THE GLOBAL MATCHED FILTER: APPLICATION TO DOA ESTIMATION WITH AN UNIFORM CIRCULAR ARRAY

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ABSTRACT

The problem of estimating the direction of arrivals (DOA's) of narrow-band sources impinging on a uniform circular array is considered. We present a method that uses as input the values of a small number of uniformly spaced beams and apply a model-fitting approach taking into account the statistical properties of the beams. The approach called the "global matched filter" fits simultaneously to the observations all the elements from a given dictionary needed to explain them. It chooses among all the so-obtained representations satisfying a given physically appealing constraint the one with minimal energy. This representation is in general quite sparse. The method drastically improves upon the conventional beamformer and has performance comparable to the best high resolution techniques. It further applies when the number of sources exceeds the number of sensors, a situation that cannot be handled by standard high resolution techniques.

1. INTRODUCTION

Estimating the directions of arrivals (DOA's) of propagating plane wawes is of interest in a number of applications including sonar, radar, mobile communications and seismology. While the widely studied uniform linear array provides coverage of 180 degrees with decreasing accuracy away from broadside, the uniform circular arrays (UCA's) provide 360^o azimuthal coverage with close to uniform accuracy. Indeed as the effective aperture analogy [1] predicts, the directional patterns synthetized with UCA's can be kept almost invariant when rotated in the plane of the array. This attractive feature led to the development of many bearing estimation techniques [2][3][4].

We propose an approach that, though it is based on beamforming techniques i.e. uses the values of a limited number of equispaced beams as input, is able to solve sources that are separated by less than the main-lobe width of the array pattern. It is based on the global matched filter concept that fits simultaneously to the observations all the elements needed to explain them. While the standard matched filter performs a correlation locally (here for one direction of arrival) the global matched filter performs all the correlations simultaneously.

For an uniform circular array (UCA) having N sensors we propose to evaluate L beams uniformlly spaced over the whole horizon and to form the L-dimensional observation vector $\hat{Y}$. We then apply the global matched filter to $\hat{Y}$. It yields a sparse representation of Y in terms of elementary responses from which one then deduces the number and characteristics of the sources that are present in the field. The performance of the approach is comparable to those of the best high resolution techniques for a quite reasonable computational cost.

As is well known for uniform linear arrays, evaluating the beams using the estimated covariance matrix of the snapshots or its "toeplerized" or "rectified" version is equivalent. One can show [12] that the same holds for any array geometry. Working behind the conventional beamformer is thus interesting because it benefits of the improved resolution induced by the rectification procedure without having to perform any preliminary projection [5][6].

2. BACKGROUND

Let us consider a planar circular array of N equispaced sensors at locations $(r\cos(k\delta), r\sin(k\delta))$ with $k \in [0, N-1]$ and $\delta = 2\pi/N$. The steering vector of the array is then:

$$d(\theta) = \left[\, e^{i\zeta\cos\theta} \;\; e^{i\zeta\cos(\theta-\delta)} \;\; \cdots \;\; e^{i\zeta\cos(\theta-(N-1)\delta)} \right]^T$$

with $\zeta = 2\pi r/\lambda$, λ being the wawelength for which the array is designed. The radius r is chosen such that the distance between two neighboring sensors equals $\lambda/2$. The array pattern when the sensor weights are all identical is then:

$$F(\theta, \theta_o) = \frac{1}{N^2}|d(\theta)^* d(\theta_o)|^2 \tag{1}$$

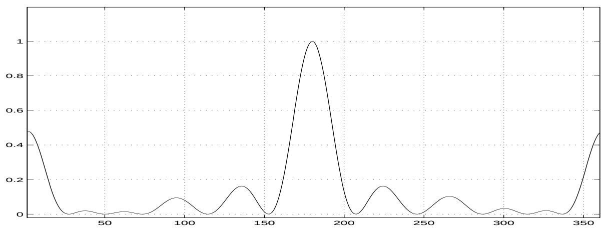

Figure 1: The beampattern for ten equispaced sensors

It is the output of the beamformer at bearing θ when there is an unique source present at bearing θ_o having unit power. For low N, this pattern depends upon both θ and θ_o and has a periodicity of $2\pi/N$. However for N as small as ten, the pattern can already be considered as invariant $F(\theta, \theta_o) = F(\theta - \theta_o, 0)$. This interesting property guarantees that the performance of the beamformer and thus of the proposed approach, is the same for all directions. Though we will use this property in the sequel to simplify the exposition it is by no means necessary for our approach to apply. The beampattern for $N = 10$ and an intersensor spacing equal to $\lambda/2$ is presented in Figure 1.

The resolution of such an UCA is considered to be close to the width of the mainlobe at halfpower i.e. 27 degrees, see Figure 1. We will improve upon this result. One can note that in the direction opposite to the main lobe there is a quite important secondary lobe. While this leads to difficulties when using the conventional beamformer output it is of no direct consequence when applying the global matched filter.

The sensor outputs are low-passed, sampled and Fourier transformed over T consecutive intervals and we denote X_k the N-dimensional vector (snapshots) containing the components at the temporal frequency of interest over time interval k. Their empirical covariance matrix:

$$\hat{R} = \frac{1}{T} \sum_{k=1}^{T} X_k X_k^* \tag{2}$$

is then also an estimate of the spectral density matrix of the sensor outputs at the considered temporal frequency whose exact value is given by:

$$R = \sum_{p=1}^{P} \alpha_p \, d\,(\theta_p)\, d(\theta_p)^* + vI \tag{3}$$

where α_p is the power of source p having bearing θ_p and v is the power of the noise assumed spatially white.

For later use, we introduce the beamformer output at bearing ϕ as being:

$$\hat{y}(\phi) = \frac{1}{N^2}\, d(\phi)^* \hat{R} d(\phi)$$

Its exact counterpart $y(\phi)$ can be rewritten, replacing $\hat{R}$ by R in (3) and using (1):

$$y(\phi) = \sum_{p=1}^{P} \alpha_p F(\phi, \theta_p) + \frac{v}{N}$$

We will later evaluate L equispaced beams and denote $\hat{y}_k$ the value of the beam at bearings ϕ_k and $\hat{Y}$ the L-dimensional column vector built upon these estimates. Similarly to $y(\phi)$ above Y can be expressed as:

$$Y = \sum_{1}^{P} \alpha_p f(\theta_p) + (\frac{v}{N})\, \mathbf{e} \tag{4}$$

where $\mathbf{e}$ is a L-dimensional vector of ones modeling the noise contribution and $f(\theta)$ represents the contribution to Y of a source at bearing θ and unit power. $\hat{Y}$ will be the input to our algorithm and in order to assess its statistical properties we will assume that the resolution of the Fourier transform is high enough (the time intervals over which the transform is taken large enough) for $\hat{R}$ (2) to be an unbiased estimate of R (3). $T\hat{R}$ can then be considered as a sample of a complex Wishart distribution with T degrees of freedom and parameter matrix R. The statistical properties of beamformer outputs are then easy to obtain [7]. For T the number of snapshots large enough, the outputs $\hat{y}_k$ of the beams can be considered to be real gaussian random variables with mean y_k and covariance matrix Σ given by:

$$\Sigma_{k,l} = E(\Delta y_k \Delta y_\ell) = \frac{1}{TN^4} |d(\phi_k)^* R d(\phi_\ell)|^2$$

The diagonal terms of this matrix are simply $E(\Delta y_k^2) = y_k^2/T$ a standard result in power spectral estimation. In the sequel we will approximate the covariance matrix of $\hat{Y}$ by an estimate of its diagonal : $\hat{\Sigma} = \text{diag}(\hat{Y})^2/T$.

3. THE GLOBAL MATCHED FILTER APPROACH

Before we apply the global matched filter to our problem we need to definite the elements upon which we want to decompose the observation vector, $\hat{Y}$. In the present situation there is no hesitation as to this choice. From (4) it follows that the building blocks we need are the $f(\theta)$-vectors introduced in (4) together with the vector $\mathbf{e}$ of one's allowing to represent the noise contribution. Let us define the set of M vectors $\{f_m,\ m \in \{0, ..., M-1\}\ \}$ of dimension L with $f_m = f(\frac{2\pi m}{M})$ (4). Our objective is to obtain a sparse representation of the observed vector $\hat{Y}$ in terms of elements of this set and the vector $\mathbf{e}$. If M is large, the discretization step $\frac{2\pi}{M}$ in bearings is small and it is possible to get an approximate reconstruction of the contribution of the P sources in $\hat{Y}$ using of the order of P elements in $\{f_m\}$. Finding such a sparse representation is however a difficult task and several solutions have been proposed, see for instance [8], [9].

The "matching pursuit" algorithm [8] is an iterative method that picks the element of the dictionary that best fits the current residual and substracts the weighted contribution of this element to form the new residual. Sparseness is not really guaranteed for this approach which is a kind of relaxation approach and works in an iterative way.

The "global matched filter" which we now present works in a global way and fits simultaneously to the signal all the necessary elements.

Let us denote F the $L \times (M+1)$ dimensional matrix whose columns are the f_m-vectors and the $\mathbf{e}$-vector, all normalized to one in ℓ_2-norm. The importance of this normalization step will appear in the sequel (see 8)). A representation of $\hat{Y}$ in terms of columns of F is of the form FX with X a vector having possibly just a few non-zero components. For a sparse representation to exist, one has to allow for reconstruction errors that also allow to erase the estimation error. The larger the allowed errors the sparser the potential representations. The simplest idea [10] is probably to ask for representations FX satisfying :

$$\|\hat{Y} - FX\|_\infty \leq \epsilon$$

where $\|R\|_\infty = \max_i \ r_i$ is the ℓ_∞ norm and ϵ is a positive bound to be fixed according to the expected value of the estimation errors. This constraint says that a representation FX is admissible if the reconstruction error nowhere exceeds ϵ. This constraint however has poor invariance properties and we do not retain it. Another possibility is to fix a bound on the residual energy :

$$\|\hat{Y} - FX\|_2^2 \leq \epsilon'$$

This is a quite natural criterion that is often used [8]. We will however retain yet another constraint with a nice physical interpretation in a beamforming context :

$$F^T(\hat{Y} - FX) \leq h \, \mathbf{e} \tag{5}$$

with $h \in \mathbb{R}^+$ and $\mathbf{e}$ an M-dimensional vector of ones. The inequality has to be taken componentwise. Note that $f_k^T(\hat{Y} - FX)$ is the output of the filter matched to a source with bearing $\frac{2k\pi}{M}$ when applied to the residual vector $R = \hat{Y} - FX$. At a representation satisfying (5), the outputs of all the filters matched to the M potential sources, when applied to the residual vector R, are smaller than a chosen threshold h.

There exist, in general, many representations satisfying these constraints and it remains to define a mean to select a sparse representation among them. We choose to minimize the energy of the representation and thus propose to solve :

$$\min_X \ \frac{1}{2} \, \|FX\|_2^2 \quad \text{subject to} \quad F^T(\hat{Y} - FX) \leq h \, \mathbf{e} \tag{6}$$

with $h \in \mathbb{R}^+$ a parameter to be tuned. At the optimum, we get the representation of $\hat{Y}$ in terms of columns of F that has lowest energy and that yields a residual vector R whose correlation with any further potential source is below a threshold h. In contrast with the matching pursuit approach which is a relaxation type algorithm, this technique selects in a *global* way (simultaneously) all the elements f_k needed to obtain the optimal model FX satisfying the constraints.

The threshold h affects the accuracy of the reconstruction in a quite sensible manner. It clearly acts as a detection threshold but, again, instead of detecting a source at a time they are detected jointly. It should be taken both small to allow for the detection of weak sources and large to reject the estimation errors, to prevent any attempt to model these errors.

At the optimum the $(M+1)$-dimensional vector X is expected to have around $2P+1$ non-zero components only : one component in front of $\frac{1}{\sqrt{L}} \, \mathbf{e}$, the column that models the noise contribution and a pair of neighboring components for each source. Indeed, since the true bearings θ_p will generically fall in between two discretization points, two columns of F will, in general, be needed to approximatively reconstruct $f(\theta_p)$.

The optimization of (6) may appear as a difficult task but using duality one can show [12] that it is equivalent to :

$$\min_{X \geq 0} \ \frac{1}{2} \, \|\hat{Y} - FX\|_2^2 \ + \ h \, \mathbf{e}^T X \tag{7}$$

Both are quadratic programs whose unique global minimum is obtained using standard and robust routines for quadratic programs available in any scientific programs library.

4. APPLICATION TO THE LOCALIZATION

4.1. Fixing the discretization steps

Besides h, we have to fix the two parameters L and M before applying criterion (6) to our problem. L is the number of beams to be evaluated and M fixes the discretization step $\frac{2\,\pi}{M}$ in bearing that separates two columns in F.

For an UCA having N sensors we recommend to evaluate $L \simeq 4N$ equispaced beams. To arrive at this figure, we note that the effective aperture of the array is $\simeq \frac{N\,\lambda}{4\pi}$, that the width of the mainlobe at halfpower (known as FWMH, [1]) is then roughly $300/N$ degrees and in order to cover the 360 degrees with about 3 samples per main-lobe one needs $4N$ beams. Though this is in general smaller than the number of real degrees of freedom (d.o.f.) of the array we believe that little information is lost. This further allows to consider the beams to be almost decorrelated.

We expect to improve the standard Rayleigh resolution by a factor 2 or 3, i.e. to solve equipowered sources separated by $100/N$ degrees (FWMH/3). For this to be easily achievable we take a step equal to about $20/N$ degrees. This allows to have about 3 zero components in the X vector in between the components representing the two equipowered sources one expects to solve and leads to $M \simeq 18\,N$ if one wants to cover the 360 degrees.

4.2. Using the statistical properties

To improve the performance of the method we take into account the statistical properties of the beams [7]. As their number is quite low, it is realistic to assume that they are close to uncorrelated random variables. As indicated at the end of section 2, an estimate of the approximate *diagonal* covariance matrix of $\hat{Y}$ is then $\hat{\Sigma} = \mathrm{diag}(\hat{Y})^2/T$ and we propose to whiten the observations before applying the criterion (6) and (7). This amounts to replace $\hat{Y}$ by $\hat{Y}_w = \hat{\Sigma}_y^{-\frac{1}{2}}\,\hat{Y}$ and F by $F_w = \hat{\Sigma}_y^{-\frac{1}{2}}\,F$ where the columns in F_w are further normalized to one.

4.3. Fixing the threshold

Finally to set the value of h we consider the constraint (5) in its whitened form :

$$F_w^T(\hat{Y}_w - F_w X) \leq h\,\mathbf{e} \tag{8}$$

Assume that X represents the exact model, the whitened residual vector $R_w = \hat{Y}_w - F_w X$ is then a gaussian vector with zero mean and covariance matrix identity. Because the columns in F_w are normalized to one, the M+1 components of $F_w^T R_w$ are then scalar gaussian random variables with zero mean and unit variance.

To make this ideal model admissible we have to fix the threshold h in (8) at a value that guarantees that the probability that the maximum of these $M+1$ standard gaussian random variables be larger than h is close to zero. Since these variables are correlated, a precise value is difficult to obtain. The algorithm is fortunately quite robust in this respect and following [11] we suggest to take $h \simeq \sqrt{\ln 2M}$.

4.4. Summary of the algorithm

For an UCA with N sensors, we evaluate $L = 4N$ uniformly spaced beams over the whole horizon and form the vector $\hat{Y}$. We built the matrix F with L-dimensional vectors f_m with a spacing of about $20/N$ degrees and a vector $\mathbf{e}$ allowing for the noise contribution.

We whiten both $\hat{Y}$ and F using the estimate $\hat{Y}$. This leads to $\hat{Y}_w = \sqrt{T}\,\mathbf{e}$. We normalize the columns of F_w and solve :

$$\min_{X \geq 0}\ \frac{1}{2}\,\|\hat{Y}_w - F_w X\|_2^2 \ +\ h\,\mathbf{e}^T X \tag{9}$$

with $h = \sqrt{\ln 2M}$ using a quadratic programming routine. For easy scenarios with high SNR's and P well separated sources, the solution vector X will have about $2P+1$ non-zero components. One for the noise contribution and the others appearing as neighboring pairs. To each pair one associates an unique source with power the sum of the 2 weights and bearing obtained by linear interpolation. For more difficult scenarios one may have to resort to some thresholding [10] to get rid of small non-zero components in X.

If one knows beforehand that sources are only present in a given sector of the horizon one can use this information and only keep in F the columns belonging to this sector while keeping all the components in $\hat{Y}$. The results appear to be unsensitive with respect to this selection. In contrast with most deconvolution approaches, no performance gains should be expected. Such prior knowledge has nevertheless two important consequences, it leads to substantial savings in computation time and considerably reduces the number of spurious sources (false alarms).

5. SIMULATION RESULTS

For an UCA with $N = 10$ sensors (see section 2) and an intersensor spacing of $\lambda/2$ the radius is about $.8\lambda$ and the effective aperture is thus 1.6λ in all directions. The performance of the array is thus similar to the one of an uniform linear array with 4 sensors but for sources around broadside. The (Raylegh) resolution of such an array (see Figure 1) is about 25 degrees and we shall improve this figure by a factor two or three. Following the recommendations indicated above, the observation vector $\hat{Y}$ is built upon $L = 40$ beams separated by 9 degrees. If no a priori knowledge on the location of the sources is assumed, we get $M = 180$, the matrix F has $180+1$ columns that cover the whole horizon. We take the threshold $h = \sqrt{\ln 2M}$ and solve (9) using the NAG E04NCF routine.

5.1. Resolution of two equipowered sources

We take $P = 2$, $\alpha_1 = \alpha_2 = v = 1$ and $\theta_1 = 20^o$, $\theta_2 = 30^o$ in (3) and simulate $T = 100$ snapshots. Note that MUSIC-like algorithms would have difficulties solving these two sources. They succeed in about one third of the realizations. The proposed approach has no difficulties as far as resolution is concerned but some spurious sources may appear especially if no prior knowledge upon the sector in which the sources are present, is assumed. The results obtained by keeping the two strongest sources, in case more than 2 sources are detected, are very close to the Cramer-Rao bounds (CRB) whether or not such prior knowledge is assumed but the percentage of realizations in which some spurious sources (in general quite weak) are present jumps from, say, 2% when the sector is known to 80 % when the whole horizon is searched [12].

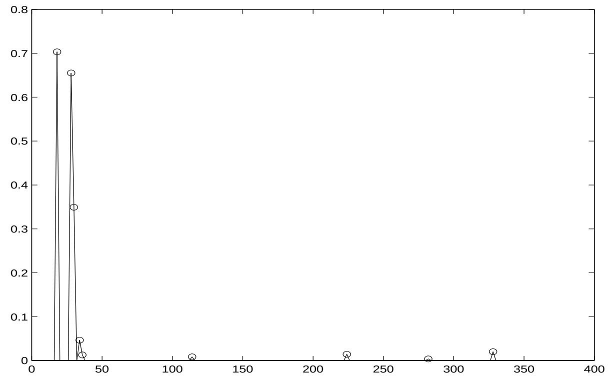

Figure 2: A typical bad output. Among the 180 weights covering 360 degrees just 9, marked by o's, are non-zero and 6 of them correspond to spurious sources.

In the first two columns of Table 1 we present the results obtained from 10000 independent trials if it is known beforehand that the true sources lie between 0 and 60 degrees. The matrix F has then 31 columns and $h = 2.02$. In the last two columns no prior knowledge is introduced, the matrix F has 181 columns and $h = 2.42$.

prior knwovledge	with	with	without	without
source number	1	2	1	2
true bearing in ^{o}.	20	30	20	30
est. bearing in ^{o}.	20.05	29.87	20.11	29.87
est. stdt dev. in ^{o}.	1.56	1.61	1.44	1.49
CR stdt dev. in ^{o}.	1.36	1.36	1.36	1.36
true amplitude	1	1	1	1
est. amplitude	.95	.94	.95	.92
est. stdt dev.	.26	.26	.25	.25
CR stdt dev.	.26	.26	.26	.26

Table 1: Results over 10000 trials with and without prior knowledge

The output of a rather bad realization with 5 false alarms (6 additional non-zero weights) is presented in Figure 2. The weakest true source at bearing 18^o has amplitude $.7$ while the strongest spurious source at bearing 35^o has 2 neighboring non-zero weights that add to .06 There a 4 other, scarcely visible, non-zero weights marked by o's.

6. CONCLUSIONS

A global matched filter approach has been proposed and applied to localizing sources using an uniform circular array. It fits simultaneously to the observations all the elements needed to explain them. It is straightforward to implement and there are few parameters to be chosen. We have indicated how to tune them for UCA's.

For a ten sensors UCA, the method allows to monitor the whole horizon (360 degrees) using just 40 beams and its performance are those of a high resolution method. For the simulated two closely spaced equipowered sources scenario, the method attains performances close to the CRB for as little as 100 snapshots.

The method also applies to scenarios where the number of sources is equal or greater than the number of sensors. Very few approaches able to handle this situation exist.

7. REFERENCES

[1] D.H. Johnson and D.E. Dudgeon. *Array Signal Processing.* Prentice Hall, SP Series, 1993.

[2] D.E.N. Davies. The handbook of antenna design. *Peregrinus.*, London, 1983. (vol. 2, ch. 12).

[3] C.P. Mathews and M. Zoltowski. Eigenstructure techniques for Angle Estimation with UCA's. *IEEE Trans. on S.P.*, vol. 42, 9, pp.2395-2407, sept. 1994.

[4] K.M. Reddy and V.U. Reddy. Analysisof spatial smoothing with uniform circular arrays *IEEE Trans. on S.P.*, vol. 47, 6, pp.1726-1730, june 1999.

[5] P. Forster and T. Ast. Rectification of cross spectral matrices for arrays of arbitrary geometry. *IEEE Proc. ICASSP*, pp. 1311-1314, Phoenix, 1999.

[6] M. Jansson and P. Stoica. Analysis of forward-only and forward-backward samples covariances. *IEEE Proc. ICASSP*, Phoenix, 1999.

[7] K.S. Miller. *Complex Stochastic Processes.* Reading, MA: Addison-Wesley, 1974.

[8] S.G. Mallat and Z. Zhang. Matching pursuit with time-frequency dictionaries. *IEEE-T-SP*, 41: 3397-3415, Dec. 1993.

[9] A.H. Tewfik et al. Optimal subset selection for adaptive signal representation. *IEEE Proc. ICASSP*, pp. 2511-2514, Atlanta, 1996.

[10] J.J. Fuchs. Extension of the Pisarenko method to sparse linear arrays. *IEEE-T-SP*, 45, 2413–2421, oct. 1997.

[11] M.R. Leadbetter, G. Lindgren and H. Rootzen. *Extremes and related properties of random processes and sequences.* Springer Verlag, 1983.

[12] J.J. Fuchs. On the application of the global matched filter to DOA estimation with UCA's. submitted to *IEEE-T-SP*, Dec. 1999.