Approximated stochastic realization and model reduction methods applied to array processing by means of state space models

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ABSTRACT

The aim of this paper is to present new methods for passive array processing. The basic idea consists in using a state-space modelling of the sensors output. This paper copes with basic problems as: unknown noise correlations, approximation by a Toeplitz matrix of lower rank, detection of small sources. The presented methods represent considerable improvements with respect to the usual ones and furthermore are quite feasible, some statistical results illustrate these claims.

1. Introduction.

A modelling of the sensors output of an equally spaced linear array by a linear system is shown to be a powerful tool, in particular to take into account fundamental hypotheses about sources and noise. A general frame for approximated stochastic realization methods is presented and connections with existing work and array processing are especially considered.

A state space modelling of the sensors output uses sources and noise properties in a summarizing way and is a powerful means to separate the noise and sources contributions even in the case of an unknown correlated noise. This last property represents a considerable improvement against usual high resolution methods. For that purpose two methods of approximated stochastic realization are considered, the links between these ones and array processing are detailed.

However a fundamental problem remains: find an estimation of the sources subspace that preserves the plane wave structure. We are now facing the basic problem which consists in extracting a low rank matrix (of sources) from an estimated (full rank) one, preserving the special structure (Toeplitz or Hankel) induced by plane wave and spatial stationarity hypotheses. The Adamjan-Arov-Krein theory is the theoretical frame, the initial problem of functional analysis is translated into another simpler and finite dimensional problem by the use of the properties of finite structures and balanced realizations. A further advantage consists in using an L₂-norm rather than an L₁-norm, much more sensitive to local variations of the spectra.

Furthermore, simulation results present comparisons between state space approaches and usual high resolution methods (as eigenvectors methods) and try to precise their advantages.

2. Additive white noise.

This case is especially simple but allows us to precise the general ideas. In the whole paper, the sensors output of a linear equispaced array constituted by n sensors are considered at a given frequency (omitted for the sequel). Under plane wave assumption, the output at the i-th sensor may be written as:

\[ y_i = \sum_{k=1}^{m} a_k \exp(-j \omega_k) + v_i \]  

where m is the number of sources, \( a_k \) and \( \omega_k \) represent respectively the random level and the deterministic phase of the k-th source, and \( v_i \) is the additive noise.

A rank m linear state space model of (1) is given in the case of an additive white noise by (1)

\[ X_{i+1} = F_i X_i \]

\[ y_i = h_i^T X_i + v_i \]

where \( F_i \) and \( h_i \) are deterministic parameters, \( X_i \) is the state vector as sensor i and \( v_i \) is an additive white noise.

The matrix \( F \) translates the phase properties of the propagation while \( h \) represents its power effects. The phases \( \omega_k \) are determined from the eigenvalues of the transition matrix \( F \) which is similar to diag(\( \exp(-j \omega_k) \)).

Let \( \Gamma_p \) be the covariances of the sensors output, i.e. \( \Gamma_p = E(y_i y_i^T) \)

\[ \Gamma_p = h_i \cdot F_i \cdot P \cdot h_i^T + \delta(p) \]  

with \( P = E(X_i X_i^T) \) and \( \delta(p) \) the white noise.

Furthermore, let \( \Gamma_s \) be the exact covariance matrix of the sensors output, then

\[ \Gamma_s = O \cdot F \cdot O^T + b^2 \cdot I_n \]  

where \( O \) is the (nxm) observability matrix.

The following property can be easily deduced from the structure of the observability matrix:

\[ O^T \cdot F \cdot O^T = O^T \]  

(1: select the (n-1) last rows, 2: select the (n-1) first rows, \( O^t \) and \( O^T \) are ((n-1)xm) matrices), and therefore:

\[ F = O^T \cdot O^T \]  

(#: denoting pseudo-inverse matrix)

Elsewhere, the covariance matrix can also be written in terms of array processing as:

\[ \Gamma_s = D \cdot \gamma \cdot D^T + b^2 \cdot I_n \]  

with \( D \) : source steering matrix

\[ D^T \cdot F = D^T \] or \( F = D^T \cdot D \) leading to property 1 ([2]), and a geometric meaning of the observability matrix.

In the asymptotic case, the eigenvalues of the transition matrix \( F \) are identical to those of the matrix \( \{(U_i^T) \# \quad U_i^T \} \) where \( U_i \) is structure.
form all the eigenvectors corresponding to the m largest eigenvalues of $\Gamma_+$. In practical situation however, the matrix $U_1$ is replaced by $U_{\hat{1}}$ an estimate (from $P_\mathbf{r}$), therefore the presented method "forgets" the special structure of the observability matrix.

3. Approximated realization methods, additive correlated noise.

In the general case, it is possible to take into account the noise correlations by modelling the sensors output as the outputs of a minimum phase innovation model, i.e.:

$$x_{i+1} = F . x_i + v_i$$

$$y_i = h_i . x_i + v_i$$  (9)

Then the approximated realization methods rely upon consideration of two vectors $Y_+$ and $Y_-$ (future and past at sensor $n/2$) defined as follows:

$$Y_+ = \{y_{n+2}^2, y_{n+2}^1, ... y_1\}$$  (9a)

$$Y_- = \{y_{n+2}^2, y_{n+2}^1, ... y_1\}$$  (9b)

The aim of the methods consists in summarizing the past in order to obtain the more efficient (for a given criterion) prediction of the future. The stochastic approach [3] determines a state vector $X$ (of a given dimension) which sums up the more pertinent part of $Y$, in order to predict $Y_+$, i.e.:

$$x = A . y, \quad \text{with } A (p \times n)$$  (11)

Theoretically one has:

$$A = [T, (F - T, h^0), T, ..., (F - T, h_p)^n-1, T]$$  (12)

Furthermore, in the case of a markovian system, the orthogonal projection of the future onto the past is given by:

$$y_+ | Y_+ = 0 . X$$  (13)

The determination of $A$ enables us to estimate $O$ and therefore the system parameters. The crucial step of these methods consists in the estimation of the matrix $A$, for that purpose various methods can be considered.

3.1. The predictive efficiency criterion (AK method).

The matrix $A$ is the matrix which minimizes the variance of the predictor error, i.e. [4]:

$$\text{Min} \{ \text{tr} [ \text{cov}(Y_+, Y_- | X)] \}$$

yielding to:

$$\text{Min} \{ \text{tr} [\Gamma_+ - H . A^* . (A . \Gamma . A^*). A . H^*] \}$$  (14)

where $H (X)$ is the energy of the gaussian vector $X$, then the estimation of $X = A . Y$, amounts to minimize the following criterion [5]:


A

Note that both criteria differ only by the functional (tr or det), the minimization of this criterion is achieved by means of elementary algebra leading to:

$$\text{Prop. 3: the solution of the problem (17) is given by:}$$

$$A = B . V_+ . \Gamma^{-1/2}$$  (18)

where $V_+$ is the matrix formed from the p right principal singular vectors of the matrix $[\Gamma_+^{-1/2}, H . \Gamma^{-1/2}]$.

Therefore the solution of (17) is nothing but the solution obtained by maximization of canonical correlations as advocated by Desai and Pal [5].

The matrix $A$ being estimated by means of Prop.2 or 3, $F$ is straightforwardly deduced by two ways since:

$$A = [T, (F - T, h^0), T, ..., (F - T, h_p)^n-1, T]$$

$$X_{i+1} = F . X_i$$

The links with classical methods of array processing are much more subtle than in §2. In the white noise case, one has [6]:

$$H = \Gamma^{-1/2}, H^* = U . D . U^*$$

therefore the matrix $H = \Gamma^{-1/2}, H^*$ plays the role of $\Gamma_+$ in Prop. 1; but this is not very relevant of the general case. For that purpose, it is more appropriate to consider the predictive efficiency criterion in terms of the intersection of two linear subspaces, i.e. [6]:

$$\text{Prop. 4: the rows of the matrix $A$ approximate a basis of } \text{Range}(H^* H) \cap \text{Range}(\Gamma_+).$$

Prop. 4 means that the rows of $A$ minimize the principal angles between $\text{Range}(\Gamma_+)$ and $\text{Range}(H^* H)$.

Under the assumption of a shortly correlated additive noise, the matrix $H$ is perturbed only by a triangular matrix, the vectors of $\text{Range}(H^* H) \cap \text{Range}(\Gamma_+)$ are mainly related to source parameters.

To sum up, the presented methods are efficient to estimate the source bearings in presence of an additive noise with unknown correlations, however they do not use fundamentally the plane wave hypothesis for the estimation of $A$. We shall now cope with this problem.

4. Optimal realization methods.

4.1. A general framework.

The approximation of a linear system by a lower order one represents an important part of control systems literature. However this field seems to be not very relevant for array processing, but the following general framework permits us to use the results of control systems theory.

The array outputs are modelled by a maximal order parametric model (e.g.: an AR model of order n), note that there is no information less since it is possible to construct the corresponding covariances ($\Gamma$) from them. It is simply another representation of the covariances. A state space model can be easily obtained from the model and it is then possible to use the model order reduction methods [7].

We want an exact approximation, preserving the structure and avoiding the classical least squares approximations (leading generally to principal component analysis). This is basically a problem of approximation. In fact, the theorem of Adjamian, Arov and Krein (AAK) is the foundation of such an
4.2 Problem formulation

Given a scalar transfert function (e.g. \( f(z) = h^* (z-1)^{-1} \)) we seek an approximation (for a certain norm) of this function by a lower order system. The use of L_2-norm is classical but is not as powerful as L_\infty-norm for our problem where sources may have very different powers. The inclusion of special structure leads to the following problem:

\textbf{Pb.1:} given \( f \in L_\infty \) (defined on \( C(0,1) \)) and a positive integer \( k \), find

\[
\text{inf} \left\{ \| f - \phi \|_{L_\infty} : \phi \in H_{\infty,k} \right\}
\]

\text{and a function} \( \phi \in H_{\infty,k} \) \text{for which the infimum is attained; } H_{\infty,k} \text{ being the set of meromorphic functions in } L_\infty \text{ which can be written as:}

\[
\phi(z) = \sum_{n=1}^\infty \frac{a_n}{z^n}
\]

This problem has been solved by AAK, its solution consists in using the Hankel matrix of a function \( f \) w.r.t. the complete basis \( (2^n) \)

then a first result is available:

\textbf{Theorem (AAK):} let \( f \in H_\infty \) \text{ which minimizes that distance being given by Nehari's theorem.}

Furthermore, the Kronecker's theorem allows us to precise the problem:

\textbf{Kronecker's theorem:} let \( f \in L_\infty \), then \( H_f \) has finite rank \( \leq k \) if and only if \( f \in H_{\infty,k} \).

The theorem of AAK gives an explicit solution to Pb. 1.

\textbf{Theorem (AAK):} let \( f \in L_\infty \) and \( k \geq 0 \). Then 

\[
\text{dist}_{L_\infty}(f, H_{\infty,k}) = \sigma_k(f) \text{ (where } \sigma_k \text{ are the singular values of } H_f \text{ ordered in decreasing value; furthermore this distance is attained at a unique function } \phi \in H_{\infty,k} \text{ and if } \phi_{k+1} \text{ is the singular vector of } H_f \text{ corresponding to } \sigma_{k+1}(H_f) \text{ then:}}
\]

\[
\phi_k(z) = f(z) - [H_f, \phi_{k+1}](z) / \phi_{k+1}(z)
\]

\text{and if } \phi_{k+1} = \Pi_k(\phi_{k+1} - f) \text{ then:}

\[
\Pi_k(\sum_{i=0}^k z^i) = \sum_{i=0}^k a_i z^i, \quad \Pi_k = I - \Pi_k \text{, one obtains}
\]

\[
\phi_k(z) = f(z) - \phi_{k+1} \frac{h_{k+1}(z)}{\phi_{k+1}(z)}
\]

\[
\mu_k(z) = \sum_{j=1}^{k+1} \frac{a_j}{z^j}
\]

with:

\[
\phi_{k+1}(z) = \sum_{i=1}^{k+1} v_{k+1} z^{-i}
\]

These results are interesting but not directly useful because they involve mainly the infinite Hankel matrix \( H_f \) and its singular vectors. The implication of finite structure allows us to solve explicitly the initial Pb. 1.

4.3 Implication of finite structure

Firstly, recall the classical equality: \( H = \mathcal{H}_\infty, \mathcal{C} \), therefore the square of the \( k \)-th singular value \( \sigma_k \) of the (infinite) matrix \( H \) is

\[
\sigma_k(H^*H) = \sigma_k(C^*C) \quad \sigma_k(H_\infty H_\infty^*)
\]

where \( P \) and \( Q \) are respectively the controllability and observability gramians. The matrices \( P \) and \( Q \) depend strongly on state-space coordinates but not the eigenvalues of the product \( P \cdot Q \), furthermore it has the great advantage to be finite-dimensional.

Let the strictly proper part of the transfert function be:

\[
f(z) = h^* (z-1)^{-1} \cdot T = d(z)
\]

where \( d(z) = \det(z-1 - F) \). By use of balanced system transformation, it is possible to obtain simpler expressions for these functions \( \mu_k(z) \) and \( \nu_k(z) \).

More precisely, let \( H = H_f \) and \( \mathcal{B} \) a balancing transform:

\[
(F^* f^* H^* F, h^* F^* H^* F) \quad (F^* h^* H^* F, T) \quad \text{(balanced triple)}
\]

Hence, if \( H = U \cdot \Sigma \cdot V \), then \( \mathcal{O}_B = U \cdot \Sigma^{1/2} \) and \( \mathcal{C}_B = \Sigma^{1/2} \cdot V \), therefore:

\[
\mu_k = z_1^{1/2} + \cdots + z_1^{-1}, \quad \nu_k = z_1^{1/2} + \cdots + z_1^{-1}, \quad e_k
\]

(26)

where \( e_k \) denotes the \( k \)-th column of the \( (n \times n) \) identity matrix.

Similarly, one obtains:

\[
\nu_k = z_1^{1/2} + \cdots + z_1^{-1}, \quad \nu_k = z_1^{1/2} + \cdots + z_1^{-1}, \quad e_k
\]

(27)

This last equality can be usefully transformed by means of the following property:

\textbf{Prop. 5:} for a balanced SISO system, the following equalities are satisfied: \( F = Q \cdot F_B \cdot Q^* \) ; \( T = Q^* \cdot h \), \( Q \) being a unitary diagonal matrix.

From (26) we see that \( \mu_k(z) \) has a rational form, i.e.:

\[
\mu_k(z) = \frac{m_k(z)}{d(z)}
\]

then (27) and Prop. 5 yield the following fundamental property:

\[
\nu_k(z) = q_1 \cdot z_1^{1}, \quad \mu_k(z) = q_1 \cdot \frac{m_k(z)}{d(z)}
\]

(28)

(29)

(30)

(31)

(32)

Furthermore it can be easily proved that the rational fraction \( q(z) / d(z) \) is actually a polynomial \( p(z) \) of degree \( < n \), as a consequence the following equality is valid:

\[
\nu_k(z) = \frac{m_k(z)}{d(z)}
\]

with:

\[
\nu_k(z) = m_k(z) - \nu_k \cdot d(z)
\]

This last form of \( \nu_k(z) \) leads to determine the coefficients of \( m(z) \) by solving the following
polynomial equation:

\[ p(z) = (z - \alpha_1) \cdot (z - \alpha_2) \cdot \ldots \cdot (z - \alpha_m) \]  

(29)

equating the coefficients of powers of \( z \) [7], one obtains the two following matricial equations (30) and (31):

\[
J \cdot S_2 \cdot J = R_\lambda - 5 \cdot J \cdot S_1 \cdot J \cdot m \quad (z^i: i = 0, \ldots, n-1)
\]

\[
S_1 \cdot J = R_\lambda - 5 \cdot S_2 \cdot J \cdot m (z^i: i = n, \ldots, 2n-1)
\]

where:

\[ n(z) = c_n + c_{n-1} \cdot z + \ldots + c_1 \cdot z \]

\[ d(z) = m_{n-1} + m_{n-2} \cdot z + \ldots + m_1 \cdot z^{n-1} \]

\[ m(z) = m_0 + m_1 \cdot z + \ldots + m_{n-1} \cdot z^{n-1} \]

\[ p(z) = p_{n-1} + p_{n-2} \cdot z + \ldots + p_0 \cdot z^{n-1} \]

and:

\[
J = \begin{bmatrix}
 J_{n-1} & 0 & \ldots & 0 \\
 0 & J_{n-2} & \ldots & 0 \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \ldots & J_0
\end{bmatrix}
\]

\[
S_1 = \begin{bmatrix}
 S_{n-1} & 0 & \ldots & 0 \\
 0 & S_{n-2} & \ldots & 0 \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \ldots & S_0
\end{bmatrix}
\]

\[
R_\lambda = \begin{bmatrix}
 R_{n-1} & 0 & \ldots & 0 \\
 0 & R_{n-2} & \ldots & 0 \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \ldots & R_0
\end{bmatrix}
\]

and \( J \) the antidiagonal matrix from (30) and (31) one obtains (33):

\[
(J \cdot S_2 \cdot J \cdot S_1 - R_\lambda - R_\lambda) \cdot m = \lambda \cdot (J \cdot S_2 \cdot J \cdot S_1 - J \cdot S_1 \cdot J) \cdot m = (m_0, m_1, \ldots, m_{n-1})
\]

which appears as a generalized eigenvalue-eigenvector equation. In order to solve (33), numerous subroutines are available. Once the coefficients of \( m(z) \) have been calculated, those of \( p(z) \) are straightforwardly deduced.

J. S_2 J. p = R_\lambda J. S_1 J. m \quad (z^i: i = 0, \ldots, n-1)

S_1 p = R_\lambda S_2 J. m \quad (z^i: i = n, \ldots, 2n-1)

Practical implementation:

1) estimation of the covariances of the sensors output (n+1 sensors); i.e. \( \gamma(0), \gamma(1), \ldots, \gamma(n) \).
2) estimation of the corresponding AR model coefficients: \( (a_1, a_2, \ldots, a_n, b_1, \ldots, b^2) \).
3) solve the generalized eigenvalue-eigenvector problem (33) with:

\[
a_1 = a_2 = \ldots = a_n = b_1 = b_2 = 1
\]

\[
c_1 = 0 = \ldots = c_{n-1} = 0 = c_n = 1
\]

select the \((k+1)\)-th generalized eigenvector (to obtain a rank \( k \) approximation) and calculate \( m_{n-k}(z) \). 4) compute the roots of \( m_{n-k}(z) \) and their arguments (note that by definition of \( H_{n-k} \) there are \( k \) roots in \( D(0,1) \)).

5. Simulation results.

Due to the lack of space, we present briefly simulation results, details are available in [6].

5.1. Approximated realization methods (ARM):

The case of an additive white noise is firstly considered. Three uncorrelated sources impinging a linear array consisted by 16 equispaced sensors (at \( \lambda/2 \)). The covariance matrix of the outputs is estimated by averaged periodogram (BT = 100). The source bearings are 56, 60, 65 degrees with respective levels -1, -11, -1 dB. The assumed sources number is 4, the results are summarized below:

<table>
<thead>
<tr>
<th>Detection of a weak source</th>
<th>MUSIC</th>
<th>ARM</th>
</tr>
</thead>
<tbody>
<tr>
<td>100 trials</td>
<td>15/100</td>
<td>40/100</td>
</tr>
</tbody>
</table>

5.3. Optimal realization method (ORM):

Simulation 1: the additive noise is spatially white. 3 sources are simulated with respective powers and bearings (30 deg., 0 dB), (36 deg., -10 dB), (45 deg., 7 dB); the number of averaged snapshots (BT) is equal to 300. After what the poles of MUSIC method and ORM are computed, the results are presented below (for ten trials):

<table>
<thead>
<tr>
<th>Detection of the weak source</th>
<th>MUSIC</th>
<th>ORM</th>
</tr>
</thead>
<tbody>
<tr>
<td>0/10</td>
<td>4/10</td>
<td>9/10</td>
</tr>
</tbody>
</table>

Note that for MUSIC method the pole associated to the weak source (-10 dB) can not be seen generally or at best is very small (\( \approx 0.5 \)), conversely the pole associated to that source is very close to the strongest ones (ORM typically 0.98 1).

Simulation 2: in order to judge ability of ORM to separate 2 close sources, two sources of same powers (0 dB) have been simulated at bearings 10 and 13 deg (BT = 300), with spatially white noise; results are summarized below:

<table>
<thead>
<tr>
<th>Separation of the two sources</th>
<th>MUSIC</th>
<th>ORM</th>
</tr>
</thead>
<tbody>
<tr>
<td>4/10</td>
<td>9/10</td>
<td>9/10</td>
</tr>
</tbody>
</table>

Simulation 3: we consider now the simulation of §5.2, the additive noise is strongly correlated. For ten trials, the sources are correctly seen at their exact bearings for ORM.

6. Conclusion.

The simulation results have demonstrated the effectiveness of a state-space modelling of the sensors output. Robustness versus noise correlations is greatly enhanced by use of ARM (w.r. "usual" high resolution method); it is their main advantage. Detection of weak source can be slightly enhanced also but estimation of sources subspace remains based upon eigenvector decomposition.

Conversely, ORM takes advantage of the estimation of sources subspace. As it can be shown in special structure and present considerable improvements for angular separation of close sources, detection of weak sources. Furthermore, the method is very robust w.r. noise correlations, its computation cost is similar to MUSIC method

Obviously the presented tools are powerful for a linear array, however they can not be directly (if it is possible) extended to a general shape.

References.