Optimization of the Observer Motion for Bearings-Only Target Motion Analysis *

J.P. Le Cadre and H. Gauvrit, IRISA/CNRS
Campus de Beaulieu
35042 Rennes, France. (e-mail: lecadre@irisa.fr)

Abstract

This paper deals with the optimization of the receiver trajectory for target motion analysis. The observations are made of estimated bearings. The problem consists in determining the sequence of controls (e.g.: the receiver headings) which maximizes a cost functional. This cost functional is generally a functional of the FIM matrix.
The determinant of the FIM matrix has all the desirable properties, the monotonicity property excepted. The analysis is thus greatly complicated. So, a large part of this paper is centered around approximations of the FIM determinants. Using them, it is shown that, under the long-range and bounded controls hypotheses, the sequence of controls lies in the general class of bang-bang controls. These results demonstrate the interest of maneuver diversity. More generally, they provide a general framework for optimizing the observer trajectory.

1. Introduction

A fundamental problem for [1] BOT tracking is the following : if the system is observable what is the accuracy of the state estimate and how to optimize the inputs of the system? In this system approach, the observer maneuvers are the system inputs. This is a very difficult problem of control since, in the first hand, the system is only partially observed, and in the second, the cost functional is non-additive. In fact, the performance of any BOT tracking algorithm essentially depends on the system inputs. Note, furthermore, that even in a passive context the input optimization constitutes the major problem.

A classic approach consists then in considering the Fisher Information Matrix (FIM) and more precisely its determinant. The choice of the determinant functional is reasonable. This is a common cost functional in the estimation literature. It is the inverse of the square of the volume of the uncertainty ellipsoid. Furthermore, we can show that, under hypotheses reasonable in the BOT context, the maximum of det(FIM) is attained when the sphericity criterion is maximum. However, as we shall see later, the det functional does not own the monotonicty property so it is not evident that adding an optimal control for the time $t+1$ to a control sequence optimal up to time t will yield a control sequence up to time $t+1$.

This explains, for a large part, the relative complexity of this problem. We shall show that using elementary multilinear algebra accurate approximations of det(FIM) may be obtained. More specifically, we shall prove that det(FIM) may be approximated by a functional involving only the successive source bearing-rates yielding thus the general form of the optimal inputs (observer maneuvers). In particular it will be shown that, under the long-range and bounded controls hypotheses, the sequence of optimal controls lies in the general class of bang-bang controls. These results demonstrates the interest of maneuver diversity. More generally, they provide a general framework for optimizing the observer trajectory by means of feedback control.

First, approximations of det(FIM) will be derived for a constant source bearing-rate. Using the same approach, these results will be extended to the case of time-varying source bearing-rates.

2. Problem statement, an historical perspective

The performance of any TMA algorithm is dramatically related to the receiver maneuvers. Optimization of the receiver maneuvers represents the main problem in TMA. It is therefore not surprising that a great deal of work has been devoted to this subject (see [1,2]). For instance, for the localization problem, rather rough approximations of the FIM (2×2) determinant suggest to

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[1] BOT means Bearings Only Tracking

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consider the following integral cost [2]

$$\mathcal{C}_T = \int_0^T \frac{\dot{\theta}(t)}{2\sigma^2 r^2(t)} dt \ . \quad (1)$$

The problem can thus be immersed in the general framework of optimal control theory. More precisely, under the (realistic) assumption of a **constant** modulus (v) of the receiver velocity, the problem consists in determining the optimal controls (i.e. the receiver heading u) for the following problem[3] :

$$\max_u \int_0^T \frac{\dot{\theta}(t)}{r^2(t)} dt$$

$$\left| \begin{array}{ll} r_x = r\cos\theta & \dot{r}_x = v\cos u \\ r_y = r\sin\theta & \dot{r}_y = v\sin u \ . \end{array} \right. \quad (2)$$

The solution to this problem is surprisingly simple i.e. :

$$\dot{u}_* = -2\dot{\theta} \ . \quad (3)$$

Actually, the above system equation is also valid for TMA (moving source), the only change is the cost functional.A first candidate cost is obtained by considering the trace of the 4×4FIM, leading to consider the following problem :

$$\mathcal{C}_{tr}(T) = \int_0^T \frac{(1+t^2)}{r_t^2} \sin^2(\theta_t - u_t)\, dt \ ,$$

for which, a solution is [3] :

$$\dot{u}_* = -3\frac{v}{r}\frac{\cos^2(u-\theta)}{\sin(u-\theta)} \ . \quad (4)$$

However, $tr(F)$ is not a very relevant functional since the Cramér-Rao bound involves the inverse of F^{-1}. In fact, both statistical (probabilty of detection) and numerical considerations (max. of the sphericity criterion) plaid for the choice of $\det(F)$ (F denoting the FIM). The problem then becomes much more complicated since an integral approximation of the cost should only provide a very poor lower bound.
A very elegant approach [4] consists then in considering only the final cost $Q = -\log\det[F(T)]$ which implies that the Hamiltonian becomes particularly simple. The originality of this approach relies on the choice of the state vector whose components are not only (r_x, r_y) but overall the components of the FIM F. In fact, the system equation becomes :

$$F_k = \Phi F_{k-1}\Phi^* + \frac{1}{\sigma^2} g_k g_k^* \ . \quad (5)$$

[2]with the following notations : θ: bearing , $\dot{\theta}$: bearing-rate, r : range, σ^2 : variance of thr bear. estimate

[3]σ^2 is independent of t

The corresponding optimal control must be solved by means of numerical methods, **but** requires the knowlege of the source trajectory. So, in order to remedy this problem, appoximations of the FIM determinant willl be carefully considered. They will allow us to derive the general form of the optimal sequence of controls.

3. Some approximations of the FIM determinant and their consequences

Consider the case of a non maneuvering source (constant velocity vector), then the calculation of the FIM is a routine exercice yielding, under the Gaussian assumption :

$$\text{FIM} = \left(\frac{\partial\Theta(\mathbf{X})}{\partial\mathbf{X}}\right)^* \Sigma^{-1} \left(\frac{\partial\Theta(\mathbf{X})}{\partial\mathbf{X}}\right) \quad (6)$$

where $\Theta(\mathbf{X})$ is the measurement vector generated by the state vector $\mathbf{X}$ and Σ is the diagonal matrix whose diagonal terms are the inverses of the variances of the measured bearings. The partial derivative matrix of the bearing vector $\Theta(\mathbf{X})$ with respect to the state vector is directly calculated yielding :

$$\frac{\partial\Theta(\mathbf{X})}{\partial\mathbf{X}} = \begin{pmatrix} \frac{\cos\theta_1}{r_1} & -\frac{\sin\theta_1}{r_1} & \frac{\cos\theta_1}{r_1} & -\frac{\sin\theta_1}{r_1} \\ \vdots & & & \\ \frac{\cos\theta_n}{r_n} & -\frac{\sin\theta_n}{r_n} & \frac{n\cos\theta_1}{r_n} & -\frac{n\sin\theta_n}{r_n} \end{pmatrix} \quad (7)$$

where $\{\theta_i\}_{i=1}^n$ represent the source bearing at the instant i and $\{r_i\}$ the source-observer distance. It is quite remarkable that the determinant of the FIM does not depend on the reference time.

The distance will be assumed to be constant (at first). Further, we assume that the diagonal noise matrix Σ is proportional to the identity (i.e. $\Sigma = \sigma^2 Id$).

We shall denote $F_{k,4}$ the FIM corresponding to a reference time k and 4 consecutive measurements, $\theta_k, \cdots, \theta_{k+3}$. Then the FIM $F_{k,4}$ takes the following form (4 measurements) :

$$F_{k,4} = (\sigma r)^{-2} \mathcal{G}_{k,4}\mathcal{G}_{k,4}^* \ .$$

where :

$$\mathcal{G}_{k,4} = (\mathbf{G}_k, \mathbf{G}_{k+1}, \mathbf{G}_{k+2}, \mathbf{G}_{k+3})$$

and $\mathbf{G}_k$ is the gradient vector of θ_k w.r.t. $\mathbf{X}_0$, i.e. :

$$\mathbf{G}_k = (\cos\theta_k, -\sin\theta_k, k\cos\theta_k, -k\sin\theta_k)^* \ . \quad (8)$$

Assuming $\mathcal{G}_{k,4}$ invertible, we have :

$$\det(F_{k,4}) = (\sigma r)^{-8} (\det \mathcal{G}_{k,4})^2$$

it is thus sufficient to calculate $\det \mathcal{G}_{k,4}$.

The calculation of $\det(\mathcal{G}_{k,4})$ is made elsewhere [2] and provide the following second order approximation of $\det(F_{k,4})$.

Prop. 1 : *Approximating the source-observer distance as constant, then the* 2^{nd} *order approximation of* $\det F_{k,4}$ *is given by :*

$$\det F_{k,4} \stackrel{2}{=} (\sigma r)^{-8}\left(6\dot{\theta}^4 - 3\ddot{\theta}^2\right)^2 . \tag{9}$$

If a 3^{rd}-order approximation of $(\cos(\theta_{k+i}), \sin(\theta_{k+i}))$ is considered then the following approximation of the determinant of $F_{k,4}$ is obtained :

$$\det F_{k,4} \simeq (\sigma r)^{-8} \cdot \left(4\dot{\theta}^4 - 3\ddot{\theta}^2 + 2\dot{\theta}\dddot{\theta}\right)^2 . \tag{10}$$

which leads to the observability criterion of Nardone and Aidala.

Obviously, our attention is not limited to four measurements per legs. So, the previous calculations will now be extended to any number of measurements. Let ℓ be the number of measurements and consider now the (4×4) FIM $F_{k,\ell}(l \geq 4)$ defined by :

$$F_{k,\ell} = (\sigma r)^{-2}\mathcal{G}_{k,\ell}\mathcal{G}^*_{k,\ell}$$

where :

$$\mathcal{G}_{k,\ell} = (\mathbf{G}_k, \mathbf{G}_{k+1}, \cdots, \mathbf{G}_{k+\ell}) \quad \ell \geq 0 . \tag{11}$$

Note that in (7.11) the source-observer distance is again assumed to be constant. Using classical properties of multilinear algebra, namely the Cauchy-Binet formula, $\det(F_{k,\ell})$ is given by the following formula :

$$\det(F_{k,\ell}) = (\sigma r)^{-8}\sum_E [\det(\mathcal{G}_E)]^2$$

where :

$$E = \{i_1, i_2, i_3, i_4\} \ s.t. \ 1 \leq i_1 < i_2 < i_3 < i_4 \leq \ell$$

and :

$$\mathcal{G}_E = (\mathbf{C}_{i_1}, \mathbf{C}_{i_2}, \mathbf{C}_{i_3}, \mathbf{C}_{i_4}) . \tag{12}$$

In (7), $\mathbf{C}_{i_j}$ stands for the i_j -th column of the matrix $\mathcal{G}$. Considering for instance, a first order expansion of the bearings θ_{k+i} (i.e. $\theta_{k+i} \stackrel{1}{=} \theta_k + i\dot{\theta}$), the calculation of $\det(F_{k,\ell})$ is reduced to the calculation of the determinants $\det(\mathcal{G}_E)$. Now each of these determinants is the determinant of a 4×4 matrix and may be calculated by using the general calculation given in [2], yielding :

Result 1:

$$\begin{aligned} \det F_{k,4} &\stackrel{1}{=} \left(\frac{\sin\dot{\theta}}{\sigma r}\right)^8 16, \\ \det F_{k,5} &\stackrel{1}{=} \left(\frac{\sin\dot{\theta}}{\sigma r}\right)^8 32\left[18 + 16\cos\left(2\dot{\theta}\right) + \cos\left(4\dot{\theta}\right)\right], \\ &\text{etc. .} \end{aligned} \tag{13}$$

The general form of this approximation of $\det(F_{k,\ell})$ is thus :

$$\det F_{k,\ell} \stackrel{1}{=} 32\left(\frac{\sin\dot{\theta}}{\sigma r}\right)^8 P_\ell\left[\cos\left(2\dot{\theta}\right), \cdots, \cos\left(4(\ell-4)\dot{\theta}\right)\right] .$$

The practical interest of the preceding results is evident since explicit forms of the FIM determinant have been obtained. We stress that these explicit forms involve only directly observable [2] parameters. More precisely $\dot{\theta}$ may be directly estimated (i.e. without any prior about the source trajectory) and even $\dot{r}/r$ may be estimated (since $\dot{r}/r = -\ddot{\theta}/2\dot{\theta}$) from the spatio-temporal data received on the sensor array. Hence, the above results allow us to optimize the observer trajectory without any prior about the source trajectory .

Another step of approximation is obtained by considering an expansion (around 0) of the polynomial P_ℓ (see the above eq.) yielding [4]:

$$32\ P_\ell(\dot{\theta}) \simeq \alpha^{-1}[\ell^3(1+\ell)^4(\ell^2+2\ell-8)(\ell^2+2\ell-3)^2(2+l)^3]$$

so that :

$$\det F_{k,\ell} \propto \ell^{16}\left(\frac{\sin\dot{\theta}}{\sigma r}\right)^8 . \tag{14}$$

Using the previous formalism, an extension of the previous results to higher order expansions of θ_{k+i} is quite straightforward but not truly enlightening.
It is more interesting to focus our attention on the effect of observer maneuvers. The following property is an extension of the previous one to this case.

Consider that the temporal evolutions of the source bearings on two successive legs are described by the two following linear models :

$$\begin{aligned} \theta_{k+i} &\stackrel{1}{=} \theta_k + i\dot{\theta}_1 \ \text{ on the } 1^{-st} \text{ leg} \\ \theta_{k'+j} &\stackrel{1}{=} \theta_{k'} + j\dot{\theta}_2 \ \text{ on the } 2^{-nd} \text{ leg} . \end{aligned} \tag{15}$$

[4] $\alpha = 54432000$

Then the following property holds ([2]) and extend the previous results :

Prop. 2:

$$\begin{aligned}\det(\mathcal{G}_E) &= (c-b)(a-b)\sin(b_1+c_1)\sin d_1\\ &+(b-d)(a-b)\sin(b_1+d_1)\sin c_1\\ &+(c-b)(d-b)\sin(c_1-d_1)\sin b_1\ ,\end{aligned} \quad (16)$$

where b_1, c_1, d_1 *have, this time, the following meanings* :

$$\begin{aligned}&b_1=(i_2-i_1)\,\dot\theta_1, c_1=(i_3-i_2)\,\dot\theta_2, d_1=(i_4-i_2)\,\dot\theta_2\\ &a=i_1, b=i_2, c=i_3, d=i_4\\ &(i_1,i_2)\in 1^{-st}\text{ leg}\quad (i_3,i_4)\in 2^{-nd}\text{ leg}\ .\end{aligned}$$

The above property allows us to approximate $\det F_{k,\ell}$ in the case of a maneuvering source and thus to investigate the effects of the receiver maneuver. In particular, the role of the bearing-rate changes clearly appears. Indeed, since the parameters $\dot\theta_1$ and $\dot\theta_2$ are usually small, we shall examine an expansion of $\det(\mathcal{G}_E)$ w.r.t. $\dot\theta_1$ and $\dot\theta_2$ around the point $(0,0)$. Then, we obtain the following types [5] of fourth-order expansions (in $\dot\theta_1$ and $\dot\theta_2$) of $\det(\mathcal{G}_E)$:

$$\begin{aligned}(\det\mathcal{G}_E)^2 &\simeq K\left(x^2y^2-2xy^3+y^4\right),\\ &\text{or}:\ K(x^2y^2-2x^3y+y^4),\\ &\text{or}:\ K(x^2y^2-2x^3y+x^4)\ ,\end{aligned} \quad (17)$$

with :

$$\begin{aligned}K &\triangleq (b-a)^2(c-b)^2(c-d)^2(d-b)^2\\ x &\triangleq \dot\theta_1\ ,\ y\triangleq\dot\theta_2\ .\end{aligned} \quad (18)$$

This result is quite fundamental for TMA and will be clarified by a geometric interpretation. Moreover, a general approximation of $\det F_{k,\ell_1,\ell_2}$ [6] is :

$$\det F_{k,\ell_1,\ell_2}\simeq\frac{1}{(\sigma r)^8}\Big[\sum_{i=1}^{5}P_i(\ell_1,\ell_2)y^{5-i}x^{i-1}\Big]\ , \quad (19)$$

where the polynomials $\{P_i(\ell_1,\ell_2)\}_{i=1}^5$ are detailed in the Appendix C. From (5.15) we note that the maximum value of $\det F_{k,\ell_1,\ell_2}$ is proportional to $\ell^{12}\dot\theta^4$ ($\ell_1\simeq\ell_2$, $\dot\theta_1\simeq-\dot\theta_2$). In fact, denoting $F_\ell(x)$ the FIM associated with a constant bearing-rate x and $F_{\ell/2,\ell/2}(x,-x)$ the FIM associated with a two-leg observer trajectory (leg 1: $\ell/2$ meas., bear. rate x; leg 2: $\ell/2$ meas., bear. rate $-x$), the previous results yield [3]:

$$\begin{aligned}\frac{\det(F_\ell(x))}{\det(F_{\ell/2,\ell/2}(x,-x)} &\simeq \frac{1}{134}\ell^4x^4\\ &\simeq \frac{1}{134}(\Delta x)^4\ ,\end{aligned} \quad (20)$$

[5] the type of the expansion only depends on the relative values of i_1, i_2, i_3, i_4

[6] ℓ_i measurements asociated with $\dot\theta_i$, $i=1,\ 2$

where Δx denotes the total bearing variation (i.e. $\Delta x=\ell x$). For usual scenarios, Δx is little in regard to 1 and ,therefore, the increase in the FIM determinant gained by optimized observer maneuvers may be rather impressive. Further, note that this gain is proportional to $(\Delta x)^4$.

The previous calculations may be easily extended to the three leg case (i.e. :$\{x,y,z\}$). As previously, $\det F_{k,\ell_1,\ell_2,\ell_3}$ is an homogeneous polynomial in (x,y,z) i.e. :

$$\det F_{\ell_1,\ell_2,\ell_3}\simeq\frac{1}{(\sigma r)^8}\Big[\sum_{i,j,k}P_{i,j,k}(\ell_1,\ell_2,\ell_3)x^iy^jz^k\Big], \quad (21)$$

with :

$$0\le\{i,j,k\}\le 4\ \text{ and }: i+j+k=4.$$

For the sake of brevity, the analytical expressions of the $P_{i,j,k}$ are not detailed (see [3]). Practically, for equal legs (i.e : $\ell_1=\ell_2=\ell_3=\ell$), the maximum value ($x=-y=z$) of $\det F_{k,\ell_1,\ell_2,\ell_3}$ is approximately $45\ell^{12}\dot\theta^4$.

4. Geometric interpretations of the properties of the FIM determinant

The preceding results advocate for a more systematic and geometric interpretation. Thus, we shall consider the determinant $\det\mathcal{G}_E$ where as previously, $E=\{i_1,i_2,i_3,i_4\}$ and $i_1<i_2<i_3<i_4$.

$$\begin{aligned}\det\mathcal{G}_E &= \det\left(\mathbf{G}_{i_1},\cdots,\mathbf{G}_{i_4}\right)\\ &= \det\left(R_1^{i_1}\mathbf{G}_k, R_1^{i_2}\mathbf{G}_k, R_1^{i_3}\mathbf{G}_k, R_1^{i_4}\mathbf{G}_k\right)\end{aligned}$$

where :

$$R_1\triangleq\begin{pmatrix}R_0 & 0\\ R_0 & R_0\end{pmatrix}\text{ and }R_0\triangleq\begin{pmatrix}\cos\dot\theta & \sin\dot\theta\\ -\sin\dot\theta & \cos\dot\theta\end{pmatrix}. \quad (22)$$

In the same spirit, the vector $\mathbf{G}_k$ may be written as :

$$\mathbf{G}_k=S_1^k\mathbf{E}$$

where :

$$S_1\triangleq\begin{pmatrix}S_0 & 0\\ S_0 & S_0\end{pmatrix}\text{ and }S_0\triangleq\begin{pmatrix}\cos\theta & +\sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}.$$

$$\theta\triangleq\theta_k/k,\ \ \mathbf{E}=(1,0,0,0)^* \quad (23)$$

Now the following property is instrumental : the matrices R_0 and S_0 commute. The matrices R_1 and S_1

then also commute and using this property $\det \mathcal{G}_E$ then becomes :

$$\det \mathcal{G}_E = \det \left(R_1^{i_1}\mathbf{E}, R_1^{i_2}\mathbf{E}, R_1^{i_3}\mathbf{E}, R_1^{i_4}\mathbf{E}\right) . \quad (24)$$

The following property has thus been proved : $\det \mathcal{G}_E$ is independent of k and θ_k. A further step yields :

$$\begin{aligned} \det \mathcal{G}_E &= \det \left(\mathbf{E}, R_1^{i_2-i_1}\mathbf{E}, R_1^{i_3-i_1}\mathbf{E}, R_1^{i_4-i_1}\mathbf{E}\right) \quad (25) \\ &= \det \left(\begin{array}{c|c|c|c} \mathbf{e} & R_0^{i_2'}\mathbf{e} & R_0^{i_3'}\mathbf{e} & R_0^{i_4'}\mathbf{e} \\ 0 & i_2' R_0^{i_2'}\mathbf{e} & i_3' R_0^{i_3'}\mathbf{e} & i_4' R_0^{i_4'}\mathbf{e} \end{array} \right) \end{aligned}$$

where :

$$\mathbf{e} \triangleq (1,0)^*, i_k' \triangleq i_k - i_1 \quad k = 2,3,4. \quad (26)$$

The calculation of $\det \mathcal{G}_E$ may then be achieved by recalling the expression of the minimal polynomial of R_0 :

$$R_0^2 = 2\cos\dot{\theta}\, R_0 - Id_2 \quad (27)$$

so that the minimal polynomial of R_1 is $[(x-\lambda)(x-\bar{\lambda})]^2$, $\lambda = \exp(i\dot{\theta})$.

The determinant $\det \mathcal{G}_E$ can thus be calculated for any subset E, yielding the general form of $\det F_{k,l}$. Further note that the vector sequence $\{\mathbf{E}, R_1^{i_2-i_1}\mathbf{E}, R_1^{i_3-i_1}\mathbf{E}, R_1^{i_4-i_1}\mathbf{E}\}$ is a part of a Krylov sequence.

The previous calculations provide interesting insights about the optimization of the observer maneuvers. Consider for instance the following determinant:

$$f(y) = \det(\mathbf{E}, R_{1,x}\mathbf{E}, R_{1,x}^2\mathbf{E}, R_{1,x}^2 R_{1,y}\mathbf{E}), \quad (28)$$

where:
$x = \dot{\theta}_1$, $y = \dot{\theta}_2$.

Let us now calculate the partial derivative $\partial f/\partial y(x)$, we obtain:

$$\frac{\partial f}{\partial y}(x) = \det(\mathbf{E}, R_{1,x}\mathbf{E}, R_{1,x}^2\mathbf{E}, R_{1,x}^2 S_{1,x}\mathbf{E}), \quad (29)$$

where $S_{1,x} = (\frac{\partial}{\partial y} R_{1,y})_{(y=x)}$, or, explicitely:

$$S_{1,x} = \begin{pmatrix} S_{0,x} & 0 \\ S_{0,x} & S_{0,x} \end{pmatrix}, \quad (30)$$

with:

$$S_{0,x} = \begin{pmatrix} -\sin x & -\cos x \\ \cos x & -\sin x \end{pmatrix}.$$

The following property is then easily proved :

$$S_{1,x} R_{1,x}^2 = R_{1,x}^2 S_{1,x} = S_{1,3x}.$$

so that:

$$\begin{aligned} \frac{\partial f}{\partial y}(x) &= \det(\mathbf{E}, R_{1,x}\mathbf{E}, R_{1,x}^2\mathbf{E}, S_{1,3x}\mathbf{E}) \quad (31) \\ &= 2\sin(2x) - \frac{\sin(4x)}{2}, \end{aligned}$$

and, therefore:

$$\begin{aligned} f(y) &\approx f(x) + (y-x)\frac{\partial f}{\partial y}(x) \quad (32) \\ &\approx c\,x^4 + (y-x)[2x + \frac{8}{3}x^3 + o(x^4)] \\ &\approx 2x(y-x). \end{aligned}$$

At this point, it is worth noting that the vector $S_{1,3x}\mathbf{E} = (-\sin 3x, \cos 3x, -3\sin 3x, 3\cos 3x)^*$ is approximately orthogonal to the vectors $\{\mathbf{E}, R_{1,x}\mathbf{E}, R_{1,x}^2\mathbf{E}\}$. This fact is typical of a four-dimensional state vector and corresponds to a diversity in maneuvers. The effect of an observer maneuver corresponds to a change from x $(\dot{\theta}_1)$ to y $(\dot{\theta}_2)$.

From the above equation it is clear that the increase of $\det(F_{x,y})$ is maximized when the term $(2x(x-y))^2$ is maximized. Since $\dot{\theta}$ is bounded, an optimal sequence of controls is necessarily a bang-bang one or, more precisely, a sequence of the form $\{\dot{\theta}_{max}, -\dot{\theta}_{max}, \dot{\theta}_{max}, \cdots\}$.

It remains to determine the optimal number of control commutations (from $\dot{\theta}_{max}$ to $-\dot{\theta}_{max}$) as well as their locations. Using the previous results, the problem may be formulated as follows. Consider a multilegs observer trajectory, then the problem consists in maximizing $\det(F_{l_1,l_2,l_3,\cdots})$ (l_i: length of the i-th leg) given below :

$$\begin{aligned} &(\sigma r)^8 \det(F_{l_1,l_2,l_3}) \approx P(l_1,\dot{\theta}_{max})\dot{\theta}^8 \\ &+P(l_1,l_2,\dot{\theta}_{max})(2\dot{\theta}_{max})^4 + P(l_1,l_3,\dot{\theta}_{max})(2\dot{\theta}_{max})^4 + \cdots . \end{aligned} \quad (33)$$

The instants of commutation may then be determined by maximizing the previous expression. The polynomials $P(l_i, l_j, \dot{\theta})$ may be obtained by means of the Cayley-Hamilton theorem.

This optimization problem may be connected with the Hadamard inequality. Indeed, let us recall a classical formulation of the Hadamard inequality:

$$\left| \begin{array}{l} \mathrm{Max}_{\{\mathbf{V}_i\}} \det(\mathbf{V}_1, \cdots, \mathbf{V}_n) \\ \text{with constraints} \quad \|\mathbf{V}_1\| = \varepsilon_1, \cdots, \|\mathbf{V}_n\| = \varepsilon_n \\ \dim(\mathbf{V}) = n. \end{array} \right. \quad (34)$$

A solution to the above problem is easily obtained by means of differential calculus (Lagrange multipliers) [3] yielding: the vectors solving the above problem are mutually orthogonal vectors satisfying the constraints. For our application, angle constraints must be added, leading to replace the above problem by [7]

$$\left| \begin{array}{l} \mathrm{Max}_{\{\mathbf{V}_i\}} \det(\mathbf{V}_1, \cdots, \mathbf{V}_n) \\ \text{with constraints} \quad \|\mathbf{V}_1\| = \varepsilon_1, \cdots, \|\mathbf{V}_n\| = \varepsilon_n \\ \cos(\mathbf{V}_1, \mathbf{V}_2) = \cdots = \cos(\mathbf{V}_{n-1}, \mathbf{V}_n) \geq \eta. \end{array} \right. \quad (35)$$

[7] $\dim(\mathbf{V}) = n$

A solution to this optimization is obtained by means of Kuhn-Tucker multipliers giving [3]: the solution vectors are the vectors maximizing the $n-1$ angles $\langle \mathbf{V}_i, \mathbf{V}_{i+1} \rangle$.

5. The effects of range variations:

Up to now, the effects of range variations have not been considered. However, the analysis is greatly simplified if we remark that the effects of range and bearing-rate variations are *geometrically* uncoupled. This follows easily by considering $\det(\mathcal{G}_E)$. Including the range, $\det(\mathcal{G}_E)$ becomes :

$$\begin{aligned} \det \mathcal{G}_E &= \det\left(\frac{1}{r_{i_1}} R_1^{i_1}\mathbf{E}, \cdots, \frac{1}{r_{i_4}} R_1^{i_4}\mathbf{E}\right), \\ &= \frac{1}{r_{i_1}} \cdots \frac{1}{r_{i_4}} \det\left(R_1^{i_1}\mathbf{E}, \cdots, R_1^{i_4}\mathbf{E}\right) \quad (36) \end{aligned}$$

From the above equality, we note that the effects of range and bearing-rate variations are uncoupled. More precise calculations can be achieved if we consider (for instance) a first order expansion of the source-receiver distance (i.e. $r_{k+i} \overset{1}{=} r_k + i\dot{r}$), then the general formula is obtained ($\dot{\theta}$ constant) :

$$\det F_{k,\ell} \simeq \frac{32\,(\sin\dot{\theta})^8}{\sigma^8 (r_0+\dot{r})^2 \cdots (r_0+6\dot{r})^2} . P_\ell\left(\dot{\theta}, \dot{r}\right)$$

where :

$$P_\ell\left(\dot{\theta}, \dot{r}\right) = P_\ell\left(\dot{\theta}\right) \cdot Q_\ell\left(\dot{r}\right)$$

with :

$$\left| \begin{array}{l} P_\ell\left(\dot{\theta}\right) = P_\ell\left[\cos\left(2\dot{\theta}\right), \cdots, \cos\left(4(\ell-4)\dot{\theta}\right)\right] \\ Q_\ell\left(\dot{r}\right) = r_0^4 + \eta_1 r_0^3 \dot{r} + \eta_2 \dot{r}_0^2 \dot{r}^2 + \eta_3 r_0 \dot{r}^3 + \eta_4 \dot{r}^4 \\ r_0 \triangleq r_k . \end{array} \right. \quad (37)$$

We thus see that the polynomial $P_\ell\left(\dot{\theta}, \dot{r}\right)$ is actually the product of the two polynomials $P_\ell\left(\dot{\theta}\right)$ and $Q_\ell\left(\dot{r}\right)$. This factorization is quite general and is simply due to the basic properties of the determinant. Therefore, the effects of range variations are easily taken into account. More precisely, it is sufficient to replace the matrix R_1 by the matrix $(1+\dot{r}/r)^{-1} R_1$ in the previous (geometric) analysis. Moreover, both $\dot{\theta}$ and $\ddot{\theta}$ are directly observable, i.e. may be estimated from the available data (i.e. the bearings $\hat{\theta}_k$).

Conclusion

Optimization of the observer maneuvers has been considered along this paper. This problem is not relevant of classical optimal control. Using basic tools of multilinear algebra, it has been proved that this functional may be accurately approximated by a functional involving only the successive source bearing-range rates. In particular, it has been shown that under the long-range and bounded controls hypotheses, the sequence of optimal control lies in the general class of bang-bang controls[8]. They demonstrate the interest of maneuver diversity. More generally, they provide us with a simple and feasible approach for optimizing the receiver trajectory.

References

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