MORE ON SPARSE REPRESENTATIONS IN ARBITRARY BASES.

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Abstract: The purpose of this contribution is to generalize some recent results on sparse representations of signals in redundant bases. The question that is considered is the following : let A be a known (n, m) matrix with $m > n$, one observes $b = AX$ where X is known to have $p < n$ nonzero components, under which conditions on A and p is it possible to recover X by solving a convex optimization problem such as a linear or quadratic program? The solution is known when A is the concatenation of two unitary matrices, we extend it to arbitrary matrices.

Keywords: Sparse representations, linear programming, quadratic programming..

1. INTRODUCTION

Let us consider a set of m n-dimensional vectors a_j with $m > n$ and let us denote A the (n, m) matrix having these vectors as columns. Any linear combinations b of these m vectors can then be written as: $b = AX$ with X a m dimensional vector of weights.

If X has just a few non-zero components it may be possible to recover the value of X from the knowledge of b and the aim of this contribution is to give condition under which this is feasible.

If the true solution is known to be sparse, one seeks a mean allowing to single out among the infinity of possible solutions the or a solution having the smallest possible number of nonzero components i.e. the solution of the following optimization problem :

$$\min_X \ \|X\|_0 \qquad \text{subject to :} \quad AX = b \qquad (P_o)$$

where $\|X\|_0$ denotes the number of non-zero components in X. This is a difficult problem that can only be solved using a combinatorial approach i.e. testing systematically all the potential combinations of columns. This approach is thus unfeasible and it is usual to consider instead the following much simpler optimization problem :

$$\min_X \ \|X\|_1 \qquad \text{subject to :} \quad AX = b \qquad (LP)$$

where $\|X\|_1 = \sum |x_i|$ denotes the ℓ_1 norm of X. The problem (LP) is easily transformed into a linear program whose solution is straightforward to obtain.

Donoho and Huo (Donoho, 2001) investigated the case where A is the concatenation of two square orthogonal matrices U_1 and U_2 : $A = [U_1 \ U_2]$, and proved that if :

$$\|X\|_0 \leq \frac{1}{M} \quad \text{with :} \ M = \sup_{i \neq j} |a_i^T a_j| \qquad (1)$$

the sparsest solution is unique. They further established that under the stronger condition

$$\|X\|_0 \leq \frac{1}{2}(1 + \frac{1}{M}) \quad \text{with :} \ M \ \text{ as above} \qquad (2)$$

the sparsest solution is also the unique solution of (LP). This result has later been refined in (Elad, 2002) but still in the case where A is the concatenation of two orthogonal bases.

Below we prove that condition (2) is indeed sufficient for arbitrary A matrices built upon vectors with unit euclidean norm. This normalization is obviously satisfied for the concatenation of orthogonal matrices.

A similar result has independently been obtained in (Gribonval, 2002) using a completely different approach.

Instead of (LP) we consider a parametrized quadratic program (QP) that encompasses (LP) as a special case and show that if $X_0 \in \{X|AX = b\}$ satisfies (2) then it is the unique minimum point of both (QP), in a sense to be defined later, and (LP). This establishes the result since if a sparser representation existed, the same reasoning would hold and one would arrive at a contradiction since both (LP) and (QP) are convex optimization problems for which all minimum points are located in a convex set that reduces to an unique point under (2).

To establish this result we merely apply more general but non explicit results presented in (Fuchs, 1997), (Fuchs, 1998) to this very specific problem. The proof we present goes through if the vectors a_j are not normalized or the weights known to be greater or equal to zero but its last part would be more intricate.

2. THE CRITERION

Let us consider the following optimization problem:

$$\min_X \; \frac{1}{2}\|AX - b\|_2^2 + h\|X\|_1 \; , \quad h > 0 \qquad (QP)$$

If one introduces new variables $x_i^+ = \max(x_i, 0)$, $x_i^- = \max(-x_i, 0)$ and replaces x_i by $x_i^+ - x_i^-$ and $|x_i|$ by $x_i^+ + x_i^-$, this unconstrained nonsmooth optimization problem is converted into a quadratic program where these new variables x_i^+ and x_i^- are constrained to be greater or equal to zero (Luenberger, 1973).

This optimization problem has thus an unique global minimum that can be obtained using standard algorithms available from any scientific program library. This criterion and similar ones have been considered for a while now (Chen, 1999),(Moal, 1998), (Fuchs, 1998), (Fuchs, 1999).

To assess the role played by h in this criterion, one can make the following remarks about the optimum X^* of (QP) as h goes from 0 to $+\infty$.

$\diamond$ For $h = 0$, one is left with $\min_X \; \|b - AX\|_2^2$, and since there are more unknowns than equations $(m > n)$, the value of the minimum is zero and it is attained for all points in a convex set (a linear manifold), some of them having at most n nonzero components.

$\diamond$ For $h = 0^+$ i.e. for h positive and arbitrarely close to zero, the solution is attained at the point(s) in the previous set having least ℓ_1-norm. Again there is a solution having at most n nonzero components.

$\diamond$ For $h \geq \; \| A^T b \|_\infty$, the optimum is attained at $X^* = 0$.

If X is sparse, one can thus expect that for h small enough, it may be possible to recover the columns that were used to build b with however a biased value of X. This bias converges to zero with h and can be corrected.

Further insight on the role of h can be gained from the dual of (QP), which is (Fuchs, 2001) :

$$\begin{aligned} &\min_X \|AX\|_2^2 \\ \text{subject to} \quad &\|A^T(AX - b)\|_\infty \leq h \quad (DQP) \end{aligned}$$

The constraint of the dual says that -at the optimum- the residues or reconstruction errors in r defined as $r = b - AX$ are such that their correlations with the columns of A never exceeds h. Note that these correlations are the outputs of the matched filter applied to the reconstruction errors. Since the ℓ_2 norm of the columns of A is equal to one, this also says that this criterion allows for reconstruction errors that are of order h. Their size can thus be fixed by tuning h.

3. OPTIMALITY CONDITIONS FOR (QP)

In order to find the conditions under which the true value of X is also the location of the unique optimum of (QP) we propose to seek the conditions under which this point satisfies the necessary and sufficient condition (NSC) for a strict global minimum of (QP).

The results we present in this section are established in the appendix. The criterion (QP) is unsconstrained, convex but not continuously differentiable. A necessary and sufficient condition (NSC) for X^* to be a global minimum of (QP) is that the vector zero be a sub-gradient of the criterion at X^* (Fletcher, 1991). To write the NSC, it is helpful to distinguish between the nonzero components and the zero components of X^*. We denote $\bar{X}^*$ the reduced dimensional vector of dimension $\|X^*\|_0$ built upon the nonzero components of X^*. Similarly $\bar{A}$ denotes the associated columns in A. One then has e.g. $AX^* = \bar{A}\bar{X}^*$. The NSC characterizing X^* are then :

$$\begin{aligned} &\bar{A}^T(b - \bar{A}\bar{X}^*) = h \;\text{sign}\bar{X}^* && (NSC_1) \\ &|a_j^T(b - A\;X^*)| \leq h \;\; \text{for} \;\; a_j \notin \bar{A} && (NSC_2) \end{aligned}$$

where $\text{sign}(x_j) = 1, \; 0, -1$ when x_j is respectively $< 0, \; = 0, \; > 0$. The first condition that concerns the nonzero components of X^* collected in $\bar{X}^*$, yields :

$$\bar{X}^* = \bar{A}^+ \, b - h \, (\bar{A}^T \bar{A})^{-1} \, \text{sgn}\bar{X}^* \qquad (3)$$

with $\bar{A}^+ = (\bar{A}^T \bar{A})^{-1} \bar{A}^T$ the pseudo-inverse of $\bar{A}$. This is only an implicit relation that does not allow to compute $\bar{X}^*$ since $\bar{X}^*$ appears on both sides. The optimum of (QP) can only be obtained through an iterative search. The second term (3) is a bias term induced by the regularization term in (QP). Once $\bar{X}^*$, and thus $\bar{A}^*$, are known, it is easy to remove the bias.

The second condition (NSC_2) concerns the zero components in X^* those associated with the columns that are not in $\bar{A}$. If the inequalities in (NSC_2) are satisfied strictly, X^* is a strict unique minimum of (QP). Both conditions will play a central role in the sequel where (NSC_2) will appear as an identifiability or separability condition,

4. THE SEPARABILITY CONDITION

We are now ready to answer the question posed in the introduction : given an $(n,\ m)$ matrix A with $m > n$ and a vector b linear combination of $p < n$ columns of A, under which condition can this linear combination be retrieved by solving (QP)?

Let us denote X_0 the exact solution, $\bar{X}_0$ the p-dimensional vector built upon the nonzero component in X_0 and $\bar{A}_0$ the (n, p) dimensional matrix built accordingly with the associated columns of A so that $b = AX_0 = \bar{A}_0 \bar{X}_0$.

For a nonzero h, the best we can expect is that X_0 and X^*, the solution to (QP), have their nonzero components at the same locations and with the same signs. From section 3 we know that this fully characterizes the optimum of (QP), which is then equal to, (see (3)) :

$$\begin{aligned} \bar{X}^* &= \bar{A}_0^+ \, b - h \, (\bar{A}_0^T \bar{A}_0)^{-1} \, \text{sign}\bar{X}_0 \\ &= \bar{X}_0 - h \, (\bar{A}_0^T \bar{A}_0)^{-1} \, \text{sign}\bar{X}_0 \qquad (4) \end{aligned}$$

It remains to check if this solution completed by zeroes satisfies (NSC_2), the second part of the NS conditions :

$$|a_j^T \, (b - \bar{A}_0 \bar{X}^*)| < h \quad \forall a_j \notin \bar{A}_0$$

Replacing $\bar{A}_0 \bar{X}^*$ by its value and introducing the vector d_0 :

$$d_0 = \bar{A}_0^{+T} \, \text{sign}\bar{X}_0 \qquad (5)$$

these conditions become :

$$|a_j^T d_0| < 1 \quad \forall a_j \notin \bar{A}_0 \qquad (6)$$

To summarize, the following conditions have thus to be satisfied for (QP) to retrieve the *true* solution :

• h has to be small enough for : $\text{sign}\bar{X}^* = \text{sign}\bar{X}_0$, in (4)

• the scenario-dependent vector $d_0 = \bar{A}_0^{+T} \text{sign}\bar{X}_0$ must be such :

$$|a_j^T d_0| < 1 \quad \forall a_j \notin \bar{A}_0$$

The first of these two conditions depends upon the magnitude of the components while the second is independent of it. The second says that there must exist two *separating* hyperplanes $H_\pm$, associated with a single vector d_0 : $H_\pm = \{a | a^T d_0 = \pm 1\}$ such that the true columns lie in these hyperplanes $\bar{A}_0^T d_0 = \text{sign}\bar{X}_0$ and the wrong ones lie in between $|a_j^T d_0| < 1$.

This is why we call the conditions (6), which are already presented in (Fuchs, 1997), (Fuchs, 1998), the separability conditions.

5. THE SPARSITY CONDITION

Let us transform the separability condition (5, 6):

$$|a_j^T \bar{A} (\bar{A}^T \bar{A})^{-1} sign(\bar{X}| < 1 \ \ \forall a_j \notin \bar{A}$$

into the more useable but also more conservative condition (2).

Remember that :

$$M = \sup_{i \neq j} |a_i^T a_j| \qquad \text{for} \quad 1 \leq i, j \leq m$$

Let us rewritte : $\bar{A}^T \bar{A} = I - H$ where H is built with the non-diagonal elements of $\bar{A}^T \bar{A}$. The diagonal elements of $\bar{A}^T \bar{A}$ are equal to one since we assume $a_j^T a_j = 1,\ \forall j$.

Since all the non-zero components in the order p square matrix H are smaller than M, its spectral radius $\rho(H)$ satisfies : $\rho(H) \leq (p-1)M$ by Gersgorin theorem (Householder, 1964).

If $p < 1 + (1/M)$, a condition implied by (2) holds, the spectral radius verifies $\rho(H) < 1$ and one can apply Neuman's lemma (Householder, 1964) to get :

$$(\bar{A}^T \bar{A})^{-1} = (I - H)^{-1} = I + \sum_{k>0} H^k$$

For any matrix B, we denote $|B|$ the matrix with entries $|b_{i,j}|$ and write $B < C$ if $b_{i,j} < c_{i,j}, \forall i, j$, one then has :

$$|(\bar{A}^T \bar{A})^{-1}| = |I + \sum_{k>0} H^k| \leq I + \sum_{k>0} |H^k|$$

$$\leq I + \sum_{k>0} |H|^k \leq I + \sum_{k>0} ME^k = (I - ME)^{-1}$$

where E is a square matrix whose elements are equal to one except those on the diagonal. We also write $E = \mathbf{1} - I$, where $\mathbf{1}$ denotes here and below a matrix of ones with adequate dimension. It is then easy to check that :

$$I - ME = (1+M)I - M\mathbf{1}$$
$$\Rightarrow (I - ME)^{-1} = \alpha I + \beta \mathbf{1}$$

with : $\alpha = \frac{1}{1+M}$ and $\beta = \frac{M}{(1+M)(1+M-Mp)}$ which in turn leads to : $|(\bar{A}^T\bar{A})^{-1}| < (\alpha+\beta)I + \beta E$.

Since : $|a_j^T\bar{A}| < M\mathbf{1^T}$, condition (6) yields :

$$|a_j^T\bar{A}(\bar{A}^T\bar{A})^{-1} sign(\bar{X})|$$
$$< M\mathbf{1^T}|(\mathbf{\bar{A}^T\bar{A}})^{\mathbf{-1}}|\mathbf{1}$$

substituting the bound of $|(\bar{A}^T\bar{A})^{-1}|$ one gets :

$$|a_j^T\bar{A}(\bar{A}^T\bar{A})^{-1} sign(\bar{X})|$$
$$< M(p(\alpha+\beta) + (p^2 - p)\beta) < 1$$

which after some easy computations becomes condition (2) if one replaces p by $\|X\|_0$.

This completes the proof of the result announced in the introduction and establishes that if h is taken small enough, the true value X_0 and the solution X^* of (QP) have their nonzero components at the same locations and with the same signs. The relation (4) indicates precisely what is meant by "small enough" and taking $h = 0^+$ garanties that $X_0 = X^*$ as is the case for the (LP) criterion.

Remark : It may be of interest to distinguish between the vectors belonging to a potential selection : $a_i \in \bar{A}_0$ and the others and define :

$$M_o = \sup_{i \neq j} |a_i^T a_j| \qquad \text{for} \quad a_i, a_j \in \bar{A}_0.$$

The condition then becomes :

$$\|X\|_0 \leq \frac{1 + M_o}{M + M_o}$$

which is weaker than (2) if $M_o < M$. In a detection-estimation context (Fuchs, 1999) where the aim is to detect which components are present in b, M_o is linked to the resolution one expects to be able to attain while M is linked to the precision with which one expects to be able to locate the components. It is then quite natural to have $M_o \ll M$ and this condition is thus less conservative than (2).

6. THE SPECIAL CASE OF THE ℓ_1 NORM

As indicated in section 2, if h in (QP) decreases to zero, the optimum of (QP) converges to the solution of (LP) and relation (4) :

$$\bar{X}^* = \bar{X}_0 - h\ (\bar{A}_0^T\bar{A}_0)^{-1}\ \text{sign}\bar{X}_0$$

indicates clearly how the tradeoff between both terms in (QP) affects the optimum.

A direct analysis of the condition under which (LP) retrieves the true solution can indeed be achieved quite easily along the lines used for the (QP) criterion. Let us do so in this section using the notations introduced above.

Let us recall the (LP) criterion:

$$\min_X \|X\|_1 \qquad \text{subject to :} \quad AX = b \qquad (LP)$$

The principal difficulty arises from the fact that the expected solution to (LP) is degenerate i.e. it has less than n nonzero components and in order to characterize the unicity of the solution one has to introduce the dual linear program :

$$\min_d d^T b \quad \text{subject to :} \quad \|d^T A\|_\infty \leq 1 \quad (DLP)$$

which as (LP) is a linear program that is not in standard form. For the sought-for vector X_0, that is entirely characterized by the knowledge of $\bar{A}_0$, to be the unique solution of (LP) one needs to be able to associate with it a solution d_0 of (DLP). The conditions for X_0, d_0 to be optimal are simply the equality of the criterions and primal and dual feasibility, i.e.:

$$\|X_0\|_1 = d_0^T b, \quad AX_0 = b, \quad \|d_0^T A\|_\infty \leq 1$$

The first of these conditions can be transformed into : $\bar{X}_0^T sign(\bar{X}_0) = d_0^T AX_0$, the second is trivialy satisfied and the last is simialr to (6).

The condition for the vector X_0 to be the unique solution of (LP) are actually :

$$\exists\ d_0 \ \ni\ \bar{A}_0^T d_0 = sign(\bar{X}_0)$$
$$|a_j^T d_0| < 1 \ \ \forall a_j \notin\ \bar{A}_0 \qquad (7)$$

This condition is similar to (6) which has to hold for (QP), but the vector d_0 for which it has to hold is now less constrained since it belongs to a (n-p) dimensional linear manifold. The vector d_0 defined in (5) is just one possible candidate, the one of least ℓ_2 norm.

This is not a minor difference and it is quite easy to build a (toy) example where (LP) works and (QP) does not, i.e. for which d_0 in (5) does not satisfy (7) while there exists a d_0 that does.

7. CONCLUSIONS

The results proposed in (Donoho, 2001) for the case where the redundant bases is limited to the concatenation of two orthonormal bases has been extended to an arbitrary set of vectors.

Using (QP) instead of (LP) further allows to handle the case where noise is present in the observations i.e. if

$$b = AX + e$$

with e a vector of white gaussian noise, for instance. In this situation, the relative magnitudes of the nonzero components in X and standard deviation of the noise model plays a crucial role and the problem of the recovery of the true decomposition of X is of a different nature. These topics have been investigated with the (QP) criterion in (Fuchs, 1998), (Fuchs, 1999) (Fuchs, 2001) and in the case of the following extended (LP) criterion :

$$\min_X \|X\|_1 \quad \text{subject to :} \quad \|AX - b\|_\infty \leq \rho$$

in (Fuchs, 1996), (Fuchs, 1997), (Fuchs, 2000).

8. APPENDIX

In this appendix we establish the optimality condition for (QP) :

$$\min_X \frac{1}{2}\|b - AX\|_2^2 + h\ \|X\|_1 \qquad \text{(QP)}$$

The criterion (QP) is convex but not continuously differentiable. It can be transformed into a quadratic program and the optimality conditions can be obtained by writing the Kuhn Tucker conditions which are necessary and sufficient for convex problems.

We use here a more direct way in writing that a necessary and sufficient condition (NSC) for X^* to be a global minimum of (QP) is that the vector 0 is a sub-gradient of the criterion at X^* (Fletcher, 1991). A vector γ is a sub-gradient of f at X^* if $f(X) \geq f(X^*) + \gamma^T(X - X^*)$. Since (QP) is non-smooth at zero only, it is worthwhile to distinguish the zero components from the non-zero components in X^*. We collect the non zero components of X^* in $\bar{X}^*$ and the associated columns of A in $\bar{A}$ so that $AX^* = \bar{A}\bar{X}^*$ and introduce the notation $\text{sign}(X)$ which is such that $\|X\|_1 = X\text{sign}(X)$.

• The sub-gradient is unique and equal to the gradient for the non-zero components in $\bar{X}^*$. By nulling the gradient of (QP) with respect to $\bar{X}$ at $\bar{X}^*$, one gets :

$$-\bar{A}^T(b - \bar{A}\bar{X}^*) + h\ \text{sign}\bar{X}^* = 0$$

which leads to the following implicit expression for the non-zero components of X^* :

$$\bar{X}^* = \bar{A}^+ b - h\ (\bar{A}^T\bar{A})^{-1}\ \text{sign}\bar{X}^*$$

where $\bar{A}^+ = (\bar{A}^T\bar{A})^{-1}\bar{A}^T$ denotes the pseudo-inverse of $\bar{A}$.

• The vector 0 is a sub-gradient for the zero components in X^*, if the criterion increases when they are taken non zero. This is the case if the absolute value of the partial derivative of the quadratic term in (QP) is smaller than h. For the j-th component of X one gets :

$$|a_j^T(b - A\ X^*)| \leq h \qquad \forall a_j\ \notin \bar{A}$$

The NSC for X^* to be a minimum of (QP) are thus :

$$a_j^T(b - AX^*) = h\ \text{sign}\bar{x_j}^* \quad \text{for} \quad x_j^* \neq 0 \quad (NSC_1)$$
$$|a_j^T(b - A\ X^*)| \leq h \quad \text{for} \quad x_j^* = 0 \qquad (NSC_2)$$

Introducing the ℓ_∞-norm, these conditions can be rewritten more compactlty as :

$$\|\ A^T\ (\hat{b} - AX^*)\ \|_\infty \leq h$$

They are the constraints of the following dual problem :

$$\min_X \|AX\|_2^2$$
$$\text{s.t.} \quad \|A^T(AX - b)\|_\infty \leq h \qquad (DQP)$$

that is equivalent to (QP) as is easily shown by using the equivalent quadratic programs (Fuchs, 2001).

For convex optimisation problems, the minimum is unique and potentially attained for all points in a convex set. A sufficient condition for the minimum of (QP) to be strict, i.e. to be attained at an unique point, is that the inequalities in (NSC_2) are strict. This follows for instance from the second order sufficient conditions to be satisfied at a strict minimum (Luenberger, 1973).

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