PHASE-ONLY MULTIDIMENSIONAL SPATIO-TEMPORAL ANALYSIS FOR MOVING SOURCES

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Abstract - The estimation of source motion parameters takes a considerable importance for many areas. Especially for underwater acoustic applications where a large amount of data is available. This paper deals with the estimation of source motion parameters directly from the spatio-temporal data. A general frame for a separate estimation of the spatial frequencies and spatial freq-rate is presented, avoiding thus the dramatic interference problems. One basic idea consists in using basically the relative phase shifts without any consideration of source powers. The other consists in considering interpolation as a basic tool for spatio-temporal analysis.

1 Introduction

The performance of any array processing can be seriously affected by the source motions [1]. The integration time, which is a paramater of fundamental importance for the performance optimization of any system, is itself limited by the basic unstationary nature of the received signals. However, for numerous applications (e.g. passive sonar) large integration time are required especially for improving the detection of weak (moving) sources. It is then necessary to incorporate the (unknown) source motion models into the source ones and to define spatio-temporal analysis using basically this extended source model.

Classical array processing are based upon a short time (spatial) analysis itself followed by the classical steps of source tracking, target motion analysis (TMA) and data association. All the array processing methods consider an instantaneous spatial contrast functional. However, a large amount of spatio-temporal data is available. It seems, then, possible to separate, detect and track the sources by using their respective trajectories. In this spatio-temporal approach, the concept of source trajectory replaces the instantaneous bearing's one.

Thus, the estimation of source trajectory parameters plays the central role. As it will be shown, the spatio-temporal data may be described by means of a 2D (multiscale) state-space model. But actually, the mono-dimensional spatio-temporal data received on the array "sees" the same spatio-temporal process, but at vavrious scales. In order to separate the estimation of spatial frequencies and spatial freq. -rates, phase-only multidimensional spatio-temporal analysis will be considered. For that purpose, the interpolation of rational function will be instrumental.

2 A spatio-temporal model of the data and consequences

The array is assumed to be linear and constituted of p equispaced sensors. The angle θ represents the bearing while the parameter λ is the wavelength. Then, the spatial frequency k is defined by $k = \cos\theta/\lambda$. For a moving source (of inst. spatial freq. k_t), the following temporal model of k_t may be considered :

$$k_t = k_0 + t\dot{k} \quad (\Delta t = 1) \tag{1}$$

The above linear model with constant parameter $\dot{k}$ is valid for a moving source evolving at medium or long range source. For close sources, it can be replaced by a local approximation, i.e. :

$$k_t = k_{t-1} + \dot{k}_{t-1} \tag{2}$$

The problem we are now dealing with is the following : how to estimate (separately) the parameters k_0 and $\dot{k}$. Note that an answer to this question will allow us to separate and detect the sources by taking into account their whole respective trajectories.

Assume now and for the rest of the paper that the array is linear with equispaced sensors (d : intersensor distance). Let $\mathbf{V}_t$ be the consecutive array snapshots at a certain frequency (omitted). Then the estimated CSM $\hat{R}_t$ is defined as follows :

$$\hat{R}_t \triangleq \mathbf{V}_t \mathbf{V}_t^* \tag{3}$$

($*$: transposition and complex conjugation).

Obviously the rank of $\hat{R}_t$ is one. Generally, this is not a Tœplitz matrix so $\hat{R}_t$ is orthogonally projected on the Tœplitz subspace (this matrix is denoted $\hat{R}_{t,T}$). Then, the following spatio-temporal data are defined :

$$1^{st} - \text{row} \left(\hat{R}_{t,T}\right) \triangleq (\hat{r}_0(t), \cdots, \hat{r}_{p-1}(t))$$

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with :

$$\hat{r}_i(t) = \frac{1}{p-1} tr\left(\hat{R}_t Z^i\right) \quad 0 \le i \le p-1 \qquad (4)$$

the symbol "tr" denoting the trace of a matrix and Z the shift matrix.

At this point, it is worth noting the exact expression of $r_m(t)$ which stands as :

$$r_m(t) = \sum_{j=1}^{s} \sigma_j^2 \exp\left[2i\pi dm\left(k_{j,0} + t\dot{k}_j\right)\right] + b^2\delta_m \quad 0 \le m \le p-1 \qquad (5)$$

(s uncorrelated sources).

By considering (5), we see that the spatial freq-rate $\dot{k}$ plays the role of the frequency is classical time series analysis. The initial freq. $k_{j,0}$ can be considered as an initial phase. It is now worth to consider a temporal filtering applied to the data $\hat{r}_m(t)$. Since the parameters $\dot{k}_j$ are usually very little, the benefits of low-pass filtering are evident. Designing $\{y(t,m)\}$ the filtered sequences (I.e. $y(t,m) = \mathcal{F}_m(\hat{r}_m(t))$), then the 2D array of spatio-temporal data can be described by the following 2D-state space model [2].

$$\left|\begin{array}{lcll} \mathbf{X}(t+1,m) & = & F_1^m \mathbf{X}(t,m) & \\ \mathbf{X}(t,m+1) & = & F_0 F_1^t \mathbf{X}(t,m) & 1 \le m \le p-1 \\ y(t,m) & = & h^* \mathbf{X}(t,m) + w(t,m) & t_0 \le t \le t_e \end{array}\right.$$

with :

$$\begin{array}{lcl} F_1 & = & \text{diag}\left(\exp\left(2i\pi d\dot{k}_1\right), \cdots, \exp\left(2i\pi d\dot{k}_s\right)\right) \\ F_0 & = & \text{diag}\left(\exp\left(2i\pi dk_{1,0}\right), \cdots, \exp\left(2i\pi dk_{s,0}\right)\right) \\ h^* & = & (1, 1, \cdots, 1) \quad i^2 = -1 \end{array}$$

and :

$$\begin{array}{l} w(t,m) = \mathcal{F}_m\left(w_0(t,m)\right), \mathbb{E}\left(w_0(t,m)\right) = 0 \\ \text{cov}\left(w_0(t,m_1), w_0(t,m_2)\right) = \frac{tr\left(R_t^{\downarrow m_1} R_t^{\downarrow m_2}\right)}{(p-m_1)(p-m_2)} \\ R_t^{\downarrow m_1} \triangleq Z^{m_1} R_t, R_t^{\downarrow m_2} \triangleq \left(Z^{m_2}\right)^t R_t \end{array} \qquad (6)$$

Let Y be the matrix whose (t,m) element is $y(t,m)$ then an equivalent description is (with the notations of [3]) :

$$Y = (\mathbf{S}, \mathbf{S}_2, \cdots, \mathbf{S}_s) \;\; \text{diag}\left(\sigma_i^2\right) (\mathbf{T}_1, \cdots, \mathbf{T}_s)^t + W$$

with : $W_{(t,m)} = w(t,m)$

$$\begin{array}{lcl} \mathbf{S}_j & = & \mathcal{P}\left(z_{j,0}\right) \triangleq \left(1, z_{j,0}, \cdots, z_{j,0}^{p-1}\right)^t \\ z_{j,0} & = & \exp\left(2i\pi dk_{j,0}\right) \end{array}$$

and :

$$\begin{array}{lcl} \mathbf{T}_j & = & \mathcal{P}'\left(\dot{z}_j\right) \triangleq \left(1, \dot{z}_j, \cdots, \dot{z}_j^T\right)^T \\ z_j' & = & \exp\left(2i\pi \dot{k}_j\right) \end{array} \qquad (7)$$

Using the "vec" operator an equivalent form is also :

$$\mathbf{Y} = \text{vec}(Y) = F\mathbf{S} + \mathbf{W}, \text{ cov}(\mathbf{W} = W)$$

with :

$$\mathbf{S} = \left(\sigma_1^2, \cdots \sigma_s^2\right)^t$$

and :

$$F = \left(\mathbf{S}_1 \otimes \mathbf{T}_1, \mathbf{S}_2 \otimes \mathbf{T}_2, \cdots, \mathbf{S}_s \otimes \mathbf{T}_s\right) \qquad (8)$$

($\otimes$: denotes the Kronecker product).

As a direct consequence, the columns of the matrix F span the signal subspace. It is then straightforward to show that the maximum likelihood estimation of the parameters $\left(k_0, \dot{k}\right)$ reverts to consider the following problem [3] :

$$\left(\hat{k}_{0,i}, \hat{\dot{k}}_i\right)_{i=1}^{s} = \arg\max tr\left[\mathbf{Y}^* W^{-1} F\left(F^* W^{-1} F\right)^{-1} F^* W^{-1}\right. \qquad (9)$$

However, a direct approach of the associated maximization problem is hopeless since it requires the use of iterative algorithms. Then, Clark and Scharf [3] consider an invertible function φ with domain Ξ and range Δ such that if the set of tuples $\xi = \left\{k_{i,0}, \dot{k}_i\right\}_{i=1}^{s}$ is an element of Ξ and $[\mathbf{a}, \mathbf{b}] = \varphi(\xi)$, then :

$$A\left(k_i\right) = \mathbf{a}^t J_{s+1} \mathcal{P}_p\left(k_i\right) = 0$$

and

$$B\left(k_i\right) = \mathbf{b}^t J_s \mathcal{P}_{p-1}\left(k_i\right) = \dot{k}_i \qquad (10)$$

where J is the exchange matrix i.e. $J = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix}$

$$\mathbf{a} = \left(a_0, a_1, \cdots, a_s\right)^t, \mathbf{b} = \left(b_0, b_1, \cdots, b_{s-1}\right)^t$$

Actually, the vectors $\mathbf{a}$ and $\mathbf{b}$ are the coefficient vectors of the two polynomial $A(z)$ and $B(z)$ with :

$$\begin{array}{lcl} A(z) & = & \prod_{i=1}^{s}\left(z - k_i\right) \\ B(z) & = & \sum_{i=1}^{s} \dot{k}_i \prod_{j=1, \neq i}^{s} \left(\frac{z - k_j}{k_i - k_j}\right) \end{array} \qquad (11)$$

($B(z)$ is a Lagrange interpolating polynomial).

Then [3], the initial problem (9) reverts in considering the following one :

$$\min_{(\mathbf{a},\mathbf{b})} \mathbf{Y} N^* \left(N W N^*\right)^+ N \mathbf{Y}$$

with :

$$N = \left(\frac{Id \otimes A}{(Id|0) \otimes B - (0|Id) \otimes \bar{I}}\right) \qquad (12)$$

(A and B determined by **a** and **b** [3]).

Even if (12) represents a considerable and promising simplification, it still requires a heavy computation effort. Therefore, another approach will now be considered.

3 Multi-dimensional estimation of the spatial freq-rates

We shall now deal with a coherent approach to the multidimensional estimation of the $\{k_j\}_{j=1}^{s}$. The proposed methods rely heavily on interpolation.

Using (6) the following interpolation formula holds for the state of the spatio-temporal sequence (STS for the sequel) $\{y(t,m)\}_t$ (m fixed) :

$$\mathbf{X}\left(t+\frac{m_0}{m},m\right)=F_1^{m_0}\mathbf{X}(t,m) \tag{13}$$

Therefore, the interpolated data $\tilde{z}(t,m)$ corresponding to the state-space model outputs (6) without noise, satisfy :

$$\tilde{z}\left(t+\frac{jm_0}{m},m\right)=h^*F_1^{jm_0}\mathbf{X}(t,m)$$

Consequently, thank to the Cayley-Hamilton theorem, there exists coefficients $a_j(t,m_0)$ s.t. :

$$\begin{aligned}\tilde{z}(t,m) &= \sum_{j=1}^{s}a_j(t,m) \\ &\quad .\ \tilde{z}\left(t-\frac{jm_0}{m},m\right) \\ &\quad 1\le m\le p-1\end{aligned} \tag{14}$$

Stress that the above relation is valid (neglecting interpolation errors) whatever the spatial index m with the unique set of coefficients $\{a_j(t,m_0)\}$. The interpolated data $\{\tilde{y}(t,m)\}$ (the interpolation factor is defined by (13), (14)) can be described by the following equation :

$$\begin{aligned}y(t,m_0) &= h_{t,m_0}^{*}A_t+w(t,m_0)\\ y(t,m_0+1) &= \tilde{h}_{t,m_0+1}^{*}A_t+\tilde{w}(t,m_0+1)\\ &\ \vdots\\ m_0\le m\le m_1\end{aligned}$$

with :

$$\begin{aligned}A_t &= (a_1(t,m_0),\cdots,a_s(t,m_0))^t\\ h_{t,m_0}^{*} &= (y(t-1,m_0),\cdots,y(t-s,m_0))\\ \tilde{h}_{t,m_0+1}^{*} &= \left(\tilde{y}\left(t-\frac{m_0}{m_0+1},m_0\right),\cdots,\tilde{y}\left(t-\frac{m_0 s}{m_0+1},m_0\right)\right)\\ &\ \vdots\end{aligned} \tag{15}$$

or in a more condensed form :

$$\mathbf{Y}_t=\tilde{\mathcal{H}}_tA_t+W_t$$

(the definition of $\mathbf{Y}_t,\mathcal{H}_t,W_t$ is direct from (15)).

A least-square solution of (15) is [12] :

$$\hat{A}_t=\left(\tilde{\mathcal{H}}_{t_0,t_e}^{*}\tilde{\mathcal{H}}_{t_0,t_e}\right)^{-1}\tilde{\mathcal{H}}_{t_0,t_e}\mathcal{Y}_t \tag{16}$$

Since the interpolation of the spatio-temporal data $y(t,m)$ represents an instrumental tool, we shall now consider a view of the interpolation problem.

4 Interpolation and identification

For this section, the notations are those of [4]. More precisely, let $\mathcal{L}_\infty$ be the space of essentially bounded functions on the unit circle and for $\hat{g}\in\mathcal{L}_\infty$ define :

$$\|\hat{g}\|_\infty=\operatorname{ess\,sup}\left\{\left|\hat{f}(z)\right|\ :\ (z)=1\right\}$$

Furthermore, let $\mathcal{H}_\infty\subset\mathcal{L}_\infty$ denote the Hardy space of bounded analytic functions in $\mathcal{D}(0,1)$. A bounded set of $\mathcal{H}_\infty$ is $\mathcal{B}_\rho,M$ defined by :

$$\begin{aligned}\mathcal{B}_{\rho,M}=\{\hat{f}\in\mathcal{H}_\infty=\hat{f}\ :\\ \text{analytic in }\mathcal{D}_\rho\text{ and}\\ \left|\hat{f}(z)\right|\le M,\forall_z\in\mathcal{D}_\rho\}\end{aligned}$$

The problem of identification in $\mathcal{H}_\infty$ assumes that the true unknown system to be identified is a stable, linear system with transfer function $\hat{h}\in\mathcal{B}_{\rho,M}$ [4], and that experimental data is given by a finite number of noisy measurements.

Let us recall now the Pick's interpolation theorem [4].

Theorem :

Let $\{z_i\}_{i=1}^n$ and $\{w_i\}_{i=1}^n$ be two sequences of complex numbers. There exists $\hat{f}\in\mathcal{B}$ s.t. $\hat{f}(z_i)=w_i=(i=1,2,\cdots,n)$ if f the hermitian (Pick) matrix :

$$(P_n)_{i,k}=\left(\frac{1-\bar{w}_iw_k}{1-\bar{z}_iz_k}\right) \tag{17}$$

of size $x\times n$ is non-negative definite. If P_n is non-negative definite, a rational interpolating function with degree at most n will exist.

Further, the following error bound is true [4].

Lemma :

Suppose $\hat{h}$ and $\hat{h}_{id}$ are both in $\mathcal{B}_{\rho,M}$ and satisfy the interpolation condition

$$\left(\hat{h}_{id}(z_k)=h(z_k),z_k=\exp(2i\pi k\pi/n)\right)$$

then :

$$\left\|\hat{h} - \hat{h}_{id}\right\| \leq 4M\rho^{-n}$$

The interpolation procedure is then based on Pick's algorithm [5].

Pick's algorithm : Let $(z_1, z_2, \cdots, z_n)$ be distinct points and $(w_1, w_2, \cdots, w_n)$ be complex numbers s.t. Pick's matrix is non-negative definite.

Step 1 :

Compute the complex numbers $w_{k,i} i = k, \cdots, n$ and $k = 2, \cdots, n$ defined by :

$$w_{k,i} = \frac{(1 - \bar{z}_{k-1} z_i)(w_{k-1,i} - w_{k-1,k-1})}{(z_i - z_{k-1})(1 - \bar{w}_{k-1,k-1} w_{k-1,i})}$$

by using initial values $w_{1,i} = w_i \quad i = 1, \cdots, n$.

Step : 2

Compute :

$$f_k(z) = \frac{w_{n-k,n-k}(1 - \bar{z}_{n-k} z) + (z - z_{n-k}) f_{k-1}(z)}{(1 - \bar{z}_{n-k} z) + \bar{w}_{n-k,n-k}(z - z_{n-k}) f_{k-1}(z)}$$

for $k = 1, \cdots, n-1$ by using the initial function :

$$f_0(z) \triangleq w_{n,n}$$

Finally, set $f(z) = f_{n-1}(z)$. Then $f(z)$ is a required function in $H_1^\infty(\mathcal{B}_{1,1})$.

5 An extension to spatial temporal analysis

The main limitation of the previous methods comes from the estimation noise $w(t,m)$ whose covariances are given in eq. 6. For the two source case, direct calculation yield [2]

$$\begin{aligned}&\lim_{p\to\infty} \operatorname{cov}\left(w_0(t,m_1), w_0(t,m_2)\right) = \\ &\rho_1^2 \exp\left(-2i\pi d(m_2 - m_1) k_{1,t}\right) \\ &+\rho_2^2 \exp\left(-2i\pi d(m_2 - m_1) k_{2,t}\right)\end{aligned}$$

$$\left(\rho_1 = \sigma_1^2/b^2, \rho_2 = \sigma_2^2/b^2\right)$$

Consequently, this is the estimation noise which mainly affect the performance and it is highly spatially correlated. In order to reduce its effects, consider now a linear combination of the interpolated data $\tilde{Y}(t,m)$ (eq. 15), then the following equation is directly deduced from (15) :

$$D_{k_0}^* \mathbf{Y}_t = \sum_{j=1}^{s} a_j(t) \mathbf{D}_{k_0}^* \tilde{\mathbf{Y}}_{t-j} + \mathbf{W}_t$$

with :

$$\begin{aligned}\mathbf{Y}_t &\triangleq (y(t,m_0), y(t,m_0+1), \cdots, y(t,m))^t \\ \mathbf{D}_{k_0} &: \text{steering associated with } k_0 \\ \mathbf{W}_t &= \mathbf{D}_{k_0}^* \mathbf{W}_t, \mathbf{W}_t = (w(t,m_0), \cdots, \tilde{w}(t,m))^t\end{aligned}$$

In order to detect a weak moving source, it is worthwhile to consider a set of beams $(k_0, k_1, \cdots k_\ell)$ whose spatial frequencies k_i are chosen in order to "isolate" a spatio-temporal sector. Then, the beam outputs can be modelled by using a unique AR model, i.e. :

$$\begin{aligned}\mathbf{D}_{k_0}^* \mathbf{Y}_t &= \sum_{j=1}^{s} a_j(t) \mathbf{D}_{k_0}^* \tilde{\mathbf{Y}}_{t-j} + \mathbf{W}_{0,t} \\ &\vdots \qquad \vdots \\ \mathbf{D}_{k_\ell}^* \mathbf{Y}_t &= \sum_{j=1}^{s} a_j(t) \mathbf{D}_{k_\ell}^* \tilde{\mathbf{Y}}_{t-j} + \mathbf{W}_{\ell,t}\end{aligned}$$

6 Conclusion

A general frame for a separate estimation of the spatial freq. and spatial freq. rates has been presented. Instrumental tools are the use of relative (temporal) phase shifts and interpolation.

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