

[1]($\mathrm{sp}(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ denotes the vector space spanned by the vectors $\{\mathbf{X}_1, \ldots, \mathbf{X}_n\}$).

[2]$\mathrm{Im}\,\mathcal{O}'$ denotes the image of $\mathcal{O}'$.

[3]The index $\langle\langle ' \rangle\rangle$ stands for the second leg.

Proposition 3 requires some comments.

COMMENT 1 The instants of velocity changes are assumed to be known.

COMMENT 2 The dimension of the source state vector $\mathbf{X}_0$ (defining the source trajectory) is equal to $\ell+2$. Practically, the estimation of $\mathbf{X}_0$ may be greatly affected as ℓ increases.

The knowledge of the instants of velocity changes seems quite restrictive but, actually, the notion of observability may be extended to these instants. More precisely, define $\boldsymbol{\theta}$ the vector of velocity changes (i.e., $\boldsymbol{\theta} = (t_1, t_2, \ldots, t_{\ell-1})^*$). Then $\boldsymbol{\theta}$ is observable if and only if:

$$\mathcal{O}(\mathbf{X}_0, \boldsymbol{\theta}) = \mathcal{O}(\mathbf{X}_0, \boldsymbol{\theta}') \Leftrightarrow \boldsymbol{\theta} = \boldsymbol{\theta}'. \tag{42}$$

Using the previous formalism (37), the observability of the velocity change can be demonstrated if H_1 and H_2 are valid.

VI. VARIANCES APPROXIMATIONS FOR 2 TMA

The computation of an analytical expression for parameter estimation variances (Appendix A) for two bearings-only TMA was carried out with a view to know the main factors on which they depend. The results yield interesting insights into the variations of the standard deviation error within different scenarios. In this section we explain three cases that allow us to compute refined approximations of the variances. All cases are based on the same scenario that is depicted in Fig. 2. The spacing between the two arrays is $2d$, and at $t = 0$ the distance between the center of the two platforms and the source is r and its azimuth is θ.

In the first case we assume that for each array the distance from the source (r_1 and r_2) is constant and the azimuth is linearly varying with time. This is quite a good approximation if and only if r is much greater than d and than the length of the source course. This gives an expression of the variances that can be computed for **any given** θ.

The second case is a little more realistic in the sense that the distance between the source and the arrays is stated to be linearly varying so that the equations of $r(t)$ and $\theta(t)$ can be written out. Although these approximations make more sense they are hardly tractable in order to obtain a formal expression of the variances for any θ.

The last case consists in writing the Fisher information matrix (FIM) directly using the coordinates of the source. In that case one has only to expand as Taylor series the inverse of the distance between the source and the arrays. This case gives an approximation of the variances which are a function of the main variables of the scenarios including r, θ, v, and γ.

A. Case 1: r is Constant

In this part one considers that the distance between the platforms and the source is constant, and that the azimuth variance estimation is also constant. First of all, let us define all the parameters.

$2d$ is the distance between the center of the arrays.
r is the distance between the source and the center of the two arrays.
r_i is the distance between the ith array and the source.
θ is the azimuth of the source at initial time for the midpoint of the two platforms.
θ_i is the azimuth of the source at initial time for the ith platform.
ν_i is the sector on which the source is seen by the ith array.
$2n+1$ is the number of bearing measurements ($t = -n, \ldots, n$).
bas_i is the baseline of the movement of the source for the ith array.
σ_i is the azimuth estimation standard deviation error for the ith array.

With these notations one can write

$$\theta_i(t) = \theta_i + (t/2n)\nu_i, \qquad i = 1,2, \quad t = -n, \ldots, n$$

$$\mathrm{bas}_i = |(2n+1)\delta t v \sin(\gamma - \theta_i)|, \qquad i = 1,2$$

$$r_1^2 = r^2 + d^2 + 2rd\sin(\theta)$$

$$r_2^2 = r^2 + d^2 - 2rd\sin(\theta)$$

$$\sin\nu_i = \mathrm{bas}_i / r_i, \qquad i = 1,2.$$

The total FIM is the sum of the FIM of the two arrays:

$$F = F_1 + F_2$$

with

$$F_1 = \sum_t \frac{1}{r_1^2\sigma_1^2} \begin{bmatrix} \Omega_1(t) & t\delta t\Omega_1(t) \\ t\delta t\Omega_1(t) & t^2\delta t^2\Omega_1(t) \end{bmatrix}$$

and

$$\Omega_1(t) = \begin{bmatrix} \cos^2\theta_1(t) & -\sin\theta_1(t)\cos\theta_1(t) \\ -\sin\theta_1(t)\cos\theta_1(t) & \sin^2\theta_1(t) \end{bmatrix}.$$

F_2 has the same expression but using $\theta_2(t)$ instead of $\theta_1(t)$. The FIM has the form:

$$F = \begin{bmatrix} f1 & f2 & f4 & f5 \\ f2 & f3 & f5 & f6 \\ f4 & f5 & f7 & f8 \\ f5 & f6 & f8 & f9 \end{bmatrix}$$

with

$$f1 = \sum_{i=1,2} \frac{1}{r_i^2 \sigma_i^2} \sum_{t=-n}^{n} \cos^2 \theta_i(t)$$

$$f2 = \sum_{i=1,2} \frac{1}{r_i^2 \sigma_i^2} \sum_{t=-n}^{n} -\sin\theta_i(t)\cos\theta_i(t)$$

$$f3 = \sum_{i=1,2} \frac{1}{r_i^2 \sigma_i^2} \sum_{t=-n}^{n} \sin^2 \theta_i(t)$$

$$f4 = \sum_{i=1,2} \frac{1}{r_i^2 \sigma_i^2} \sum_{t=-n}^{n} t\,\delta t \cos^2 \theta_i(t)$$

$$f5 = \sum_{i=1,2} \frac{1}{r_i^2 \sigma_i^2} \sum_{t=-n}^{n} -t\,\delta t \sin\theta_i(t)\cos\theta_i(t)$$

$$f6 = \sum_{i=1,2} \frac{1}{r_i^2 \sigma_i^2} \sum_{t=-n}^{n} t\,\delta t \sin^2 \theta_i(t)$$

$$f7 = \sum_{i=1,2} \frac{1}{r_i^2 \sigma_i^2} \sum_{t=-n}^{n} t^2\,\delta t^2 \cos^2 \theta_i(t)$$

$$f8 = \sum_{i=1,2} \frac{1}{r_i^2 \sigma_i^2} \sum_{t=-n}^{n} -t^2\,\delta t^2 \sin\theta_i(t)\cos\theta_i(t)$$

$$f9 = \sum_{i=1,2} \frac{1}{r_i^2 \sigma_i^2} \sum_{t=-n}^{n} t^2\,\delta t^2 \sin^2 \theta_i(t).$$

These trigonometric expressions are expanded into Taylor series with respect to ϵ about the value 0, and the sum is evaluated. The inverse of this matrix yields the variance estimation of the position and velocity parameters. All these computations can be done thanks to MAPLE an interactive computer algebra system. Unfortunately this computation yields too large expressions which can be evaluated only for a certain θ. That means that the coefficients of the different orders of the expansion of these trigonometric expressions have an internal representation in the computer that cannot be handled to compute the inverse of the matrix. Approximations for $\theta = 0$ or $\theta = \pi/4$ can be obtained, but a general expression involving θ as is cannot be computed due to the numerical burden.

Since the source range is much greater than the total length of the arrays, the standard deviation error on the bearing estimates are stated equal for all the platforms. With this simplification the inverse of the FIM can be computed, and the Cramer-Rao lower bound (CRLB) of each element of the state vector can be found.

For $\theta = 0$ the terms $f1$ to $f9$ can be expanded up to order 6 and the inverse of the FIM is computed. The following expressions are obtained after having kept the higher order term of the numerator and the denominator of the diagonal terms of the inverse of the FIM:

$$\mathrm{var}(\hat{r}_x) = \frac{(r^2 + d^2)^2}{2(2n+1)r^2}\sigma^2 \tag{43}$$

$$\mathrm{var}(\hat{r}_y) = \frac{(r^2 + d^2)^2}{2(2n+1)d^2}\sigma^2 \tag{44}$$

$$\mathrm{var}(\hat{v}_x) = \frac{3(r^2 + d^2)^2}{2n(n+1)(2n+1)r^2\,\delta t^2}\sigma^2 \tag{45}$$

$$\mathrm{var}(\hat{v}_y) = \frac{3(r^2 + d^2)^2}{2n(n+1)(2n+1)d^2\,\delta t^2}\sigma^2. \tag{46}$$

Since none of these equations depends on the speed of the source, when the source increase its velocity, this approximation becomes less and less accurate. If one try to compute the Taylor series of the terms $f1$ to $f9$ to a higher order, the analytical calculation the inverse of the FIM becomes intractable.

For $\theta = \pi/4$ at the mid-point of the observation interval the following expressions were obtained:

$$\mathrm{var}(\hat{r}_x) = \frac{r^4 + d^4 + 4r^2d^2}{(2n+1)r(r + \sqrt{r^2 - 2d^2})}\sigma^2 \tag{47}$$

$$\mathrm{var}(\hat{r}_y) = \frac{r(r^4 - 3d^4)}{(2n+1)d^2(r + \sqrt{r^2 - 2d^2})}\sigma^2 \tag{48}$$

$$\mathrm{var}(\hat{v}_x) = \frac{r^4 + d^4 + 4r^2d^2}{n(n+1)(2n+1)d^2(r + \sqrt{r^2 - 2d^2})\,\delta t^2}\sigma^2 \tag{49}$$

$$\mathrm{var}(\hat{v}_y) = \frac{r(r^4 - 3d^4)}{n(n+1)(2n+1)d^2(r + \sqrt{r^2 - 2d^2})\,\delta t^2}\sigma^2. \tag{50}$$

In all those expressions the factor σ^2 is present. This is the estimation variance of the central bearing θ. This variance is computed with the Woodward formula (5). In this formula the array length L can be expressed as d (the platform interspace) multiplied by a constant.

With the last remark in mind it can be seen that the terms relative to the x component are proportional to r^2/d^2 and those relative to the y component vary as r^4/d^4. In the same way it can be seen that the position variances behave like $1/n$ ($2n+1$ is the number of bearing measurements), and the velocity variances vary as $1/n^3$. This means that the longer the integration time, the better the estimation will be, especially for the velocity. The source movement has been integrated in the estimator, so it is not limited by the source velocity and long-time integration can be performed without any loss of generality.

B. Case 2: r and θ Linearly Varying

In the previous section it has been seen that considering r as a constant is a convenient but restrictive hypothesis. In this case we develop the algorithm assuming that $r(t)$ and $\theta(t)$ are linearly varying.

The parameters $\theta(t)$, $r(t)$, v, and γ have been defined

$$r(t) = \sqrt{(r_{x0}+tv_x)^2+(r_{y0}+tv_y)^2} = \sqrt{r_x^2(t)+r_y^2(t)}$$

$$\theta(t) = \arctan\left(\frac{r_x(t)}{r_y(t)}\right)$$

$$v = \sqrt{v_x^2+v_y^2}$$

$$\gamma = \arctan\left(\frac{v_x}{v_y}\right).$$

One has to compute the derivative of $\theta(t)$ and $r(t)$ with respect to t

$$\frac{\partial\theta(t)}{\partial t} = \frac{v_x r_y(t) - v_y r_x(t)}{r_y^2(t)}\cos^2\theta(t)$$

$$= \frac{vr(t)(\sin(\gamma)\cos(\theta(t)) - \cos(\gamma)\sin(\theta(t)))}{r^2(t)\cos^2(\theta(t))}$$

$$= \frac{v}{r(t)}\sin(\gamma-\theta(t)). \tag{51}$$

In the same way:

$$\frac{\partial r(t)}{\partial t} = \frac{v_x r_x(t) + v_y r_y(t)}{r(t)}$$

$$= v\cos(\gamma-\theta(t)). \tag{52}$$

Therefore, the first-order expansions of $r(t)$ and $\theta(t)$ obtained for $t\in[-T;T]$ are the following:

$$\theta(t) \stackrel{1}{=} \theta(0) + t\frac{v}{r(0)}\sin(\gamma-\theta(0)) \tag{53}$$

$$r(t) \stackrel{1}{=} r(0) + tv\cos(\gamma-\theta(0)). \tag{54}$$

Thanks to these hypothesis the baseline is easily expressed as

$$\text{baseline} = |2r(0)\sin(((2n+1)\delta tv/r(0)\sin(\gamma-\theta(0)))|. \tag{55}$$

With these expressions and the notations of the preceding subsection, the FIM can be computed for the two platform case and its inverse may be calculated in order to have an analytic formulation of the variances. But there is still the problem of dimensionality of the calculation involved. Since the computation for an arbitrary θ cannot be implemented, it must be fixed at the beginning of the calculation. For $\theta = 0$ the following expressions hold

$$\text{var}(\hat{r}_x) = \frac{r^2+d^2}{2(2n+1)(1-\cos^2(d/r))}\sigma^2 \tag{56}$$

$$\text{var}(\hat{r}_y) = \frac{r^2+d^2}{2(2n+1)\cos^2(d/r)}\sigma^2 \tag{57}$$

$$\text{var}(\hat{v}_x) = \frac{3(r^2+d^2)}{2n(n+1)(2n+1)(1-\cos^2(d/r))}\sigma^2 \tag{58}$$

$$\text{var}(\hat{v}_y) = \frac{3(r^2+d^2)}{2n(n+1)(2n+1)\cos^2(d/r)}\sigma^2. \tag{59}$$

After a second-order expansion of the cosine functions, the equations obtained with the first case emerge. If the cosines are kept as is, this approximation is better than the first one when the velocity of the source increases and when the number of bearing measurements increases too. It means that when the baseline gets higher this second analytical approximation of the variances fits better to the reality than those obtained with the first case.

C. Case 3

In this third case we deal with the equations that define the temporal evolution of $r_x(t)$ and $r_y(t)$ to compute the FIM. This matrix that was expressed in terms of sines and cosines, can be computed using the elements of the state vector, using the already known relations:

$$r_x(t) = r(t)\sin(\theta(t))$$
$$r_{x1}(t) = r_{x1}(0) + t\delta t v_x = r\sin(\theta) + d + t\delta t v_x$$
$$r_{x2}(t) = r_{x2}(0) + t\delta t v_x = r\sin(\theta) - d + t\delta t v_x$$
$$r_y(t) = r(t)\cos(\theta(t)) = r\cos(\theta) + t\delta t v_y.$$

The total FIM is still the sum of the FIM relative to the different arrays:

$$F = F_1 + F_2$$

with

$$F_1 = \sum_t \frac{1}{r_1(t)^4\sigma_1^2}\begin{bmatrix}\Omega_1'(t) & t\delta t\Omega_1'(t)\\ n\delta t\Omega_1'(t) & t^2\delta t^2\Omega_1'(t)\end{bmatrix}$$

and

$$\Omega_1'(t) = \begin{bmatrix} r_{x1}^2(t) & r_{x1}(t)r_y(t)\\ r_{x1}(t)r_y(t) & r_y^2(t)\end{bmatrix}.$$

F_2 has the same expression but using $r_{x2}(t)$ instead of $r_{x1}(t)$. The FIM has the the form:

$$F = \begin{bmatrix} f1 & f2 & f4 & f5\\ f2 & f3 & f5 & f6\\ f4 & f5 & f7 & f8\\ f5 & f6 & f8 & f9\end{bmatrix}$$

with

$$f1 = \sum_{i=1,2} \frac{1}{\sigma_i^2} \sum_{t=-n}^{n} \frac{r_{xi}^2(t)}{r_i^4(t)}$$

$$f2 = \sum_{i=1,2} \frac{1}{\sigma_i^2} \sum_{t=-n}^{n} \frac{r_{xi}(t) r_y(t)}{r_i^4(t)}$$

$$f3 = \sum_{i=1,2} \frac{1}{\sigma_i^2} \sum_{t=-n}^{n} \frac{r_y^2(t)}{r_i^4(t)}$$

$$f4 = \sum_{i=1,2} \frac{1}{\sigma_i^2} \sum_{t=-n}^{n} \frac{t\,\delta t r_{xi}^2(t)}{r_i^4(t)}$$

$$f5 = \sum_{i=1,2} \frac{1}{\sigma_i^2} \sum_{t=-n}^{n} \frac{t\,\delta t r_{xi}(t) r_y(t)}{r_i^4(t)}$$

$$f6 = \sum_{i=1,2} \frac{1}{\sigma_i^2} \sum_{t=-n}^{n} \frac{t\,\delta t r_y^2(t)}{r_i^4(t)}$$

$$f7 = \sum_{i=1,2} \frac{1}{\sigma_i^2} \sum_{t=-n}^{n} \frac{t^2\,\delta t^2 r_{xi}^2(t)}{r_i^4(t)}$$

$$f8 = \sum_{i=1,2} \frac{1}{\sigma_i^2} \sum_{t=-n}^{n} \frac{t^2\,\delta t^2 r_{xi}(t) r_y(t)}{r_i^4(t)}$$

$$f9 = \sum_{i=1,2} \frac{1}{\sigma_i^2} \sum_{t=-n}^{n} \frac{t^2\,\delta t^2 r_y^2(t)}{r_i^4(t)}.$$

These expressions are expanded into Taylor series with respect to $r(0)$ about the infinity up to the order 6. The expansions of the $\{f_i\}_{i=1}^9$ are quite simple because the numerator and the denominator are polynomials of the variable $r(0)$. As in the preceding subsection σ_1 and σ_2 will be stated equal to simplify the computation. The inversion of the FIM can be done even if θ is kept as a parameter and the following result is obtained

$$\mathrm{var}(\hat{r}_x) = \frac{r^4}{2(2n+1)d^2} \tan^2(\theta)\sigma^2 \tag{60}$$

$$\mathrm{var}(\hat{r}_y) = \frac{r^4 \cos^2(\theta)}{2(2n+1)d^2\cos^2(\theta)}\sigma^2 \overset{\theta \neq \pi/2}{=} \frac{r^4}{2(2n+1)d^2}\sigma^2 \tag{61}$$

$$\mathrm{var}(\hat{v}_x) = \frac{45 r^4 \sin^2(\theta)}{2\,\delta t^2 n(n+1)(2n+1)[15d^2\cos^2(\theta) + (2n-1)(2n+3)\,\delta t^2 v^2 \sin^2(\theta-\gamma)]}\sigma^2 \tag{62}$$

$$\mathrm{var}(\hat{v}_y) = \frac{45 r^4 \cos^2(\theta)}{2\,\delta t^2 n(n+1)(2n+1)[15d^2\cos^2(\theta) + (2n-1)(2n+3)\,\delta t^2 v^2 \sin^2(\theta-\gamma)]}\sigma^2. \tag{63}$$

Unfortunately the computation cannot be done more precisely. If the expressions $f1$ to $f9$ are expanded to an order over 6, the calculation of the inverse of the FIM induces long equations that cannot be simplified. If these expressions are used for $\theta = 0$ or $\theta = \pi/2$, the results obtained are 0, ∞, or the wrong answer. These answers hold for these particular bearings because in each equation the numerator or the denominator (or both of them) are proportional to the sine or the cosine of the bearing. For these particular values of θ the following approximations are obtained.

For $\theta = 0$:

$$\mathrm{var}(\hat{r}_x) = \frac{[15d^2 + 3(3n^2+3n-1)\,\delta t^2 v^2 \sin^2(\gamma)]r^2}{2(2n+1)[15d^2 + (2n-1)(2n+3)\,\delta t^2 v^2 \sin^2(\gamma)]}\sigma^2 \tag{64}$$

$$\mathrm{var}(\hat{r}_y) = \frac{r^4}{2(2n+1)d^2}\sigma^2 \tag{65}$$

$$\mathrm{var}(\hat{v}_x) = \frac{[3d^2 + n(n+1)\,\delta t^2 v^2 \sin^2(\gamma)]r^2}{n(n+1)(2n+1)\,\delta t^2 d^2}\sigma^2 \tag{66}$$

$$\mathrm{var}(\hat{v}_y) = \frac{45 r^4}{2\,\delta t^2 n(n+1)(2n+1)[15d^2 + (2n-1)(2n+3)\,\delta t^2 v^2 \sin^2(\gamma)]}\sigma^2. \tag{67}$$

For $\theta = \pi/2$:

$$\mathrm{var}(\hat{r}_x) = \frac{3r^6}{n(n+1)(2n+1)\,\delta t^2 v^2 \cos^2(\gamma) d^2}\sigma^2 \tag{68}$$

$$\mathrm{var}(\hat{r}_y) = \frac{3(3n^2+3n-1)r^2}{2(2n-1)(2n+1)(2n+3)}\sigma^2 \tag{69}$$

$$\mathrm{var}(\hat{v}_x) = \frac{[45d^2 - (2n-1)(2n+3)\,\delta t^2 v^2 \cos^2(\gamma)]r^4}{2n(n+1)(2n-1)(2n+1)(2n+3)\,\delta t^4 v^2 d^2 \cos^2(\gamma)}\sigma^2 \tag{70}$$

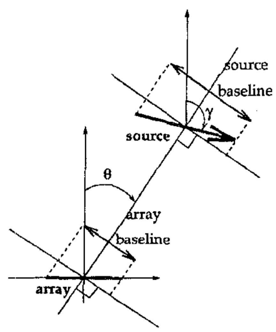

Fig. 3. Baseline definition for array and source.

$$\mathrm{var}(\hat{v}_y) = \frac{3r^4}{2n(n+1)(2n+1)\,\delta t^2 d^2}\sigma^2. \qquad (71)$$

When $\theta = \pi/2$ it can be easily observed that when the heading of the source (γ) is $\pi/2$ (the source moves in the array axis) the elements of the state vector relative to y are not observable. This is the result that has been demonstrated in Section IV (Fig. 3).

Some comparisons between the real value of the CRLB and its approximation given by (60)–(63) have been conducted. It should be noted that when the particular expressions for $\theta = 0$ or $\theta = \pi/2$ fits better to the real value than the general expression, the latter is still quite efficient (no more than 40% relative error for reasonable scenarios). Here are two examples for $\theta = 0$. The main parameters are set as follows: $r = 15$ km, $2d = 2000$ m, $v = 20$ ms^{-1}, $\gamma = 0$ rad, $2n + 1 = 101$, $\delta t = 1$ s. The curves on Fig. 4 (respectively, Fig. 5) represent the relative error on the estimation variances of the different elements of the state vector when the heading varies from $-\pi$ to π (respectively, when the number of integration $2n + 1$ varies from 20 to 1000).

VII. ANALYTICAL EXPRESSIONS OF VARIANCES FOR MTMA

Once an efficient method to compute the FIM and its inverse has been found for the two-platform case, it can be easily extended to the multiplatform case ($m \geq 2$, m number of platforms). In the generic figure (Fig. 6) there are $2m + 1$ arrays separated one from each other by the distance d. The central array is the origin of the space, that is the array from where r and θ are computed. γ is still the heading of the source and v its velocity.

The computation of an analytical formula of the estimation variances for MTMA is of great interest, especially when the arrays are the result of the division of one large array. In that case it may be interesting to know how the number of subarrays influence the variances of the estimation of the elements of the state vector.

If the last method explained is used in order to calculate the FIM, the terms $f1$ to $f9$ have the same expression except that the sum over the platforms is changed:

$$\sum_{i=1,2} \longrightarrow \sum_{i=-m}^{m}.$$

In addition, the expression of $r_{xi}(t)$ is modified:

$$r_{xi}(t) = r_{x0} + id + t\,\delta t v_x = r\sin(\theta) + t\,\delta t v_x.$$

The expressions $f1$ to $f9$ are expanded into Taylor series with respect to $r(0)$ about the infinity. σ_1 and σ_2

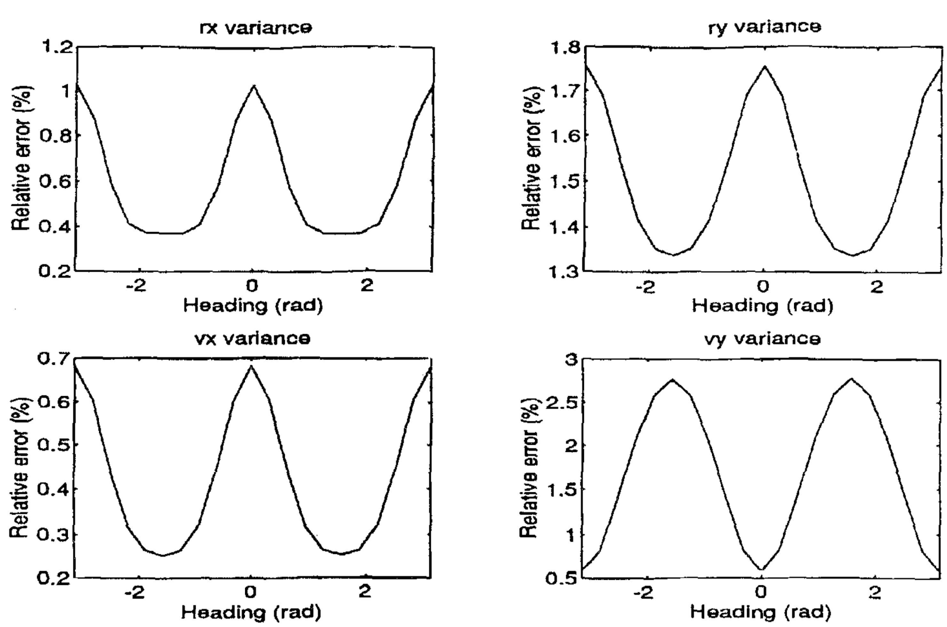

Fig. 4. Relative error on the variances between the real value and the result of the approximation for $\theta = 0$ rad, $r = 15$ km, $2d = 2000$ m, $v = 20$ ms^{-1}, $2n + 1 = 101$, $\delta t = 1$ s.

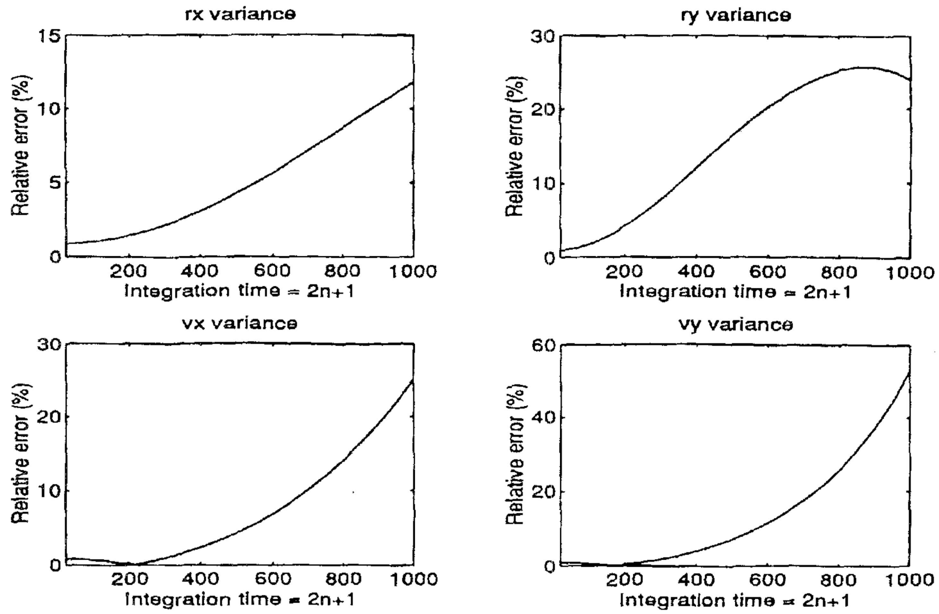

Fig. 5. Relative error on the variances between the real value and the result of the approximation for $\theta = 0$ rad, $r = 15$ km, $d = 2000$ m, $v = 20$ ms^{-1}, $\gamma = 0$ rad, $\delta t = 1$ s.

are be stated equal to simplify the computation. The inversion of the FIM can be done even if θ is kept as a parameter and the following result are obtained

$$\mathrm{var}(\hat{r}_x) = \frac{3r^4}{(2n+1)m(2m+1)(m+1)d^2}\tan^2(\theta)\sigma^2 \tag{72}$$

$$\mathrm{var}(\hat{r}_y) = \frac{3r^4}{(2n+1)m(2m+1)(m+1)d^2}\sigma^2 \tag{73}$$

$$\mathrm{var}(\hat{v}_x) = \frac{45r^4\sin^2(\theta)}{\delta t^2 n(n+1)(2n+1)(2m+1)[5m(m+1)d^2\cos^2(\theta)+(2n-1)(2n+3)\delta t^2 v^2\sin^2(\theta-\gamma)]}\sigma^2 \tag{74}$$

$$\mathrm{var}(\hat{v}_y) = \frac{45r^4\cos^2(\theta)}{\delta t^2 n(n+1)(2n+1)(2m+1)[5m(m+1)d^2\cos^2(\theta)+(2n-1)(2n+3)\delta t^2 v^2\sin^2(\theta-\gamma)]}\sigma^2. \tag{75}$$

Two very important terms appear in these formulas: the array and the source baseline. These parameters have the following formulation:

$$\mathcal{A}_{\mathrm{bas}} = 2md|\cos\theta| = \mathcal{L}_{\mathrm{tot}}|\cos\theta| \tag{76}$$

$$\mathcal{S}_{\mathrm{bas}} = (2n+1)v\,\delta t|\sin(\theta-\gamma)| \tag{77}$$

where $\mathcal{L}_{\mathrm{tot}}$ is the total length of the arrays. These factors does not appear exactly but via two approximations of their square:

$$m(m+1)d^2\cos^2\theta \simeq \tfrac{1}{4}\mathcal{A}_{\mathrm{bas}}^2 \tag{78}$$

$$(2n-1)(2n+3)v^2\delta t^2\sin^2(\theta-\gamma) \simeq \mathcal{S}_{\mathrm{bas}}^2. \tag{79}$$

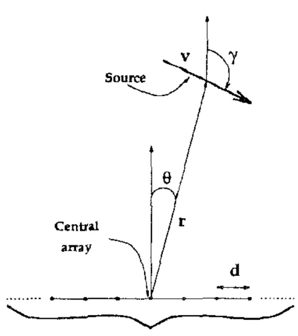

Fig. 6. Typical simulation for MIMA.

With this approximations the formulations of the variances can be rewritten as

$$\mathrm{var}(\hat{r}_x) = \frac{12r^4\sin^2\theta}{(2n+1)(2m+1)\mathcal{A}_{\mathrm{bas}}^2}\sigma^2 \tag{80}$$

$$\mathrm{var}(\hat{r}_y) = \frac{12r^4\cos^2\theta}{(2n+1)(2m+1)\mathcal{A}_{\mathrm{bas}}^2}\sigma^2 \tag{81}$$

$$\mathrm{var}(\hat{v}_x) = \frac{180r^4\sin^2\theta}{\delta t^2 n(n+1)(2n+1)(2m+1)[5\mathcal{A}_{\mathrm{bas}}^2+4\mathcal{S}_{\mathrm{bas}}^2]}\sigma^2 \tag{82}$$

$$\mathrm{var}(\hat{v}_y) = \frac{180r^4\cos^2\theta}{\delta t^2 n(n+1)(2n+1)(2m+1)[5\mathcal{A}_{\mathrm{bas}}^2+4\mathcal{S}_{\mathrm{bas}}^2]}\sigma^2. \tag{83}$$

These equations provide interesting insights [8].

1) All these equations are proportional to σ^2, which means that the quality of the bearing measurements conditions the quality of the position and velocity estimates.

2) The position estimates improve with observation interval, but the velocity estimates improve with its third power. This means that integration time is of great importance especially for the estimation of the source velocity.

3) The position estimates accuracy does not depend, with this first-order approximation, on the source baseline. In fact, if more terms are computed on the numerator and the denominator this baseline appears. This means that as long as the source baseline is much less than its range, it does not modify the position estimates variances.

4) In the case where one large array is divided into multiple subarrays, bearing estimation accuracy is essentially proportional to m^3 and inversly proportional to $4m^2d^2\cos^2(\theta)$, the square of the effective baseline of the total array. The source baseline ($\mathcal{S}_{\mathrm{bas}}$) is defined by $(2n+1)\delta tv|\sin(\theta-\gamma)|$. In this situation the position variances vary as $m^2(r/\mathcal{A}_{\mathrm{bas}})^4$, and the velocity estimates error vary as $m^2r^4/(\mathcal{A}_{\mathrm{bas}}^2(\mathcal{A}_{\mathrm{bas}}^2+\mathcal{S}_{\mathrm{bas}}^2))$.

5) In the case where identical arrays are added one after the other, the bearing estimation variance is not sensitive to m. Thus, in this situation the position variances vary as $m^{-1}(r/\mathcal{A}_{\mathrm{bas}})^4$, and the velocity estimates error vary as $m^{-1}r^4/(\mathcal{A}_{\mathrm{bas}}^2(\mathcal{A}_{\mathrm{bas}}^2+\mathcal{S}_{\mathrm{bas}}^2))$.

As for the two-platform case, these formulas are not precise enough or simply wrong for $\theta = 0$ or $\pi/2$. So for these bearings a separate computation has to be done. A new approximation of $\mathcal{S}_{\mathrm{bas}}$ appears in the numerator of the expression of the variance of r_x at $\theta = 0$ and of r_y at $\theta = \pi/2$:

$$(3n^2+3n-1)\delta t^2v^2\sin^2(\theta-\gamma) \simeq \tfrac{3}{4}\mathcal{S}_{\mathrm{bas}}^2. \tag{84}$$

For $\theta = 0$, the following expressions are obtained ($\mathcal{A}_{\mathrm{bas}}$ and $\mathcal{S}_{\mathrm{bas}}$ are used instead of their approximations):

$$\mathrm{var}(\hat{r}_x) = \frac{[5\mathcal{A}_{\mathrm{bas}}^2+9\mathcal{S}_{\mathrm{bas}}^2]r^2}{(2n+1)(2m+1)[5\mathcal{A}_{\mathrm{bas}}^2+4\mathcal{S}_{\mathrm{bas}}^2]}\sigma^2 \tag{85}$$

$$\mathrm{var}(\hat{r}_y) = \frac{12r^4}{(2n+1)(2m+1)\mathcal{A}_{\mathrm{bas}}^2}\sigma^2 \tag{86}$$

$$\mathrm{var}(\hat{v}_x) = \frac{3[\mathcal{A}_{\mathrm{bas}}^2+\mathcal{S}_{\mathrm{bas}}^2]r^2}{n(n+1)(2n+1)(2m+1)\mathcal{A}_{\mathrm{bas}}^2}\sigma^2 \tag{87}$$

$$\mathrm{var}(\hat{v}_y) = \frac{180r^4}{\delta t^2 n(n+1)(2n+1)(2m+1)[5\mathcal{A}_{\mathrm{bas}}^2+4\mathcal{S}_{\mathrm{bas}}^2]}\sigma^2. \tag{88}$$

It can be pointed out that at $\theta = 0$ the array baseline is the total length of the arrays. For $\theta = \pi/2$ the following formulations are obtained

$$\mathrm{var}(\hat{r}_x) = \frac{144r^6}{(2n+1)(2m+1)\mathcal{L}_{\mathrm{tot}}^2\mathcal{S}_{\mathrm{bas}}^2}\sigma^2 \tag{89}$$

$$\mathrm{var}(\hat{r}_y) = \frac{9\mathcal{S}_{\mathrm{bas}}^2r^2}{(2n+1)(2m+1)\mathcal{S}_{\mathrm{bas}}^2}\sigma^2 \stackrel{\gamma\neq\pi/2}{=} \frac{9r^2}{(2n+1)(2m+1)}\sigma^2 \tag{90}$$

$$\mathrm{var}(\hat{v}_x) = \frac{9[5\mathcal{L}_{\mathrm{tot}}^2-4\mathcal{S}_{\mathrm{bas}}^2]r^4}{\delta t^2 n(n+1)(2n+1)(2m+1)\mathcal{L}_{\mathrm{tot}}^2\mathcal{S}_{\mathrm{bas}}^2}\sigma^2 \tag{91}$$

$$\mathrm{var}(\hat{v}_y) = \frac{9\mathcal{S}_{\mathrm{bas}}^2r^4\sigma^2}{\delta t^2 n(n+1)(2n+1)(2m+1)\mathcal{L}_{\mathrm{tot}}^2\mathcal{S}_{\mathrm{bas}}^2} \stackrel{\gamma\neq\pi/2}{=} \frac{9r^4\sigma^2}{\delta t^2 n(n+1)(2n+1)(2m+1)\mathcal{L}_{\mathrm{tot}}^2}. \tag{92}$$

At this point Weinstein's results can be compared with ours. As they are expressed in polar coordinate, for $\theta = 0$, r_x and v_x have to be compared with $r\theta$ and v_θ, in the same way r_y and v_y have to be compared with r and v_r. Weinstein's has the following notation for its results.

M	Number of sensors ($2m+1$ in our results)
$MS(\omega)/N(\omega)$	Postbeamforming signal-to-noise ratio (SNR)
L	Length of total array
c	Sound celerity
T	Obsevation time.

With these notations, Weinstein's analytical formulations of the variances are the following:

$$\mathrm{var}(\hat{\theta}) = \frac{12c^2}{\frac{T}{\pi}\frac{M+1}{M-1}\int_0^{\infty}\frac{\omega^2[MS(\omega)/N(\omega)]^2}{1+MS(\omega)/N(\omega)}d\omega}\left(\frac{1}{L\sin(\theta)}\right)^2 \tag{93}$$

$$\mathrm{var}(\hat{r}) = \frac{720c^2}{\frac{T}{\pi}\frac{(M+1)(M^2-4)}{(M-1)^3}\int_0^{\infty}\frac{\omega^2[MS(\omega)/N(\omega)]^2}{1+MS(\omega)/N(\omega)}d\omega}\left(\frac{r}{L\sin(\theta)}\right)^4 \tag{94}$$

$$\mathrm{var}(\hat{v}_\theta)|\underset{\text{known}}{(\theta,r)} = \frac{144c^2}{\frac{T^3}{\pi}\frac{M+1}{M-1}\int_0^{\infty}\frac{\omega^2[MS(\omega)/N(\omega)]^2}{1+MS(\omega)/N(\omega)}d\omega}\left(\frac{r}{L\sin(\theta)}\right)^2 \tag{95}$$

$$\mathrm{var}(\hat{v}_r)|\underset{\text{known}}{(\theta,r)} = \frac{8640c^2}{\frac{T^3}{\pi}\frac{(M+1)(M^2-4)}{(M-1)^3}\int_0^{\infty}\frac{\omega^2[MS(\omega)/N(\omega)]^2}{1+MS(\omega)/N(\omega)}d\omega}\left(\frac{r}{L\sin(\theta)}\right)^4. \tag{96}$$

The integral term represents the wideband processing, and it must be compared with our narrowband analysis term $(1+p\rho)/p^2\rho^2$ that appears in the expression of the σ^2. In these formulas the source baseline does not appear because this is a short-time TMA. In our results, if identical arrays are concatenated one after the other, the term σ^2 is not sensitive to m. This leads to conclude that for $\theta = 0$ the elements of the states vector one wanted to compare vary exactly as expected. Only the coefficients are different due to the different approaches

$$\mathrm{var}(\hat{r}_x) \text{ and } \mathrm{var}(\widehat{r\theta}) \quad \text{vary as} \quad \frac{1}{n}\frac{1}{m}\left(\frac{r}{\mathcal{A}_{\text{bas}}}\right)^2$$

$$\mathrm{var}(\hat{r}_y) \text{ and } \mathrm{var}(\hat{r}) \quad \text{vary as} \quad \frac{1}{n}\frac{1}{m}\left(\frac{r}{\mathcal{A}_{\text{bas}}}\right)^4$$

$$\mathrm{var}(\hat{v}_x) \text{ and } \mathrm{var}(\hat{v}_\theta) \quad \text{vary as} \quad \frac{1}{n^3}\frac{1}{m}\left(\frac{r}{\mathcal{A}_{\text{bas}}}\right)^2$$

$$\mathrm{var}(\hat{v}_y) \text{ and } \mathrm{var}(\hat{v}_r) \quad \text{vary as} \quad \frac{1}{n^3}\frac{1}{m}\left(\frac{r}{\mathcal{A}_{\text{bas}}}\right)^4.$$

This means that it is equivalent to consider only the bearings measurements than utilizing the time delay (bearing) and the time delay rate (the so-called Doppler compression), to perform multiple platform TMA with short integration time (one bearing estimation).

VIII. DISCUSSION

Multiple platform TMA has been considered. The observability of the system has been carefully studied and general conclusions about single and multiple platform TMA have been obtained. Analytical approximations of source position and velocity estimation error variances have been derived for long integration time, giving thus the main parameters on which they depend. These results corroborate and extend those of [8] for long integration times, and agree with those of E. Weinstein [11] for short integration times (and $\theta = 0$).

A. Fisher Information Matrix

Being able to compute a lower bound on the variance of any unbiased estimator is extremely useful. This bound is known as the CRLB. As we wish to estimate a vector parameter ($\mathbf{X} = \{x_1,\ldots,x_n\}$), its CRLB will allow us to place a bound on the variance of each element of the vector. The CRLB of the ith element of the vector parameter is found as the $[i,i]$ element of the inverse of a matrix:

$$\mathrm{var}(\hat{x}_i) \geq [F^{-1}(\mathbf{X})]_{i,i}$$

where $F(\mathbf{X})$ is the FIM. The latter is defined by

$$[F(\mathbf{X})]_{i,j} = -E\left\{\frac{\partial^2 \ln p(\boldsymbol{\theta}\mathbf{X})}{\partial x_i \partial x_j}\right\} \qquad 1 \leq i,j \leq n.$$

In evaluating $F(\mathbf{X})$ we use the true value of $\mathbf{X}$. From (7) one can write out

$$F = \left(\frac{\partial \boldsymbol{\theta}(\mathbf{X})}{\partial \mathbf{X}}\right)^* \Sigma^{-1} \left(\frac{\partial \boldsymbol{\theta}(\mathbf{X})}{\partial \mathbf{X}}\right)$$

where $\boldsymbol{\theta}(\mathbf{X})$ is the measurement vector generated by the state vector $\mathbf{X}$, and Σ is the diagonal matrix of the inverse of the variances of the measured bearings. The partial derivative of the bearing vector with respect to the state vector is

$$\frac{\partial \boldsymbol{\theta}(\mathbf{X})}{\partial \mathbf{X}} = \begin{pmatrix} \frac{\partial \theta_1}{\partial r_x} & \frac{\partial \theta_1}{\partial r_y} & \frac{\partial \theta_1}{\partial v_x} & \frac{\partial \theta_1}{\partial v_y} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \theta_n}{\partial r_x} & \frac{\partial \theta_n}{\partial r_y} & \frac{\partial \theta_n}{\partial v_x} & \frac{\partial \theta_n}{\partial v_y} \end{pmatrix}.$$

So the $[i,j]$ element of the FIM is computed via the following sum:

$$(F)_{i,j} = \sum_{t=1}^{n} \frac{1}{\sigma_{\theta_t}^2} \frac{\partial^2 \theta_t(\mathbf{X})}{\partial X_i \partial X_j}.$$

In the multiple platform case the vector $\boldsymbol{\Theta}$ replaces $\boldsymbol{\theta}$. The former is built by replacing the scalar observation $\{\theta_t\}$ by a vectorial one:

$$\boldsymbol{\Theta}_t = \{\theta_{1,t}, \ldots, \theta_{m,t}\}.$$

The FIM is still defined by

$$F = \left(\frac{\partial \boldsymbol{\Theta}(\mathbf{X})}{\partial \mathbf{X}}\right)^* \Sigma^{-1} \left(\frac{\partial \boldsymbol{\Theta}(\mathbf{X})}{\partial \mathbf{X}}\right).$$

The expression of the $[i,j]$ element of the FIM is now the following:

$$\begin{aligned}(F)_{i,j} &= \sum_{t=1}^{n} \sum_{p=1}^{m} \frac{1}{\sigma_{\theta_{p,t}}^2} \frac{\partial^2 \theta_{p,t}(\mathbf{X})}{\partial X_i \partial X_j} \\ &= n \sum_{p=1}^{m} \sum_{t=1}^{n} \frac{1}{\sigma_{\theta_{p,t}}^2} \frac{\partial^2 \theta_{p,t}(\mathbf{X})}{\partial X_i \partial X_j} \\ &= \sum_{p=1}^{m} (F_p)_{i,j}\end{aligned}$$

where F_p is the FIM of the pth platform. This equation shows strikingly that the total FIM is the sum of the FIM of all the platforms.

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