Temporal integration for array processing

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Abstract

This paper deals with temporal integration for array processing in the passive array context. Its main objective is to improve the detection of weak and slowly varying sources, in a first time by optimization of classical methods and in a second one by developing new array processing methods.

1 Introduction

The fundamental aims of passive array processing (especially in the sonar context) consist in detection, localization and tracking of (weak) moving sources. The problem is complicated by the presence of strong sources, propagation effects, etc.
Classical array processing (beamforming and others) are based upon a short time analysis, itself followed by a post-processing including source tracking and T.M.A. (target motion analysis). This kind of methods present some advantages ; however, it is fundamentally irrelevant to the detection of (slowly) varying sources.
The recent developments of array processing methods, e.g. high resolution methods [1], have demonstrated the benefits of temporal integration. Nevertheless, these benefits are limited by the intrinsic non-stationary nature of the received signals, mainly due to source motion and propagation effects. Thus, a too large integration time can even produce dramatic effects in terms of detection, angular resolution, etc.
Despite its considerable practical importance, this problem has not been the subject of an extensive literature [2]. Roughly speaking, the source motion spreads its spatial spectrum, which induces two major effects : lower performances in detection and angular resolution. This spatial spreading can be easily quantified under simple hypotheses ; it is then possible to determine the optimal integration time. Despite the very restrictive hypotheses needed for the calculation, an analytical expression of the array performances is of a great practical interest. A comparison with simulation results proves their validity.
A simplified model for source in motion has been developed, involving only a restricted number of parameters (initial bearing, spatial frequency speed). Using this model, the calculation of the Fisher information matrix (FIM) reveals interesting conclusions. This model is one of the basic tools.
The more difficult task consists in the development of array processing methods including the motion model to the source model itself. The modelling of the sensor outputs by a linear (non-stationary) system appears to be a powerful tool and allows us to take advantages of the basic hypotheses relative to source motion.
Finally, this problem fully justifies the use of parametric methods in the array processing area and lead to develop a new class of methods.

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2 A simplified model of source motion

Consider a source in motion (linear and uniform) and denote v the constant speed of the source θ_n and L_n respectively the bearing and distance *w.r.t.* the array at instant t_n ; then by using elementary calculations the following approximation is obtained :

$$\cos\theta_n \simeq \cos\theta_{n-1} + [\cos\beta - \cos\theta_{n-1}\cos(\theta_{n-1}-\beta)]\cdot\frac{v\Delta t}{L_{n-1}} \quad . (1)$$

β is the source heading)
The previous equation may be changed in order to obtain a first order approximation at $t = t_0$, let :

$$\cos\theta_n \cong \cos\theta_0 + (\cos\beta - \cos\theta_0\cos(\theta_0-\beta))\frac{v\Delta T n}{L_0} = k_0 + \dot{k}_0 n \quad . (2)$$

The factor $\dot{k}_0$ is, in eq.2, a constant. Denoting k_n (spatial frequency) the terms $\cos\theta_n$ in eq.1 and (2), they are rewritten in a simpler form as :

$$k_n = k_{n-1} + \dot{k}_n \cdot \text{ (1') and } k_n = k_0 + n\cdot\dot{k}_0 \text{ (2')}$$

Eq. 2' is valid only for far sources but lead to simple models of the received signals while eq.1' is valid even for short, medium range sources but needs more complicated analysis.

3 Effects of source motion for classical array processing

For the sake of simplicity, the analysis is achieved after Fourier transformation. Let $\hat{\Gamma}$ the empirical estimated covariance matrix of the sensor outputs after DFT, i.e. :

$$\hat{\Gamma} = \frac{1}{N}\sum_{k=0}^{N-1} X_k \cdot X_k^*$$

Then :

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$$\mathbb{E}\left[\hat{\Gamma}\right] = \Gamma = \frac{1}{N}\sum_{k=0}^{N-1}\mathbb{E}\left[X_k X_k^*\right] = \frac{1}{N}\sum_{k=0}^{N-1}\Gamma_{xk} \qquad (3)$$

Denote now $P(f,\theta)$ the asymptotic spatial density of the received signals at the frequency f (omitted for the sequel), i.e $P(f,\theta) = D_\theta^* \cdot \Gamma \cdot D_\theta$, then classical calculations yield the following result for a unique source in motion (eq.2') :

$$\begin{aligned} P(f,\theta) &= \frac{\sigma}{N}\sum_{l=-(p-1)}^{p-1}(p-|l|)exp\left(-i\frac{2\pi dl}{\lambda}\left(\cos\theta - \cos\theta_0 - \dot{k}\frac{N-1}{2}\right)\right) \\ & x\frac{\sin\left(\frac{\pi dlN\dot{k}}{\lambda}\right)}{\sin\left(\frac{\pi dl\dot{k}}{\lambda}\right)} + bp \qquad (4) \\ \lambda &= c/f \end{aligned}$$

with :

- d : intersensor distance ; p : sensor number
- b : noise power ; N : number of averaged snapshots

Eq.4 exhibits a special kernel which shows two consequences of source motion :

1. enlarged main lobe induced by source motion
2. averaged spatial frequency k_M defined by :

$$k_M = \frac{1}{\lambda}\left(\cos\theta_0 + \dot{k}\frac{N-1}{2}\right) \qquad (5)$$

The practical effects of eq.4 are degraded performances in terms of source detection, angular resolution (eq.4) and bias (eq.5). These effects, furthermore, are increasing as the sensor number becomes greater. Thus, the choice of the integration time (N) is much more crucial when p is great.

In the unique source case, the exact expression of P_d and P_{fa} (resp. probability of detection and false alarm) have been calculated [3] ; furthermore, it has been shown that the following gaussian approximation is valid for $N \geq 10$:

$$\Lambda = \hat{G}(\theta_M) = \frac{1}{N}\sum_{k=0}^{N-1}D_{\theta M}^* X_k X_k^* D_{\theta M} \underset{H_0}{\overset{H_1}{\lessgtr}} \mu$$

and

$$\begin{cases} \mathbb{E}\left[\Lambda\right] = \frac{1}{N}\sum_{k=0}^{N-1}Tr\left[D_{\theta M}D_{\theta M}^*\Gamma_{xk}\right] = m \\ \mathbb{E}\left[\Lambda^2\right] - \mathbb{E}[\Lambda]^2 = \frac{1}{N^2}\sum_{k=0}^{N-1}Tr\left[\left(D_{\theta M}D_{\theta M}^*\Gamma_{xk}\right)^2\right] = \sigma^2 \\ \Lambda = \mathcal{N}(m,\sigma^2) \end{cases} \qquad (6)$$

For a moving source, the curve representing $P_D(N)$ exhibits a very special behavior since it begins by increasing (as in the stationary case) and then decreases fastly. This behavior is more dramatic as p is great. The examination of the ROC curves of the classical quadratic receiver Λ (eq.6) shows that the degradation induced by source motion amounts to decrease the signal to noise ratio up to 20dB (for a large array and great values of N).

The study of the variance of the estimated averazed bearing is treated in a similar way, giving the following expression of var $\left(\hat{\theta}_M\right)$:

$$\text{var}\left(\hat{\theta}_M\right) = \frac{1}{N^2}\frac{\sum_{n=0}^{N-1}Tr\left((A\Gamma_{xn})^2\right) + \left[\sum_{n=0}^{N-1}Tr\left(A\Gamma_{xn}\right)\right]^2}{\left[\mathbb{R}_e\left(V_{\theta M}^* R D_{\theta M}\right) + U_{\theta M}^* R U_{\theta M}\right]^2}$$

with :

$$\begin{aligned} A &= \frac{C+C^*}{2} & V_\theta &= \frac{\partial^2 D_\theta}{\partial^2\theta} \\ C &= U_{\theta M}D_{\theta M}^* & R &= \frac{1}{N}\sum_{n=0}^{N-1}\Gamma_{xn} \\ U_\theta &= \frac{\partial D_\theta}{\partial\theta} \end{aligned} \qquad (7)$$

The curve representing var $\left(\hat{\theta}_M\right)$ as a function of the parameter N (integration time) exhibits the same behavior that the P_D curves. It begins by decreasing and then fastly increase.

The analytical expressions of P_d, P_{F_a} (eq.6) and var $\left(\hat{\theta}_M\right)$ (eq.7) allow us to determine the optimal integration time N for a source whose parameters are known (k_0 and $\dot{k}$ for instance). For a small value of $p (p = 8)$ it may be rather important, one obtains typically $N = 400$; but for larger values of $p (p = 100)$, N must be chosen rather low (typically $N = 30$).

The effects of source motion for H.R. methods can be investigated by the same analysis but the calculations are much more complicated. Very roughly speaking, the source motion induces an higher rank of the source covariance matrix. The consequences for H.R. methods of this phenomenon are source splitting, non-detection of weak sources, etc.

The calculation of the FIM reveals interesting conclusions for the estimation of the parameters k_0 and $\dot{k}$, it is given by the following formula :

$$\begin{aligned} FIM(1,1) &= -\mathbb{E}\left[\frac{\partial^2 L(\theta)}{\partial^2 k_0}\right] = \frac{N\rho^2 4\Pi^2 d^2 p^2\left(p^2-1\right)}{6(1+\rho p)} \\ FIM(2,2) &= -\mathbb{E}\left[\frac{\partial^2 L(\theta)}{\partial^2 \dot{k}}\right] = \frac{\rho^2 4\Pi^2 d^2 p^2\left(p^2-1\right)N(N+1)(2N+1)}{36(1+\rho p)} \\ FIM(1,2) &= -\mathbb{E}\left[\frac{\partial^2 L(\theta)}{\partial k_0 \partial \dot{k}}\right] = \frac{\rho^2 p^2\left(p^2-1\right)4\Pi^2 d^2 N(N+1)}{12(1+\rho p)} \end{aligned} \qquad (8)$$

The above expression of the FIM shows that var $\left(\dot{k}\right)$ is proportional to N^{-3} whereas var $(k)\alpha\ N^{-1}$. However, this result must be compared with the value of $\dot{k}$ which is usually very small (for instance $10^{-4} rd/s$ for medium range sources).

4 Simultaneous estimation of the parameters k_0 and $\dot{k}$

The most classical and natural approach consists in focused beamforming or equivalently to define the density $P\left(k,\dot{k}\right)$ by :

$$P\left(k,\dot{k}\right) = \frac{1}{N}\sum_{n=0}^{N-1}\left|X_n^* \cdot D_{k_0,\dot{k},n}\right|^2 \qquad (9)$$

where the source steering vector $D_{k_0,\dot{k},n}$ is the vector $D_{k_0+n\dot{k}}$.

The method consists then in computation of the density $P\left(k_0,\dot{k}\right)$ and in seeking the values of $\left(k_0,\dot{k}\right)$ maximizing the density. Despite its conceptual simplicity, the above method suffers of severe drawbacks wich are : computation cost and, overall, numerous spurious peaks. This last drawback is actually induced by the simultaneous analysis in k and $\dot{k}$. In order to remedy this problem, we shall develop a separated state space model.

5 State-space model of the data

The data (correlations) are in a 2D-array and indexed by the time t and the space m . Then, the data $y(t,m)$ may be represented by a 2D-state model, i.e. :

$$\begin{cases} X(t+1,m) &= F_1^m X(t,m) \qquad 1 \le m \le p \\ X(t,m+1) &= F_0 F_1^t X(t,m) \\ y(t,m) &= h^* X(t,m) + \omega(t,m) \end{cases}$$

with :

$$\begin{aligned} F_1 &= \text{diag } \left(exp\left(2i\pi d\dot{k}_1\right), \cdots, exp\left(2i\pi d\dot{k}_s\right)\right) \\ F_0 &= \text{diag } \left(exp(2i\pi dk_{1,0}), \cdots, exp(2i\pi dk_{s,0})\right) \\ h^* &= (1,1,\cdots,1) \end{aligned} \quad . (10)$$

Thanks to the classical properties of Fourier transforms the noise $\omega(t,m)$ is uncorrelated in time, its covariance may be calculated [4] and is given by the following formula :

$$\mathbb{E}\left[\omega(t,m_1)\,\omega^*(t,m_2)\right] = \frac{Tr\left(\Gamma x t^{\downarrow m_1} \Gamma x t^{\uparrow m_2}\right)}{(p-m_1)(p-m_2)} \quad . (11)$$

with : $\Gamma_{X_t}^{\downarrow m}$ is the matrix Γ_{X_t} translated m times down.
This model can be extended to multidimensional analysis, consider then the vector $\mathcal{Y}(t)$ defined by :

$$\mathcal{Y}(t) = (y(t,1), y(t,2), \cdots, y(t,p))^t$$

then the model (10) can be extended to :

$$\begin{cases} \mathcal{X}(t+1) &= \mathcal{F} \cdot \mathcal{X}(t) \\ \mathcal{Y}(t) &= \mathcal{H}^* \cdot \mathcal{X}(t) + W(t) \end{cases}$$

with :

$$\begin{aligned} \mathcal{F} &= \text{diag}\left[F_1, \cdots, F_1^P\right]; \mathcal{H} = \begin{pmatrix} h^* & & 0 \\ & h^* & \\ & & \ddots \\ 0 & & h^* \end{pmatrix} \\ W(t) &= (w(t,1), \cdots, w(t,p))^t \end{aligned} \quad . (12)$$

Obviously, other models can be considered, ; another state space model will be used for non-stationary analysis.

6 SVD-based algorithms

We are now dealing with the estimation of the matrices F_0 and F_1. The general scheme consists in a separated estimation of the matrices F_1 and F_0, followed by an association procedure of the estimated parameters.
The basic method uses the following decomposition of the Hankel matrix $\mathcal{H}_m = \text{Hank } (y(t,m))$ [4] :

$$\mathcal{H}_m = \theta_m \cdot \mathcal{X}_m$$

where θ_m is the observability matrix and $\mathcal{X}_m$ the state matrix, i.e. :

$$\begin{aligned} \theta_m^t &= \left(h^*, h^* F_1^m, \cdots, h^* F_1^{ml}\right) \\ \mathcal{X}_m^t &= (X(1,m), \cdots, X(c,m)) \end{aligned} \quad . (13)$$

Actually, an observability matrix $\hat{\theta}_m$ is deduced from $\hat{\mathcal{H}}_m$ by a SVD procedure and an estimated $\hat{F}_1$ is then obtained. This analysis can be extended to the multidimensionnal model (12) without difficulty. The variance of the parameters $\dot{k}$ have been estimated [4] and are approximated by :

$$\text{var } \left(\hat{\dot{k}}_j\right) \cong \sigma_w^2(m) \left\|\frac{\partial \hat{\dot{k}}_j}{\partial \hat{h}}\right\|^2 \quad . (14)$$

Where the variance of the estimation noise $\sigma_w^2(m)$ have been calculated by eq.11, the other factor of the right term of (14) is calculated by a classical SVD-perturbation analysis [4].
Roughly, the term $\sigma_w^2(m)$ grows as $(p-m)^2$ whereas the geometric term $\left\|\frac{\partial}{\partial \hat{h}} \hat{\dot{k}}_j\right\|^2$ decreases as m^2. Thus, there is an optimum choice of m and, more generally, it is possible to optimize the spatio-temporal configuration [4].

7 AR-model approaches

For a fixed spatial indice m, eq.10 leads to an AR modelling. This model is estimated, in a first time, by classical methods. Then it is followed by an interpolation procedure, i.e. we define the interpolated polynomial (from m_k to m_1) by :

$$\hat{\tilde{P}}_{m_k}^{m_1}(z) = \prod_{j=1}^{N_s} \left(z - \hat{z}(j,m_k)^{\frac{m_1}{m_k}}\right) \quad . (15)$$

and, then, an interpolated polynomial :

$$\hat{\tilde{P}}_{m_k}^{m_1}(z) = z^{N_s} - \hat{\tilde{a}}_{m_k}^{m_1}(1) z^{N_s - 1} \cdots - \hat{\tilde{a}}_{m_k}^{m_1}(Ns) \quad . (16)$$

These polynomials are then "fusionned" by a least square method. More specifically, the covariance matrices $\tilde{T}_{m_k}^{m_1}$ are deduced from (15) thanks to Gohberg's formula, i.e. :

$$\begin{aligned} \tilde{T}_{m_k}^{m_1} &= \left(\tilde{\sigma}_{m_k}^{m_1}\right)^2 \left(\tilde{A}_{m_k}^{m_1} \tilde{A}_{m_k}^{*m_1} - \tilde{\bar{B}}_{m_k}^{m_1} \tilde{\bar{B}}_{m_k}^{*m_1}\right)^{-1} \\ \tilde{A}_{m_k}^{m_1} &= \sum \tilde{a}_{m_k}^{m_1}(i) \cdot Z^i, \tilde{B} = \sum \tilde{a}_{m_k}^{m_1}(i) \cdot Z^{P-i}, Z\text{: shifting matrix} \end{aligned} \quad . (17)$$

The covariance matrices provide then a unique AR model by means of a least square procedure. This model resumes all the spatio-temporal data, it can itself reduced to a lower rank by an optimal approximation.
This method present good performances [4] for far sources whose trajectories (seen by the array) can be correctly modelled by a straight line (eq.2'). Unfortunately, the performances are affected by medium and low range sources.

8 Non stationary AR-models

In order to avoid the problems induced by near sources, a non stationary AR-model [5] is considered, i.e. :

$$\begin{cases} X_{t+1} &= F \cdot X_t + W_t \\ Y_t &= h_t^* \cdot X_t + n_t \end{cases} \quad . (18)$$

with :

$$Y_t = y(t,m); h_t^* = (y(t-1,m),\cdots,y(t-q,m)) \quad .(19)$$

$$X_t = (a_1(m,t),\cdots,a_q(m,t)) \quad .(20)$$

$$1 \leq m \leq p$$

In the eq.18, the matrices F and $Q_t = \text{cov}\,(W_t)$ are deduced from eq.1' by using the Leverrier's algorithm or hypotheses about source motion. The estimation of the state vector X_t (AR model) is itself estimated by means of a Kalman filter, the poles of the associated AR model yield the estimated parameters $\hat{\dot{k}}_n$. The method can be extended to spatio-temporal analysis by means of the following state space model :

$$\begin{cases} X_{t+1} &= F \cdot X_t + W_t \\ Y(t,1) &= h_{t,1}^* \cdot X_t + n_1(t) \\ Y(t,2) &= h_{t,2}^* \cdot \tilde{X}_2^1(t) + n_2(t) \\ \vdots & \end{cases}$$

with :

$$\begin{cases} h_{t,1} &\equiv (y(t-1,1),\cdots,y(t-q,1)) \\ h_{t,2} &\equiv (y(t-1,2),\cdots,y(t-q,2)) \\ \tilde{X}_2^1(t) &\equiv X_t \text{ interpolated from 2 to 1 by eq.15} \end{cases} \quad .(21)$$

The above model is itself followed by a Kalman filter and smoothing. This is only a partial model for the 2D-array ; extensions of this kind of methods are under investigation.

The first line of Eq. 18 (state eq.) plays a crucial role for source tracking and must be carefully considered. Actually, the following model of source motion (spatial frequency) is quite acceptable :

$$\dot{k}_{t+1} = 2\dot{k}_t - \dot{k}_{t-1} + w_t, \qquad w_t \,: \mathcal{N}(O,\sigma^2) \qquad .(22)$$

Eq. 22 corresponds to a slowly varying source and for most of the cases σ^2 must be chosen very little (typically $\sigma^2 \alpha 10^{-10}$).

Eq. 18 is classicaly modelled by a n.times differentiation [5], for instance if $n = 2$ (as in eq. 2) then F is :

$$F = \begin{pmatrix} 2I & -I \\ I & 0 \end{pmatrix}, \quad \text{with} \quad W_t = \mathcal{N}\left(0, \begin{pmatrix} \tau^2 & & \\ & \ddots & \\ & & \tau^2 \end{pmatrix}\right) \quad .(23)$$

The value of τ^2 must be itself chosen very little w.r.t. the value of the observation noise. As suggested in [5], the adequate value of τ^2 may be estimated by a maximum likehood procedure, fortunately the choice of τ^2 is not too critical.

By using eq. 22 it is possible to specialize eq. 23 to the case of slowly varying sources. This task may be investigated in two major ways :

1. First order linearization of AR coefficients :

$$da_i(t) = \sum_{j=1}^{S} \frac{\partial}{\partial \dot{k}_j} a_i(t).d\dot{k}_j \qquad .(24)$$

   In eq. 24, the term $\frac{\partial}{\partial \dot{k}_j}.a_i(t)$ is itself calculated by means of the leverrier algorithm. Obviously, eq. 24 may be extended to a second order expansion. The terms $d\dot{k}_j$ are themselves calculated by using eq. 22.

2. Use of a symmetric function of the roots :

   The AR coefficients $\{a_k\}$ may be expressed as symmetric functions of the roots, i.e.:

$$a_k = (-1)^k . \sum z_{i,1} z_{i,2} \cdots z_{i,k}$$

with

$$z_{i,1} = exp\left(-2i\pi \dot{k}_{i,1}\right) \qquad .(25)$$

A recursion w.r.t. the $a_k(t)$ may then be derived from eq. 22 assuming independent increments w_t for each source, more specifically one obtains for instance for the first order :

$$a_k(t+1) = a_k(t) + (n-k).w_t \qquad .(26)$$

9 Simulation results

Numerous simulations have been performed, the results are available in the reports [3] and [4]. As an example consider the following simultation : two sources with rectilinear and constant speed motion, one source (the greater) is very close to the array whereas the second is far. Their respective evolution laws in terms of spatial frequency are illustrated by fig. 1.

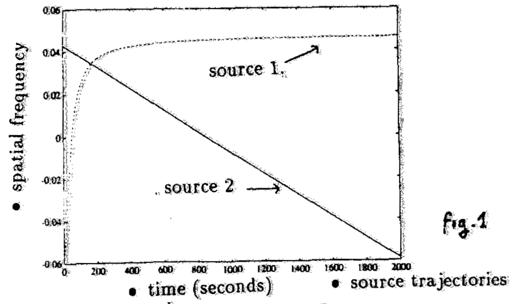

fig.1

The exact value of $\dot{k}_2$ is $-5.0875.10^{-5}$. The respective levels of the sources are : 0 dB (close source) and $-20, -30dB$ for the far source. The array processing is described in §2,5,7, it assumes validity of eq. (2') for the far source.

The results are (one trial) : 16 sensors

$$\hat{\dot{k}}_2 = -5.24.10^{-5}(-20dB) \,; \hat{\dot{k}}_2 = -3.4.10^{-5}(-30dB)$$

Conclusion

The effects of source motion have been studied ; then we have developed array processing methods including these (unknown) motions in the source models. The great variety of source trajectories lead to model them by a markovian model. This last way, associated with interpolation-fusion procedures, seems quite promising and feasible. It can lead to a new class of array processing.

References

[1] D.H. Johnson, "The application of spectral estimation to bearing estimation problems". Proc. of IEEE, vol.70, no9, p.1018-1028, sept. 1982.

[2] B. Champagne, "Optimum space-time processing in non-stationary environments". Ph. D, University of Toronto, 1990.

[3] O. Zugmeyer et J.P. Le Cadre, "Méthodes d'intégration temporelle en traitement d'antennes". Rapport INRIA, no566, IRISA, 35042 Rennes, France

[4] O. Zugmeyer et J.P. Le Cadre, "Méthodes paramétriques pour la détection de sources en mouvement". Rapport INRIA, no567, IRISA, 35042 Rennes, France

[5] G. Kitagawa and W. Gersh, "A smoothness priors time-varying...". IEEE Trans on A.C., vol.AC-30, no1, Janv. 1985, p.48-56.