# ON THE OPTIMAL CONTROL OF POLYNOMIAL DYNAMICAL SYSTEMS OVER $\mathbb{Z}/_{p\mathbb{Z}}$

Hervé Marchand[1], Michel Le Borgne[2]

[1] Current address is University of Michigan, Dept. of Electrical Eng. & Computer Science, 1301 Beal avenue, Ann Arbor, Michigan 48109-2122, USA, e-mail : marchand@eecs.umich.edu
[2] Current address is IRISA / Université de Rennes I, F-35042 RENNES, France, e-mail : leborgne@irisa.fr

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## Abstract

*This paper describes how a polynomial equationnal approach of the theory of logic discrete event systems leads to efficient algorithms for the synthesis of supervisory controllers. Even if traditional control objectives can be considered in our framework (invariance, reachability or attractivity), we address here the synthesis of optimal controller with control objectives specified as a minimization of a given cost function over the states through the trajectories of the system.*

## Introduction

The main purpose of real-time applications is to control reactive systems for which control and physical parts are often merged. Thus, different theories dedicated to the control of discrete event systems had emerged since the 80's [1, 2]. Usually, the starting point of these theories is: given a model for the system and the control objectives, a controller must be derived by various means such that the resulting behavior of the closed loop system meets the control objectives. In the Ramadge and Wonham theory [1], the physical model is modeled as finite state automaton. The behavior of the plant is then described in terms of the language generated by the automaton. The control of the physical model is then performed by inhibiting some events belonging to a set of controllable events, while the other events can not be prevented from occuring. In our case, the physical model is decribed as a polynomial dynamical system over $\mathbb{Z}/_{p\mathbb{Z}}$ [1], with p prime [3]. In our framework, we use an input/output approach (however systems defined as finite state automata, like in Ramadge and Wonham framework [1], can also be considered within this framework). The physical model is then represented by a polynomial dynamical system while the control of the system is performed by restricting the controllable input values to values suitable for the control goal. This restriction is obtained by incorporating new algebraic equations to the initial system, using various algebraic operations.

Usually, control objectives are expressed as *invariance*, *reachability* and *attractivity* of a given properties. However some control objectives could not be expressed as traditional objectives. They relate more to the way to get to a logical goal, than to the goal to be reached. The optimal control, very popular in classical control theory can solve these problems. This new kind of control has emerged in the nineties. Some papers concerning the optimal control theory can be found in [4, 5, 6]. In this paper, the optimal controller synthesis problem for polynomial dynamical systems is presented. It involves constructing a controller which is able to choose a sequence of inputs that will transfer the polynomial dynamical system from a set of initial states to a given set of final states while minimizing a cost function over the states through the trajectories of the system.

The remainder of this paper is organized as follows: the first section is dedicated to the presentation of polynomial dynamical systems and to an overview of of traditional control problems. In the second part, the optimal control problem is presented. Some examples of possible uses of the optimal control theory are finally presented in the third part.

## 1 Control of Polynomial dynamical systems

The first point concerns the choice of the model on which the control will be performed. We have chosen to represent the system by a polynomial dynamical system (PDS), which can be seen as an implicit equational representation of an automaton. A PDS can then be used as formal basis for verification and optimal controller synthesis purpose.

### 1.1 A Polynomial dynamical system

Formally, the general form of a PDS over $\mathbb{Z}/_{p\mathbb{Z}}$ is the following :

$$S = \begin{cases} X' & = & P(X,Y,U) \\ Q(X,Y,U) & = & 0 \\ Q_0(X) & = & 0 \end{cases} \quad (1)$$

where $X, Y, U, X'$ are vectors of variables in $\mathbb{Z}/_{p\mathbb{Z}}$ and $dim(X) = dim(X') = n$. The components of vectors


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[1] $\mathbb{Z}/_{p\mathbb{Z}}$ denotes the Galois field with $p$ elements $\{0, .., p-1\}$ with usual multiplication and addition modulo $p$.

$X$ and $X'$ represent the states of the system and are called *state variables*. $Y$ is a vector of variables in $\mathbb{Z}/_{p\mathbb{Z}}$, with $dim(Y) = m$, called *uncontrollable event variables*, whereas $U$ is a vector of *controllable event variables*, with $dim(U) = l$,. For simplicity, we can consider that the uncontrollable event variables are emitted by the system to the controller, and the controllable event variables are emitted by the controller to the system. The first equation is the *state transition equation*. It captures the dynamical behavior of the system. The second equation is called the *constraint equation*, it specifies which event may occur in a given state. The last equation defines the set of initial states. In the sequel, $x \in (\mathbb{Z}/_{p\mathbb{Z}})^n$, $y \in (\mathbb{Z}/_{p\mathbb{Z}})^m$, $u \in (\mathbb{Z}/_{p\mathbb{Z}})^l$, will denote a particular instantiation of the vectors $X$,$Y$,$U$. The behavior of such a PDS is the following: at each instant $t$, given a state $x_t$ and an admissible $y_t$, we can choose some $u_t$ which is admissible *i.e.*, such that $Q(x_t, y_t, u_t) = 0$. In this case, the system evolves into state $x_{t+1} = P(x_t, y_t, u_t)$. A *trajectory* $(x_i, y_i, u_i)_{i\in\mathbb{N}}$, initialized at $(x_0, y_0, u_0)$ is a sequence $(x_i, y_i, u_i)$, such that the pair $Q(x_i, y_i, u_i) = 0$ and $x_{i+1} = P(x_i, y_i)$.

### 1.2 The control of a polynomial dynamical system

Due to the distinction between the event variables (controllable and uncontrollable status), the events $(y_t, u_t)$ include an uncontrollable component $y_t$ and a controllable one $u_t$. We have no direct influence on the $y_t$ part which depends only on the state $x_t$. On the other hand, we have full control over $u_t$ and we can choose any value of $u_t$ which is admissible, i.e. such that $Q(x_t, y_t, u_t) = 0$. The chosen value determines the next state $x_{t+1}$, and indirectly influences the possible values for $y_{t+1}$. A PDS $S$ can be controlled by first selecting a particular initial state $x_0$ and then by choosing suitable values for $u_1, u_2, \ldots, u_n, \ldots$. Different strategies can be chosen to determine the control values. Here, we will only consider static control policies. This means that the value of the control $u_t$ is instantaneously computed from the value of $x_t$ and $y_t$. Such a controller is called a *static controller*. Formally, it is a system of two equations:

$$\begin{cases} C(X,Y,U) = 0 \\ C_0(X) = 0 \end{cases} \tag{2}$$

where the equation $C_0(X) = 0$ determines initial states satisfying the control objectives and the other one describes how the instantaneous controls are choosen; when the controlled system is in state $x$, and when an event $y$ occurs, any value $u$ such that $Q(x, y, u) = 0$ and $C(x, y, u) = 0$ can be chosen.

### 1.3 Traditional control objectives

We illustrate the use of this present framework for solving a particular traditional control synthesis problem. Suppose we want to ensure the *invariance* of a set of states $E$.

Let us introduce the operator $\widetilde{pre}$, defined by: for any set of states $F$,

$$\begin{aligned} \widetilde{pre}\ (F) \quad = \quad & \{x \in (\mathbb{Z}/_{p\mathbb{Z}})^n \ / \ \forall y \text{ admissible} \\ & \exists u, Q(x,y,u) = 0 \ \text{ and } P(x,y,u) \in F\} \end{aligned}$$

Roughly speaking, $\widetilde{pre}\ (F)$ computes the set of states from which it is always possible to reach $F$ in one transition by choosing a suitable controllable event $u$ when an event $y$ occured. Consider now the sequence $(E_i)_{i\in\mathbb{N}}$ defined by:

$$\begin{cases} E_0 &= E \\ E_{i+1} &= E_i \cap \widetilde{pre}\ (E) \end{cases} \tag{3}$$

The sequence (3) is decreasing. Since all sets $E_i$ are finite, there exists a $j$ such that $E_{j+1} = E_j$. The set $E_j$ is then the greatest control-invariant subset of $E$. Let $g_j$ be the polynomial which has $E_j$ as solution, then

$$\begin{cases} C_0(X) &= g_j \\ C(X,Y,U) &= P^*(g_j) \end{cases} \tag{4}$$

where $P^*(g)$ is the polynomial which has as solutions the set $\{(x, y, u) \ / \ P(x, y, u) \ \textit{is solution of}\ g\}$ (cf [7]), is an admissible feed-back controller and the controlled system $S_C : S + (C_0, C)$ such that

$$S = \begin{cases} X' &= P(X,Y,U) \\ Q(X,Y,U) &= 0 \\ C(X,Y,U) &= 0 \\ Q_0(X) &= 0 \\ C_0(X) &= 0 \end{cases} \tag{5}$$

verifies the invariance of the set of states $E$. Furthermore, using the same methods, we are able to ensure *attractivity*, *reachability* of a set of states. For more details about this kind of control, the reader could refer to previous works [8].

We can also consider control objectives which are conjunction of basic properties of state trajectories. However, basic properties cannot, in general, be combined in a modular way. For example, a *safety* property (*i.e.*, an invariant property) puts restrictions on the set of state trajectories which may be not compatible with an *attractivity* property. Finally, the synthesis of a controller insuring both properties (invariance of a set of states and at the same time attractivity of an other one, for example) must be effected by considering both properties *simultaneously*[2] not by combining a controller insuring safety with a controller insuring attractivity independently.

However, many properties of discrete event system cannot be stated with the help of static relations. For example if we want a signal $y$ never to take the same

[2]By simultaneously, we mean that the fix point computations (similar to (3)), used to compute the unique controller that ensures the invariance of a set of states and at the same time attractivity of an other one) are mixed in a single fix point computation.

value two consecutive times, it must satisfy the relation $\forall n,\ y_{n+1} - y_n \neq 0$. Such properties nead a dynamical controller to be enforced. The main idea is to extend the order of the initial system so that the initial control objective is reduced to a static control control objective for the new system [9].

## 2 Optimal Control

Although it is very popular in classical control theory, optimal control is a new approach for DES, which has emerged in the nineties. This approach was motivated by the fact that some control objectives could not be expressed as traditional objectives (invariance, attractivity, temporal logic). The graph theoretical formulation of the optimal control problem, for a class of DESs represented by automata, was given by Sengupta and Lafortune [4] and a supervisory optimal control by Lin [10], whereas Ionescu [5] presented in an optimization method for a system specified with a temporal logic. Passino and Antaklis [6] have associated a cost with every state transition of a DES and examined the optimization in respect of an event cost function. They use the A* algorithm to perform the controller synthesis computation. Other work in this area can be found in [11, 12, 13].

In this section, the optimal controller synthesis problem for PDSs is presented. It involves constructing a controller which is able to choose a sequence of inputs that will transfer the system from a set of initial states to a given set of final states while minimizing a cost function. We assume that all the events are controllable (for notational simplicity); that means that there is no perturbation. The control strategy is defined as follows. A cost is attached to each control variable. In our approach, we recall that a control is a vector of controllable variables. It is then possible to have synchronization between different controls. To be more general, a cost is also attached to each state. Finally, the optimal control strategy is based on dynamical programming [14]. An overview of the optimal controller synthesis problem under perturbations will be introduced in section 2.3

### 2.1 The optimal controller synthesis problem for PDS's

Since, in this section, we only consider the case where there is no perturbation, the PDS can be rephrased as follows:

$$\left\{\begin{array}{rcl} X' & = & P(X,U) \\ Q(X,U) & = & 0 \\ Q_0(X_0) & = & 0 \end{array}\right. \quad (6)$$

where $X$ (respectively) $U$ is the vector of state variables in $(\mathbb{Z}/p\mathbb{Z})^n$ (respectively vector of commands in $(\mathbb{Z}/p\mathbb{Z})^l$). In more usual terms, this defines the dynamical system

$$\left\{\begin{array}{rcl} x_{i+1} & = & P(x_i,u_i) \\ Q(x_i,u_i) & = & 0 \\ Q_0(x_0) & = & 0 \end{array}\right. \quad (7)$$

Now we are given a set $\mathcal{X}_{init}$ of initial states (i.e. the solutions of the polynomial $Q_0(X_0)$), and a set $\mathcal{X}_f$ of final states. A *valid control sequence* of the system is then a sequence of controls $u_0, \ldots, u_{n_f}$ which transfers the system from one of the initial states to one of the final states:

$$\mathcal{X}_{init} \ni x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} \ldots \xrightarrow{u_{n_f-1}} x_{n_f-1} \xrightarrow{u_{n_f}} x_{n_f} \in \mathcal{X}_f \quad (8)$$

where $n_f$ is the first hitting time of $\mathcal{X}_f$, for the considered valid control sequence.

The cost of a trajectory $s_x = (x_0, \ldots, x_{n_f})$ associated to the corresponding valid control sequence $s_u = (u_0, \ldots, u_{n_f-1})$ is defined as follows:

$$C(s_x, s_u) \stackrel{def}{=} \sum_{i=0}^{n_f} c''(x_i) + \sum_{i=0}^{n_f-1} c'(u_i) \quad (9)$$

where $c''(x_i)$ (resp. $c'(u_i)$) is the cost attached to the state $x_i$ (resp. the event $u_i$). Thus we wish to minimize (9) subject to (7,8). This is achieved by computing backward recursively the *value function,* following a variant of the Bellman principle [14]. The actual computation of this is described next.

### 2.2 Optimal controller synthesis

This section outlines a computational method for synthesizing control equations which will force the system to evolve from an initial set of states $\mathcal{X}_{init}$ to a final set of states $\mathcal{X}_f$ with a minimal cost. Finally, the various stages of the algorithm are described in this section. Problem (9,7,8) is indeed a time invariant finite horizon problem, since the final instant $n_f$ is a first hitting time. Hence one should expect a time invariant value function together with a time invariant controller, we calculate both next.

#### 2.2.1 Optimal value function computation

First, we compute a value function $V_{\min}(x)$, the minimal cost, for a given admissible control sequence and initial state $x$, to reach $\mathcal{X}_f$. To this end, we consider a sequence of value functions $(V_i)_{i\in\mathbb{N}}$, initialized by:

$$\left\{\begin{array}{rcll} V_0(x) & = & c''(x) & \text{for } x \in \mathcal{X}_f \\ & = & \infty & \text{for } x \notin \mathcal{X}_f \end{array}\right. \quad (10)$$

This sequence of value functions is updated by backtracking from the final states to the initial states. This is described now in terms of our framework. Set $\mathcal{X}_0 = \mathcal{X}_f$. In a first step, we compute the set of states from which $\mathcal{X}_0$ can be reached in one transition.

$$\mathcal{X}'_1 = \{x \in (\mathbb{Z}/p\mathbb{Z})^n \ /\ \exists u, Q(x,u) = 0 \Rightarrow P(x,u) \in \mathcal{X}_0\}$$

To each state of $\mathcal{X}'_1$ is attached the cost $V_1(x)$:

$\forall x \in \mathcal{X}'_1$,

$$V_1(x) = \min\{V_0(x), c''(x) + \min_{u/Q(x,u)=0}\{c'(u) + V_0(P(x,u))\}\}$$

Let us now assume that we are at iteration $i$, then $\mathcal{X}_{i+1}$ and $V_{i+1}$ are computed as follows:

$$\left\{\begin{array}{rcl} \mathcal{X}'_{i+1} & = & \{x \in (\mathbb{Z}/_{p\mathbb{Z}})^n \ / \\ & & \exists u, Q(x,u) = 0 \Rightarrow P(x,u) \in \mathcal{X}_i\} \\ \mathcal{X}_{i+1} & = & \mathcal{X}'_{i+1} \bigcup \mathcal{X}_i \\ V_{i+1}(x) & = & \widehat{V},\ \forall x \in \mathcal{X}'_{i+1} \end{array}\right. \tag{11}$$

where

$$\widehat{V} = \min\{V_i(x), c''(x) + \min_{(u/Q(x,u)=0)} \{c'(u) + V_i(P(x,u))\}\}$$

If a cost has already been computed (in a previous step), then this cost is compared with the new one, and the minimum is taken. This way cycles on states can be considered.

It is easy to see, that $\forall i \in \mathbb{N},\ V_i(x) \geq V_{i+1}(x)$. Since $V_i$ has values in $\mathbb{N}$, there exists a $k$ such that $V_{k+1} = V_k\ \forall x \in (\mathbb{Z}/_{p\mathbb{Z}})^n$ (note that, in this case, we also have $\mathcal{X}_{k+1} = \mathcal{X}_k$). This is our $V_{\min}(x)$ value function. If $V_{\min}(x) < +\infty$ and $x \in \mathcal{X}_{init}$, then $x$ is a valid initial state.

This description of the algorithm provides us directly with an efficient implementation using both ADD (Arithmetic Decision Diagrams) and BDD (Binary Decision Diagrams) technologies (see [9] for details). Using such implementation, the actual computation of $\mathcal{X}_i$ and $V_i$ are performed without state space enumeration.

### 2.2.2 Control strategy

The classical way to recover the (time invariant) feedback control from Bellman recursion (11) would consist in keeping a table providing, for each state $x$, the optimal controls $u$ if any. In our case this would result in combinatorial explosion. Instead, we shall regard the value function $V_{\min}$, computed using (11), as specifying a preorder relation on the set of states, from which a controller could be computed.

Using the preceding notations, for a given state $x$ and associated set $\{u_1, \ldots, u_k\}$ of admissible controls, the pair $(x, u)$ is said to be *preferred* to the other pairs $(x, u_i)$, if and only if

$$\begin{array}{l} \forall u' \in \{u_1, \ldots, u_k\}, \\ \quad c'(u') + V_{\min}(P(x,u')) \geq c'(u) + V_{\min}(P(x,u))\ , \end{array}$$

i.e., we set:

$$(x,y) \succeq (x',y') \Leftrightarrow c_{xu}(x,u) \geq c_{xu}(x',u') \tag{12}$$

with $c_{xu}(x,u) = V_{\min}(P(x,u)) + c'(u)$.

Using algebraic methods rely on the ADD [9], it is possible to re-express the corresponding order relation as a polynomial relation and further to synthesize the control equations.

Relation (12) is then translated in terms polynomial relations. To this purpose, we introduce the sets $A_1, A_2, \ldots, A_{k_{max}} \in \mathbb{Z}/_{p\mathbb{Z}}[X,U]$, where $k_{max}$ is the maximum of the value function $V_{\min}$, such that,

$$\begin{array}{l} \forall i \in [1..k_{max}], \\ A_i \ = \ \{(x,u) \in (\mathbb{Z}/_{p\mathbb{Z}})^{n+m} / c_{x,u}(x) = i\}\ . \end{array} \tag{13}$$

Using the $(A_i)_{i \in [0,\ldots,kmax]}$, relation (12) becomes:

$$\begin{array}{c} (x,u) \succeq (x',u') \\ \Leftrightarrow \\ \exists i \in [0,..,k_{max}],\ (x,u) \in A_i \Rightarrow (x',u') \in \bigcup_{j=i}^{n} A_j\ . \end{array}$$

Let $g_0, \ldots, g_i$ be the principal generators [3] of the sets $A_0, \ldots, A_i$ then the preorder relation $\succeq$ defined by (12) can be expressed as polynomial relation:

$$(x,u) \succeq (x',u') \Leftrightarrow R_{\succeq}(X,X',U,U') \ = \ 0$$

where

$$R_{\succeq}(X,X',U,U') \ = \ \prod_{i=1}^{n}\{(1 - g_i(X,U))(\prod_{j=i}^{n}(g_j(X',U')))\}$$

$R_{\succeq}(x,x,u,u') = 0$ means that, in a given state $x$, the trajectory initialized in $x$ with the control $u$ will have a smaller cost than the one with the control $u'$. Finally, the control equations are then given by:

$$\begin{array}{c} R(X,U) = 0 \\ \Leftrightarrow \\ \forall U' \in (\mathbb{Z}/_{p\mathbb{Z}})^l, (Q(X,U') = 0 \Rightarrow R_{\succeq}(X,X,U,U') = 0) \end{array}$$

In other words, the control chooses, for a state $x$, an admissible control which makes it evolve to the state which is maximal for the relation $R$. The controller of the system is then provided by the following polynomial relation:

$$R(X,U) = \forall elim_{U'}[(1 - Q(X,U'))(R_{\succeq}(X,X,U,U'))].$$

where the solutions of the polynomial $\forall elim_{X'}(P(X,X'))$ is the set which is equal to $\{x \ / \ \forall x,\ (x,x') \text{ is solution of } P\}$.

The controlled system is then obtained by adding the controller $R(X,U)$ to the initial system (6) and by restricting the set of initial states $\mathcal{X}_{init}$:

$$S_C \ = \ \left\{\begin{array}{rcl} X' & = & P(X,U) \\ Q(X,U) & = & 0 \\ R(X,U) & = & 0 \\ I_{max}(X_0) & = & 0 \end{array}\right.$$

where the solution of $I_{max}(X_0)$ is the subset of initial states $\mathcal{X}_{init}$ with a minimal cost, i.e., such that:

$$I_{max}(x) = 0 \Leftrightarrow \forall x' \in \mathcal{X}_{init}, V_{min}(x) \leq V_{min}(x')$$

In other words, we remove from the initial set of states $\mathcal{X}_{init}$ all the states for which there exists a state with a greater cost.

[3] By principal generator of a set $E$, we mean the polynomial for which its solutions are the set $E$. In our framework, this polynomial always exists.

### 2.3 Optimal control with perturbations

If the PDS has uncontrollable event variables, the computation of the value function $V_{\min}$ is quite different. In fact, we cannot minimize directly the cost of uncontrollable events. We take a minmax game theoretic approach. The computation of the value function is then realized by taking, for a given state $x$, the maximal cost for the admissible uncontrollable event, and for this pair $(x, y)$, by choosing the control with the minimal cost.

As for the case with no perturbation, the computation of $V_{\min}$ is realized by backtracking from the final states to the initial states:

$$\left\{ \begin{array}{lcl} \mathcal{X}'_{i+1} & = & \{x \in (\mathbb{Z}/p\mathbb{Z})^n \ /\forall y \text{ admissible} \\ & & \exists u, Q(x,y,u) = 0 \Rightarrow P(x,y,u) \in \mathcal{X}_i\} \\ \mathcal{X}_{i+1} & = & \mathcal{X}'_{i+1} \bigcup \mathcal{X}_i \\ V_{i+1}(x) & = & \widehat{V}, \quad \forall x \in \mathcal{X}'_{i+1} \end{array} \right.$$

where

$$\begin{array}{lcl} \widehat{V} & = & \min\{V_i(x)\ ,\ c''(x) + \max_{y/Q'(x,u)=0}\{c_y(y) + \\ & & \min_{(u/Q(x,y,u)=0)}\{c'(u) + V_i(P(x,y,u))\}\}\} \end{array}$$

Then control synthesis is quite similar to the one developed in the previous section.

## 3 Example

In this section, we briefly mention some examples which illustrate the use of optimal control.

First, the optimal control theory can be used to perform an excursion of minimal duration from the set of initial states. To synthesize such a controller, we attach to each event a cost equal to one. The final states are taken identical to the initial states. By computing the value function $V_{\min}$, we attach to the initial state the minimal number of transitions which is necessary to come back to one of the set of initial states. This kind of control could be useful to perform quickest resetting of a system when some external event is sent to the system, by a human operator for example.

Alternatively, suppose we want to ensure both the invariance of a set of states $E$ and the optimal control of the system $S$ from the initial states into $E$. We recognize here the notion of optimal attraction of a set of states $E$ as introduced by Brave and Heymann in [12]. The computation of such a controller consists of two steps.

In the first step, we synthesize the controller $(C_0, C)$ that ensures the invariance of the set of states $E$. If such a controller exists, then according to section 1.3, the set of states $F$ such that $x \in F \Leftrightarrow C_0(x) = 0$ is invariant for the controlled system $S_c = S + (C_0, C)$. From this point, in the second step, we can compute the optimal controller $(C_1, C')$, that will drive the system $S$ from the initial set of states to $F$, according to a given cost function over the states and the event of the system $S$.

Let us now consider the following controlled system $S'_C$.

$$\left\{ \begin{array}{rcl} Q(X,Y,U) & = & 0 \\ X' & = & P(X,Y,U) \\ C_1(X) & = & 0 \\ C'(X,Y,U) * C_0(X,Y,U)^{p-1} & = & 0 \\ C(X,Y,U) * (1 - C_0(X,Y,U)^{p-1}) & = & 0 \end{array} \right.$$

It is straightforward to see that the controller $C'$ is active if and only if the system is not in $F$, *i.e.*, $C_0(X,Y,U)^{p-1} = 1$ (at the same time the controller $C$ is not active) and one time the system has reached the set of states $F$ ($C_0(X,Y,U) = 0$), the controller that ensures the invariance of $E$ becomes active and $C'$ becomes inactive since $C_0$ is equal to zero.

We can also compose two (or more) optimal control synthesis problems. Suppose that the system is initialized in a set of states $\mathcal{X}_0$, and that the first goal to achieve is to reach the set of states $\mathcal{X}_1$ with a minimal cost. Suppose now that once the goal is achieved, the system must come back to the set of states $\mathcal{X}_0$. In order to perform both goals, we first compute a controller $C_1$ which ensures the first goal for the system $S$. A second controller $c_1$ which ensures the second goal for the system $S$ is then computed. Finally by composing the two controller $C_1$ et $C_2$ in the following way, we obtain a controlled PDS which achieves the global goal.

$$S_c = \left\{ \begin{array}{rcl} X' & = & P(X,Y,U) \\ G' & = & f(G) \\ Q(X,Y,U) & = & 0 \\ Q'(G,X,Y,U) & = & 0 \\ (1-G) * C_1(X,Y,U) & = & 0 \\ (1+G) * C_2(X,Y,U) & = & 0 \\ \mathcal{X}_0(X_0) & = & 0 \\ f_0(G_0) & = & 0 \end{array} \right.$$

where $G$ is a new state variable, which is equal to 1 when the system tries to achieve the first goal and is equal to -1 when the system tries to achieve the second goal.

## 4 Conclusion

In this paper, we have shown the usefulness of control theory concepts for the class of polynomial dynamical systems over a finite Galois field (in particular $\mathbb{Z}/p\mathbb{Z}$). Even if traditional control can be performed, we showed that using the same algebraic framework, optimal control synthesis problem can also be performed.

In our framework, the optimal control consists in constructing a controller which is able to choose a sequence of inputs that will transfer the system from a set of initial states to a given set of final states while minimizing a cost function. The computation of such a controller can be split into two different stages. During the first one, we compute a value function $V_{\min}(x)$, the minimal cost, for a given admissible control sequence and initial state $x$, to reach the final set of states. The value function $V_{\min}$

is then regarded as specifying a preorder relation on the set of states, from which a controller can be computed.

However, the use of order relation to express control objectives cabe separately considered and gives access to another class of static optimal control This kind of control can be used to synthesize control objectives which relate more to the way to get to a logical goal, than to the goal to be reached. Morever, the controller coming from traditional control objectives are not deterministic, in the sense that for a given state and an admissible uncontrollable event fired, different controllable events can be chosen by the controller. Therefore, in order to obtain explicit control laws over the controllable event variables, this new kind of control objectives must encompass traditional control objectives (see [15] for more details). Finally, we can notice that the optimal control theory presented in this paper has been implemented for a particular class of polynomial dynamical system over $\mathbb{Z}/_{3\mathbb{Z}}$. This kind of system results from the translation of a SIGNAL program [16, 3] (not presented here). We then have a powerful environment to describe the model for real-time data-flow system, on which control can be performed.

The theory of polynomial dynamical systems over $\mathbb{Z}/_{p\mathbb{Z}}$ deserves much more research. One issue is the control under partial observations or in a slightly different domain the control of implicit non-deterministic polynomial dynamical systems. Some other perspectives concern the synthesis of fault tolerance controllers or the synthesis of controllers with control objectives expressed as properties that depend on the behavior of numerical variables.

## References


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