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Recovery of Exact Sparse Representations in the Presence of Bounded Noise

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***Abstract*—The purpose of this contribution is to extend some recent results on sparse representations of signals in redundant bases developed in the noise-free case to the case of noisy observations. The type of question addressed so far is as follows: given an (n, m)-matrix A with $m > n$ and a vector $b = Ax_o$, i.e., admitting a sparse representation x_o, find a sufficient condition for b to have a unique sparsest representation. The answer is a bound on the number of nonzero entries in x_o. We consider the case $b = Ax_o + e$ where x_o satisfies the sparsity conditions requested in the noise-free case and e is a vector of additive noise or modeling errors, and seek conditions under which x_o can be recovered from b in a sense to be defined.**

The conditions we obtain relate the noise energy to the signal level as well as to a parameter of the quadratic program we use to recover the unknown sparsest representation. When the signal-to-noise ratio is large enough, all the components of the signal are still present when the noise is deleted; otherwise, the smallest components of the signal are themselves erased in a quite rational and predictable way.

***Index Terms*—Basis pursuit, global matched filter, mixed ℓ_1–ℓ_2 norm minimization, nonsmooth optimization, quadratic program, redundant dictionaries, sparse representations.**

I. Introduction

Sparse approximation is the problem of finding a representation of a signal as a linear combination of a small number of elements from an overcomplete set of vectors or signals often called a dictionary or a redundant basis. Indeed, several problems are of interest depending on the context. One may seek the sparsest exact representation of the signal in terms of the elements or the representation of a given complexity that minimizes a certain approximation error or the sparsest representation that yields an approximation error smaller than a specified threshold.

Recently, some theoretical results concerning the first of these questions have been obtained. Given an (n, m)-matrix A with $m > n$ and a vector b that indeed admits an exact sparse representation, say $b = Ax_o$, it has been shown that if the number of nonzero entries in x_o is smaller than a given bound, then x_o is the unique sparsest representation. Here and quite generally in the text the subscript "$_o$" is used to designate the sparse representation we aim to recover, it will also be associated with other quantities to emphasize their link with x_o. We hope that there will be no confusion with x_i which denotes the ith component of the vector x and exists only for $i \geq 1$. Since searching for the sparsest representation is a nonpolynomial (NP) hard problem [1] that can only be solved by exhaustive search, one is tempted to replace the true search for the sparsest solution

$$\min_x \ \|x\|_0 \quad \text{subject to} \quad Ax = b \tag{P_0}$$

with $\|x\|_0$ the number of nonzero entries in x, by the easy-to-solve linear program:

$$\min_x \ \|x\|_1 \quad \text{subject to} \quad Ax = b \tag{LP}$$

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i.e., to minimize the ℓ_1 norm of x instead of the sparsity itself. Here and in what follows we denote by $\|x\|_k$ the ℓ_k norm of a vector x, defined as

$$\|x\|_k = \left\{ \sum_1^m |x_j|^k \right\}^{1/k}, \qquad \text{for } k \geq 1.$$

Notice that we denote this optimization problem by (LP) although it is not in the standard form of a linear program. The problem is then to determine sufficient conditions for the two criteria to have the same unique solution. This problem has been initiated in [2] and developed since then by several other authors, e.g., [3]–[7].

We consider an extension of this problem. We assume the signal to have an exact sparse representation that could be recovered in the absence of noise, but that we observe it in additive noise: $b = Ax_o + e$ with e a bounded perturbation vector, and seek conditions under which x_o can be recovered partially or completely from the observation of b by solving a convex optimization problem such as a linear or a quadratic program. If one seeks an x that leads to an exact reconstruction of b, it will have generically at least n nonzero components. To get a sparse representation, one therefore has to allow for reconstruction errors. The best one can expect is that the optimum solution of the program and x_o have their nonzero components at the same locations, with the same signs, but of course slightly different values. That is, we want to recover the support of the true sparse expansion with slightly biased weights, the bias converging to zero as the energy of the noise diminishes.

This problem has received some attention recently [8]–[10]. If e is assumed to be Gaussian instead of bounded and no parsimony invoked, it is a difficult detection problem for which numerous *ad hoc* solutions have been proposed [11] over the last decades in the linear regression literature. It is known as the "subset selection" or "selection of variables" problem. In [12], the Gaussian noise case is considered in the present parsimonious representations context and some parts of this contribution can be seen as an application of [12] to the bounded noise case.

This correspondence is organized as follows. In Section II, we further specify the problem and introduce the algorithm we shall consider. In Section III, we briefly present some known results concerning this algorithm that we need in the later sections. The global behavior of this algorithm is investigated in Section IV both in the case where a sparse representation exists and in the general case of a noisy observation. It is this acute analysis that allows to establish in Section V the precise conditions under which a sparse representation observed in bounded additive noise can be recovered. Conclusions are presented in Section VI. They include a summary of the contribution and its practical implications as well as comparisons with other approaches and other results, and potential further research directions.

II. Problem Formulation

A. Noiseless Case

Consider a set of m n-dimensional vectors of unit Euclidean norm $\{a_j\}$, with $m \geq n$, and denote by A the (n, m) matrix having these vectors as columns. Any linear combination b of these m vectors can then be written as $b = Ax$ with x an m-dimensional vector of weights. If $b = Ax_o$, with x_o having just a few nonzero components, it may well be the unique and the sparsest representation of b. In order to define when this is the case, we introduce the mutual coherence [2]

$$M = \max_{1 \leq i \neq j \leq m} \left| a_i^T a_j \right| \tag{1}$$

where a^T denotes the transpose of the vector a of the dictionary whose atoms or components are the columns of A. The smaller M, the less coherent are the components of the dictionary and $M = 0$ if and only if the columns are orthogonal. It is worthwhile to know that there are indeed redundant dictionaries with $m \simeq n^2$ components and mutual coherence $M \simeq \frac{1}{\sqrt{n}}$ [13], [14].

It has been shown in [2]–[5] that if

$$\|x_o\|_0 < \frac{1}{2}\left(1 + \frac{1}{M}\right) \tag{2}$$

then x_o is the unique sparsest representation of b and that it can be recovered by solving (LP) as defined above. In [5], it is further shown that this is also true when (LP) is replaced by

$$\min_x \frac{1}{2}\|Ax - b\|_2^2 + h\|x\|_1, \qquad h > 0 \tag{QP}$$

for sufficiently small h. The problem (QP) is a nonsmooth convex optimization problem that can be transformed into a quadratic program, hence, its denomination. Indeed for $h = 0^+$ (QP) and (LP) are equivalent but, as we shall see later, (QP) is a quite natural way to handle the more general situation where one observes $b = Ax_o + e$ with e a noise or perturbation vector.

B. Noisy Case

In the sequel, we will investigate the case where one observes $b = Ax_o + e$ with $\|e\|_2 < \epsilon$. We will assume further that $\|x_o\|_0$ satisfies not only (2) but a more stringent condition. Taking a bounded perturbation allows to conduct a deterministic analysis and leads to generic results. In [12], the same parsimonious model is considered but, with e a Gaussian noise, the question of recovering the true support of x_o is then a complex detection problem and only weak asymptotic results can be expected.

When $b = Ax_o + e$, the optimum of (LP) has generically n nonzero entries and no sparser solution exists. One can, however, expect that for e small enough, the $\|x_o\|_0$ components of the optimum the largest in absolute value correspond to the nonzero entries of x_o, while the others are induced by the noise. Applying some pruning to the components of the optimum to declare equal to zero those that are below some threshold could thus lead to the recovery of the true expansion. In the linear regression analysis literature, where the number of components m is generally smaller than the number of observations, n, this is known as "elimination of variables" and is based on statistical inference.

We do not pursue this approach since we consider the case where $m > n$ and little can be said about the way the degenerate optimum $x^* = x_o$ of (LP) in the noise-free case evolves as b drifts away from Ax_o. This is due to the fact that for $e = 0$, the optimum of the dual of (LP) is undetermined and it is thus impossible to know *a priori* which additional components of the optimum of (LP), the primal, will become nonzero. The same remark applies to the following extension of (LP) considered in [15], [16] that allows for reconstruction errors of bounded magnitude in ℓ_∞:

$$\begin{aligned} \min_x \quad & \|x\|_1 \\ \text{subject to:} \quad & \|Ax - b\|_\infty \leq B_\infty \end{aligned}$$

with $\|x\|_\infty = \max_i |x_i|$. A more promising criterion is $\min \|x\|_1$ subject to $\|Ax - b\|_1 \leq B_1$, but the natural extension to the present ℓ_2-bounded-noise context is indeed

$$\begin{aligned} \min_x \quad & \|x\|_1 \\ \text{subject to:} \quad & \|Ax - b\|_2^2 \leq B_2^2. \end{aligned} \tag{3}$$

This is a convex optimization problem that is equivalent to

$$\min_x \frac{1}{2}\|Ax - b\|_2^2 + h\|x\|_1, \qquad h > 0 \tag{QP}$$

for an adequately chosen parameter $h > 0$. Notice that we denote this optimization problem (QP) although it is not in the standard form of a quadratic program. This criterion and similar ones have been considered for a while now [6]–[9], [17]–[19].

Indeed if h in (QP) is taken equal to the inverse of twice the Lagrange multiplier of the constraint in (3) at the optimum, then (3) and (QP) have the same optimum. There is an implicit and unknown relation between B_2 and h. Now (QP) is in turn identical to

$$\begin{aligned}\min_x \quad & \|Ax\|_2^2 \\ \text{s.t.} \quad & \|A^T(Ax-b)\|_\infty \leq h. \end{aligned} \tag{DQP}$$

The two programs (QP) and (DQP) can be transformed into quadratic programs that are equivalent by duality [5], [20]. Of course, (DQP) can be rewritten

$$\begin{aligned}\min_y \quad & \|y\|_2^2 \\ \text{s.t.} \quad & \|A^T(y-b)\|_\infty \leq h \end{aligned} \tag{DQP$'$}$$

and the optimum of (DQP) or (QP) deduced from the Lagrange multipliers of the lower dimensional problem (DQP$'$). In the sequel, we mainly concentrate on (QP) which is the most convenient for our analysis but as far as the implementation is concerned any equivalent form can of course be used.

III. Quadratic Programming Approach

In this section, we state some properties of the optimum of

$$\min_x \frac{1}{2}\|Ax-b\|_2^2 + h\|x\|_1, \qquad h > 0 \tag{QP}$$

which are established in [5] and we will use them in later sections. The optimum of (QP) will be denoted either x^* or $x(h)$ depending on the context, the notation $x(h)$ being used when the emphasis is on h. We will often need to distinguish between the nonzero components and the zero components of x^*. We denote $\bar{x}^*$ or $\bar{x}(h)$ the reduced dimensional vector built upon the nonzero components of x^*. Similarly, $\bar{A}$ denotes the associated columns in A. One then has, e.g., $Ax^* = \bar{A}\bar{x}^*$. We will also use the notation $\bar{\bar{A}}$ for the remaining columns in A and thus decompose A as $A = [\bar{A}\ \bar{\bar{A}}]$.

A. Preliminary Remarks

Problem (QP) is a nondifferentiable convex optimization problem parameterized by the positive scalar h.

When $h = 0^+$, (QP) is equivalent to (3) with $B_2 = 0$ which in turn is then identical to (LP). As h increases, one can get an intuition about the way the optimum of (QP) evolves by looking at the equivalent criterion (DQP). For $h > \|A^Tb\|_\infty$, $x = 0$ is an admissible point in (DQP) that is thus the optimum. One thus expects that, as h increases from 0^+ to $\|A^Tb\|_\infty$, the number of nonzero components in the optimum decreases down to zero.

More precisely, the interval $]0^+, \|A^Tb\|_\infty[$ can be divided into subintervals characterized by the fact that within each such subinterval the number of nonzero components of the optimum of (QP) is constant. We will establish below that while this evolution of the number of nonzero components is not necessarily monotonic at the beginning (for small h) it always is so once the optimum is sparse enough (for large h). This is indeed a valuable feature in a selection of variables context.

Also note that even in the sparse noiseless case, the optimum point $x(h)$ of (QP) achieves a compromise between the two terms of the criterion $\|Ax-b\|_2^2$ and $\|x\|_1$. One therefore has $Ax(h) \neq b$ and thus $x(h) \neq x_o$. The best one can expect is that $x(h)$ and x_o have their nonzero components at the same locations, i.e., have the same support, and with the same signs. If this holds, we say that (QP) allows to recover x_o [5].

In the noisy case, the discrepancy between $x(h)$ and x_o will not only be due to this compromise between the two terms of the criterion but also to the presence of the noise e.

B. Optimality Conditions

In order to investigate the conditions under which it is possible to recover x_o, we need to state and detail the optimality conditions of (QP). For this purpose, we introduce the subdifferential of $\|x\|_1$ [21], a set of vectors called the subgradients, denoted $\partial\|x\|_1$

$$\begin{aligned}\partial\|x\|_1 &= \{u \mid u^Tx = \|x\|_1, \|u\|_\infty \leq 1\} \\ &= \{u \mid u_i = \mathrm{sign}(x_i) \text{ if } x_i \neq 0 \text{ and } |u_i| \leq 1 \text{ otherwise}\}\end{aligned}$$

where x_i denotes the ith component of x, $\mathrm{sign}(x_i) = 1$ when $x_i > 0$, and $\mathrm{sign}(x_i) = -1$ when $x_i < 0$. In the scalar case, it amounts to saying that the derivative of the function $|x|$ for $x = 0$ is any real in $[-1, 1]$. The following result then holds [21].

Theorem 1: The point x^* is a global minimum of (QP) if and only if the vector zero is a subgradient of the criterion at x^*:

$$\exists u \in \partial\|x\|_1 \text{ such that } A^T(Ax^* - b) + hu = 0. \tag{NSC}$$

□

To write this necessary and sufficient condition (NSC) in a more usable way, we use the notation $\bar{x}^*$ and $\bar{A}$ defined at the beginning of Section III, i.e., we distinguish between the nonzero components and the zero components of x^*. For the rows in (NSC) associated with the nonzero entries $\bar{x}^*$, the subgradient is unique and equal to $\mathrm{sign}(\bar{x}^*)$, while for the other rows the subgradient takes any value in $[-1, 1]$. The necessary and sufficient conditions become

$$\bar{A}^T(b - \bar{A}\bar{x}^*) = h\ \mathrm{sign}(\bar{x}^*) \tag{NSC$_1$}$$

$$\left|a_j^T(b - \bar{A}\bar{x}^*)\right| \leq h, \qquad \text{for } a_j \notin \bar{A}. \tag{NSC$_2$}$$

One can further establish that if $\bar{A}$ is full rank, which is always the case if the number of columns in $\bar{A}$ satisfies (2) [6], [7], and if the inequalities in (NSC$_2$) are strictly satisfied, then the minimum point of (QP) is unique or strict. Hence, the following corollary to Theorem 1 [5].

Corollary 1: Sufficient conditions for x^* to be a strict minimum point of (QP) are

1) $\bar{A}^T(b - \bar{A}\bar{x}^*) = h\ \mathrm{sign}\ (\bar{x}^*)$ (4)

2) $\left|a_j^T(b - \bar{A}\bar{x}^*)\right| < h, \qquad \text{for } a_j \notin \bar{A}$ (5)

3) $\bar{A}$ full rank

with $\bar{x}^*$ and $\bar{A}$ as defined above. □

The first of these relations then leads to an implicit expression of the optimum

$$\bar{x}(h) = \bar{A}^+b - h(\bar{A}^T\bar{A})^{-1}\ \mathrm{sign}(\bar{x}(h)) \tag{6}$$

where $\bar{A}^+ = (\bar{A}^T\bar{A})^{-1}\bar{A}^T$ is the Moore–Penrose inverse of $\bar{A}$ also known as the pseudoinverse.

In (6), we use $x(h)$-notation since the emphasis is on h. This is an implicit relation since $\bar{x}(h)$ is present on both sides. It is valid only for the limited domain around the current value of h for which the signs of all the components of $\bar{x}(h)$ computed according to the expression given in the second member of (6) remain unchanged. Indeed, as h varies, as soon as a component in $\bar{x}(h)$ becomes zero, the decomposition of (NSC) into (NSC$_1$) and (NSC$_2$) has to be modified. We take a close look at the behavior of $\bar{x}(h)$ in Section IV.

C. Separability and Sparsity Condition

For later use, we briefly state some results developed in [5]. The idea is to use Corollary 1 to establish that if x_o satisfies (2), then to solve (QP) with $b = Ax_o$ and sufficiently small h allows to recover x_o. Let us decompose, in a similar way as above, x_o into $\left[\bar{x}_o^T\ 0\right]$ and A into $[\bar{A}_o\ \bar{\bar{A}}_o]$, Theorem 2 in [5] then states the following.

Theorem 2: The solution x_o of $Ax = b$, with $\bar{A}_o$ and $\bar{x}_o$ as defined above and $\bar{A}_o$ a full rank matrix, can be recovered from the unique optimum point of (QP) if

$$1)\quad \left|a_j^T d_o\right| < 1, \qquad \forall\, a_j \notin \bar{A}_o \text{ with } d_o = \bar{A}_o^{+T}\ \mathrm{sign}(\bar{x}_o) \tag{7}$$

$$2)\quad h \in]\,0, h_1[,\quad \text{the domain in which}$$
$$\mathrm{sign}\left\{\bar{x}_o - h\left(\bar{A}_o^T \bar{A}_o\right)^{-1}\ \mathrm{sign}(\bar{x}_o)\right\} = \mathrm{sign}(\bar{x}_o) \tag{8}$$

with $\bar{A}_o^+ = \left(\bar{A}_o^T \bar{A}_o\right)^{-1} \bar{A}_o^T$, the pseudoinverse of $\bar{A}_o$. □

Condition (7) is condition (NSC_2) rearranged after substitution of (6), while condition (8) is the translation of the remarks made at the end of the previous section to the present context where h is close to zero. The final step consists of proving that (7) is implied by (2), which is the object of Theorem 3 in [5].

Theorem 3: If the columns in A are normalized to one in ℓ_2 norm, then (2) implies (7), i.e.,

$$\|x_o\|_o < \frac{1}{2}\left(1 + \frac{1}{M}\right) \Rightarrow \left|a_j^T d_o\right| < 1, \qquad \forall\, a_j \notin \bar{A}_o \tag{9}$$

where $d_o = \bar{A}_o^{+T}\ \mathrm{sign}(\bar{x_o})$. □

Combining then Theorems 2 and 3 leads to the announced recovery result. The proof of Theorem 3 relies on Gershgorin theorem and Neumann's lemma [22]. In the sequel, we will need the following intermediate result [5].

Lemma: With $\|x_o\|_o = p$ and the notations defined above, one has

$$\left|\left(\bar{A_o}^T \bar{A}_o\right)^{-1}\right| \le \alpha I + \beta \mathbf{1},$$
$$\text{with } \alpha = \frac{1}{1+M} \quad \text{and} \quad \beta = \frac{M}{(1+M)(1+M-Mp)}$$

where I is the identity matrix, $\mathbf{1}$ a matrix of adequate dimension whose components are all equal to one and where for a matrix B, we denote $|B|$ the matrix with entries $|b_{i,j}|$ and write $B < C$ if $b_{i,j} < c_{i,j}, \forall\, i, j$. □

IV. Behavior of the Optimum Solution of (QP)

We are now ready to analyze the behavior of the solution of (QP) as h increases from zero to infinity. We first make some preliminary remarks when no assumption is made on the observation vector b. We later specialize to the case where there exists a true sparse representation satisfying (2) and then pursue with the case where this same representation is observed in additive noise. This analysis has some similarities with a method developed in [25].

A. Preliminary Remarks

We solve (QP) for a fixed h and analyze what happens to its optimum solution as h is increased.

From Theorem 1, we know that, associated with the optimum x^*, there exists a vector u, a subgradient of $\|x\|_1$ at x^*, that satisfies $A^T(b - Ax^*) = hu$.

As explained at the beginning of Section III, we decompose x^* into $\bar{x}^*$ and $\bar{\bar{x}}^* = 0$, and accordingly, the matrix A and the vector u. Replacing $\bar{u}$ by $\mathrm{sign}(\bar{x}^*)$, the preceding relation becomes

$$\bar{A}^T(b - \bar{A}\bar{x}^*) = h\ \mathrm{sign}(\bar{x}^*)$$
$$\bar{\bar{A}}^T(b - \bar{A}\bar{x}^*) = h\ \bar{\bar{u}}\ .$$

We are interested in how $\bar{x}^*$ and $\bar{\bar{u}}$ evolve as h increases because the present decomposition is only valid as long as no component in $\bar{x}^*$ becomes zero and no component in $\bar{\bar{u}}$ becomes equal to ± 1. From the first of these two relations, we deduce $\bar{x}^*$ which we denote $\bar{x}(h)$ to emphasize its dependency on h. We introduce this expression into the second relation to finally obtain after some rearrangements

$$\bar{x}(h) = \bar{A}^+ b - h(\bar{A}^T \bar{A})^{-1}\ \mathrm{sign}(\bar{x}(h))$$
$$\bar{\bar{u}}\ (h) = \frac{1}{h}\bar{\bar{A}}^T\ b^\perp + \bar{\bar{A}}^T\ \bar{A}^{+T}\ \mathrm{sign}(\bar{x}(h)) \tag{10}$$

where $b^\perp = (I - \bar{A}\bar{A}^{+T})b$ is the projection of b on the subspace orthogonal to the range of $\bar{A}$. Since $\|\bar{x}(h)\|_1 = \mathrm{sign}(\bar{x}(h))^T\, \bar{x}(h)$, it follows from the first of these two relations that for small δh

$$\|\bar{x}(h + \delta h)\|_1 = \|\bar{x}(h)\|_1 - \delta h\ \mathrm{sign}(\bar{x}(h))^T(\bar{A}^T \bar{A})^{-1}\ \mathrm{sign}(\bar{x}(h))$$

which shows that the ℓ_1-norm of $\bar{x}(h)$ decreases as h increases. If in the second relation we introduce, as we did in the statement of Theorem 2, the vector $d = \bar{A}^{+T}\ \mathrm{sign}(\bar{x}(h))$, it becomes

$$\bar{\bar{u}}\ (h) = \frac{1}{h}\bar{\bar{A}}^T\ b^\perp + \bar{\bar{A}}^T\ d. \tag{11}$$

One can then check that

$$\|\ \bar{\bar{u}}\ (h)\|_\infty = \left\|\frac{1}{h}\bar{\bar{A}}^T\ b^\perp + \bar{\bar{A}}^T\ d\right\|_\infty \le 1, \qquad \left\|\bar{\bar{A}}^T\ d\right\|_\infty < 1$$
$$\Rightarrow \|\ \bar{\bar{u}}\ (h + \delta h)\|_\infty < 1, \qquad \forall\, \delta h > 0 \tag{12}$$

where the verification is easy for each individual $a_j \in \bar{\bar{A}}$. The relations (10)–(12) will be used below to investigate the evolution of the optimum $x(h)$ of (QP) as h increases from 0^+ to infinity.

Remark: The implication (12) brings to light the importance of the vector d in our context. It was introduced in the statement of Theorem 2 and somehow indicates whether a representation is recoverable or not. Note that in Theorem 2, since the vector d is attached with the representation x_o, it inherits of the subscript "$_o$" and becomes d_o. It appears that a representation is essentially defined by $\bar{A}$, the columns it uses, and $\mathrm{sign}(\bar{x})$, the signs of the corresponding weights. This is also precisely what is needed to build $d = \bar{A}^{+T}\mathrm{sign}(\bar{x})$. One associates with this single vector d [5], [17] two parallel separating hyperplanes $H_\pm$, defined by: $H_\pm = \{a | a^T d = \pm 1\}$ and Theorem 2 then says that (QP) allows to recover the representation if the columns of A not in $\bar{A}$ (those in $\bar{\bar{A}}$) lie strictly between these two hyperplanes (7). Related work has been reported recently [23]. If the emphasis is put on sparse representations, it follows that a representation is recoverable if the sparsity condition (2) implies the separability condition (7) (see Theorem 3).

B. Sparse Noiseless Case

Let $b = Ax_o$ with x_o satisfying the parsimony condition (2). Solving (QP) for $h = 0^+$ yields then an optimum $x(0^+)$ that allows to recover x_o. This is established in [5] and follows from Theorems 2 and 3. For h small enough, relations (10) and (11) become

$$\bar{x}_o(h) = \bar{A}_o^+ b - h(\bar{A}_o^T \bar{A}_o)^{-1}\mathrm{sign}(\bar{x}_o(h))$$
$$= \bar{x}_o - h(\bar{A}_o^T \bar{A}_o)^{-1}\mathrm{sign}(\bar{x}_o)$$

$$\bar{\bar{u}}_o(h) = \frac{1}{h}\bar{\bar{A}}_o^T b_o^\perp + \bar{\bar{A}}_o^T d_o = \bar{\bar{A}}_o^T d_o \tag{13}$$

with $d_o = \bar{A}_o^{+T}\,\mathrm{sign}(\bar{x}_o)$. Note that $b_o^\perp = 0$ follows from $\bar{A}_o^+ b = \bar{x}_o$. Note that the subscript "$_o$" has been appended here since the quantities are associated with the representation x_o.

The vector $\bar{\bar{u}}_o(h)$ is thus independent of h for small h. This is not the case for the components in $\bar{x}_o(h)$ in (13). As h increases, $\|\bar{x}_o(h)\|_1$ decreases and a first component of $\bar{x}_o(h)$ will become zero. As in Theorem 2, we denote h_1 the corresponding value of h. At this point, one re-decomposes the optimum into $\left[\bar{x}_1(h)^T\ 0\right]^T$, A into $A = [\bar{A}_1\ \bar{\bar{A}}_1]$, and u accordingly. One then has, for h slightly larger than h_1

$$\begin{aligned} \bar{x}_1(h) &= \bar{A}_1^+ b - h(\bar{A}_1^T\bar{A}_1)^{-1}\mathrm{sign}(\bar{x}_1(h)) \\ \bar{\bar{u}}_1(h) &= \frac{1}{h}\bar{\bar{A}}_1^T b_1^\perp + \bar{\bar{A}}_1^T d_1 \end{aligned} \tag{14}$$

with $b_1^\perp$ the projection of b on the subspace orthogonal to the range of $\bar{A}_1$ and d_1 defined similarly to d_o above. As h increases beyond h_1, both components in $\bar{\bar{u}}_1(h)$ (14) and those in $\bar{x}_1(h)$ vary and one has to check which event happens first at say $h = h_2$, a component in $\bar{\bar{u}}_1(h)$ becoming equal to ± 1 or a component in $\bar{x}_1(h)$ becoming zero.

Since $\bar{x}_1(h)$ satisfies (2), it follows from Theorem 3 that $\|\bar{\bar{A}}_1^T d_1\|_\infty < 1$ and hence, from implication (12) that $\bar{\bar{u}}_1(h)$ will remain strictly inside $[-1, 1]$. At $h = h_2$, a component of $\bar{x}_1(h)$ will thus become zero and this will further diminish the number of nonzero components in $x(h)$. The same reasoning can then be re-applied until $x(h) = 0$ for $h \geq \|A^T b\|_\infty$. We have thus established the following.

Proposition 1: If x_o satisfies the parsimony condition (2), (QP) allows to recover it for h sufficiently small, i.e., for $h \in]0^+, h_1[$. If h is increased beyond h_1, the number of nonzero components in the optimum $x(h)$ of (QP) decreases monotonically. □

One can further check that $x(h)$ is a continuous function of h and that $\|x(h)\|_1$ is monotonically decreasing down to zero. Note that the implication in (12) is quite generally applicable and actually holds for any b and any h as soon as the corresponding optimum $x(h)$ satisfies (2).

C. General Case

We now consider an arbitrary b, which includes of course the case where $b = Ax_o + e$ with $\|e\|_2 \leq \epsilon$ and x_o satisfies the parsimony condition (2). Solving (QP) for $h = 0^+$ is equivalent to solving the linear program (LP) whose solution is generically obtained at a basic feasible solution [21]. We will further assume that this optimum is unique. This means that $x(0^+)$ has n nonzero components and that in the associated partition of $A = [\bar{A}_o\ \bar{\bar{A}}_o]$, $\bar{A}_o$ is invertible. This assumption (if any) is extremely weak since for it to be false means that there exists a column a_j in $\bar{\bar{A}}_o$ such that the $(n, n+1)$-dimensional matrix $[\bar{A}_o\ a_j]$ topped by the $(n+1)$-dimensional row $\left[\mathrm{sign}\left(\bar{x}_o(0^+)\right)\ 1\right]$ yields a square order $(n+1)$ *singular* matrix. Since, due to the presence of noise in b, the columns in $\bar{A}_o$ are essentially chosen randomly in A, the probability that there exists a column $a_j \in \bar{\bar{A}}_o$ that makes this matrix singular should be close to zero.

With the notations of the previous section, one has

$$\begin{aligned} \bar{x}_o(h) &= \bar{A}_o^{-1} b - h\left(\bar{A}_o^T\bar{A}_o\right)^{-1}\mathrm{sign}\left(\bar{x}_o(0^+)\right) \\ \bar{\bar{u}}_o(h) &= \frac{1}{h}\bar{\bar{A}}_o^T b_o^\perp + \bar{\bar{A}}_o^T d_o = \bar{\bar{A}}_o^T d_o. \end{aligned} \tag{15}$$

Note that now $b_o^\perp = 0$ because the range of $\bar{A}_o$ is the whole space. For h sufficiently small, $\bar{\bar{u}}_o(h)$ is thus again independent of h and as h increases a component in $\bar{x}_o(h)$ will become zero at $h = h_1$ since $\|\bar{x}_o(h)\|_1$ decreases. Assume that the component which vanishes is the last one, x_{on}, which is associated with column a_n of $\bar{A}_o$. One then re-decomposes the optimum into $\left[\bar{x}_1(h)^T 0\right]^T$, A into $A = [\bar{A}_1\ \bar{\bar{A}}_1]$, and u accordingly. Using the formula giving the partitioned inverse [24] of $\bar{A}_o$ written as $\bar{A}_o = [\bar{A}_1\ a_n]$, the last component x_{on} of $\bar{x}_o(h)$ in (15), that becomes zero for $h = h_1$, is equal to

$$x_{on}(h) = \frac{1}{\left\|a_{n1}^\perp\right\|^2}\left\{a_n^T b_1^\perp + h a_n^T d_1 - h\,\mathrm{sign}(x_{on})\right\} \tag{16}$$

where $b_1^\perp$ and $a_{n1}^\perp$ are the projections of b and a_n on the subspace orthogonal to the range of $\bar{A}_1$ and d_1 is defined as above. For $h = h_1$, one has the following two expressions for $\bar{\bar{u}}_1(h_1)$:

$$\bar{\bar{u}}_1(h_1) = \begin{bmatrix} \mathrm{sign}(x_{on}) \\ \bar{\bar{A}}_o^T d_o \end{bmatrix} = \frac{1}{h_1}\bar{\bar{A}}_1^T b_1^\perp + \bar{\bar{A}}_1^T d_1.$$

As h increases beyond h_1, both

$$\bar{\bar{u}}_1(h) = \frac{1}{h}\bar{\bar{A}}_1^T b_1^\perp + \bar{\bar{A}}_1^T d_1$$

and

$$\bar{x}_1(h) = \bar{A}_1^+ b - h\left(\bar{A}_1^T\bar{A}_1\right)^{-1}\mathrm{sign}(\bar{x}_1(h_1))$$

are varying and it is now possible that a component in $\bar{\bar{u}}_1(h)$ becomes equal to ± 1 before a further component of $x(h)$ becomes zero. Using (16), one can verify that it will not be the first component of $\bar{\bar{u}}_1(h)$, the one that just entered $\bar{\bar{u}}_1(h)$. Let us assume it is the component of $\bar{\bar{u}}_1$ associated with, say, the column a_2 of $\bar{\bar{A}}_1$ that becomes equal to, say, 1 for $h = h_2$. This means that $\frac{1}{h_2}a_2^T b_1^\perp + a_2^T d_1 = 1$ and, for later use, note that since this quantity would become greater than $+1$ for larger h, one has $a_2^T b_1^\perp < 0$ and $a_2^T d_1 > 1$.

At $h = h_2$, one re-decomposes A and $\bar{A}_2 = [\bar{A}_1\ a_2]$. The new nonzero components in $x(h)$ included in $\bar{x}_2(h)$, say $x_{22}(h)$, has then the following expression, using the partitioned form of $\bar{A}_2$:

$$\begin{aligned} x_{22}(h) &= \frac{1}{\|a_2^\perp\|^2}\left\{a_2^T b_1^\perp + h a_2^T d_1 - h\,\mathrm{sign}(x_{22}(h))\right\} \\ &= \frac{1}{\|a_2^\perp\|^2}\left\{a_2^T b_1^\perp + h a_2^T d_1 - h\right\} \end{aligned}$$

which is equal to zero for $h = h_2$ by definition of h_2, and, as follows from $a_2^T b_1^\perp < 0$ and $a_2^T d_1 > 1$, becomes strictly positive as h increases. It will thus certainly not become zero during the next interval $h \in]h_2, h_3]$.

This step completes the analysis of the behavior of the optimum $x(h)$ of (QP) as h increases. In summary, one can say that, as h increases, $\|x(h)\|_1$ decreases steadily, but this does not hold for $\|x(h)\|_0$, the number of nonzero components in $x(h)$. This number, which never exceeds n, can sometimes increase. This last event however cannot happen once $\|x(h)\|_0$ satisfies (2) as we have seen in Section IV-B. Note that since (2) is only a sufficient condition, this nice rationally expected behavior can start for larger values of $\|x(h)\|_0$.

V. Recovery Properties of (QP)

In the previous section, we investigated the general behavior of the optimum $x(h)$ of (QP). We will now analyze more precisely its properties. We establish the following.

Theorem 4: Let b be equal to $b = Ax_o + e$ with $\|e\|_2 \leq \epsilon$ and $p = \|x_o\|_0 \leq \frac{\beta}{M}$ with $\beta \leq \frac{1}{2}$.

Then, if one solves (QP) with

$$h > \left(1 + \frac{\beta}{1 + M - 2\beta}\right)\epsilon$$

the support of the optimum $x(h)$ is either identical to, or contained in, the support of x_o.

And if one solves (QP) with

$$h = \left(1 + \frac{\beta}{1+M-2\beta}\right)\epsilon^+$$

and if

$$x_m > \left(\frac{1}{1+M-\beta} + \frac{1}{1+M-2\beta}\right)\epsilon$$

where x_m is the absolute value of the nonzero component of x_o with the smallest absolute value, then one recovers the true representation, i.e., $x(h)$ and x_o have their nonzero components at the same locations and with the same signs.

Proof: Solving (QP), one knows from Theorem 1 and Corollary 1 that $x(h)$ is the optimum if and only if $\exists u \in \partial\|x\|_1$ such that $A^T(b - Ax(h)) = hu$, (NSC). We decompose x_o into $x_o^T = \left[\bar{x}_o^T 0\right]$, with $\bar{x}_o$ the p-dimensional vector of the nonzero components of x_o, and proceed similarly with the matrix A, the optimum $x(h)$ and the vector u. One then has, for instance, $b = Ax_o + e = \bar{A}_o\bar{x}_o + e$. Note that this decomposition is driven by x_o and not $x(h)$ and that, e.g., $\bar{u}$ is not necessarily equal to $\mathrm{sign}(\bar{x}(h))$.

To establish the first part of the theorem, we need to show that $\| \bar{\bar{u}} \|_\infty < 1$ since this implies that $\bar{\bar{x}}\ (h) = 0$ as follows from the definition of $\partial\|x\|_1$.

Decomposing the relation (NSC) into two parts, the first $\bar{A}_o^T(b - \bar{A}_o\bar{x}(h)) = h\bar{u}$ leads to

$$\bar{x}(h) = \bar{A}_o^+ b - h\left(\bar{A}_o^T\bar{A}_o\right)^{-1}\bar{u} = \bar{x}_o + \bar{A}_o^+ e - h\left(\bar{A}_o^T\bar{A}_o\right)^{-1}\bar{u}$$

with some components potentially equal to zero. The second part of (NSC) $\bar{\bar{A}}_o^T(b - \bar{A}_o\bar{x}(h)) = h\,\bar{\bar{u}}$ becomes after substitution of $\bar{x}(h)$

$$\bar{\bar{A}}_o^T(e^\perp + h\delta_o) = h\,\bar{\bar{u}}, \qquad \text{with } \delta_o = \bar{A}_o^{+T}\bar{u}$$

where $e^\perp = \left(I - \bar{A}_o\bar{A}_o^+\right)e$. To establish $\| \bar{\bar{u}} \|_\infty < 1$ amounts thus to prove that

$$\left|a_j^T(e^\perp + h\delta_o)\right| < h, \qquad \forall\, a_j \in \bar{\bar{A}}_o.$$

Remember that by definition one always has $\|u\|_\infty \le 1$.

Now $\left|a_j^T e^\perp\right| \le \epsilon$ since $\|e\|_2 \le \epsilon$ and $\|a_j\|_2 = 1$. Using the lemma of Section III-C, the definition of M, and the fact that $\|\bar{u}\|_\infty \le 1$ one has

$$\begin{aligned}\left|a_j^T\delta_o\right| &\le \left|a_j^T\bar{A}_o\right| \left|\left(\bar{A}_o{}^T\bar{A}_o\right)^{-1}\right| |\bar{u}| \\ &\le M\mathbf{1}^T\left|\left(\bar{A}_o{}^T\bar{A}_o\right)^{-1}\right|\mathbf{1} \\ &\le M(\alpha p + \beta p^2) \\ &\le \frac{Mp}{1+M-Mp}.\end{aligned}$$

On the other hand

$$\begin{aligned}\left|a_j^T(e^\perp + h\delta_o)\right| &\le \left|a_j^T e^\perp\right| + h\left|a_j^T\delta_o\right| \\ &\le \epsilon + h\frac{Mp}{1+M-Mp}.\end{aligned}$$

Thus, $h > \epsilon + h\frac{Mp}{1+M-Mp}$ implies $|a_j^T(e^\perp + h\delta_o)| < h$ and this holds precisely if the announced bound for h is satisfied. This completes the proof of the first part of the theorem.

For the proof of the second part of the theorem, we need to prove as above that $\| \bar{\bar{u}} \|_\infty < 1$ and that under the additional condition on x_m no component of $\bar{x}(h)$ is zero and $\mathrm{sign}(\bar{x}(h)) = \mathrm{sign}(\bar{x}_o)$. To establish that $\| \bar{\bar{u}} \|_\infty < 1$, one proceeds exactly as above, the proof goes through without modification. It remains to establish that $\mathrm{sign}(\bar{x}(h)) = \mathrm{sign}(\bar{x}_o)$. To this end, we start from

$$\bar{x}(h) = \bar{A}_o^+ b - h\left(\bar{A}_o^T\bar{A}_o\right)^{-1}\bar{u} = \bar{x}_o + \bar{A}_o^+ e - h\left(\bar{A}_o^T\bar{A}_o\right)^{-1}\bar{u}.$$

Using the lemma of Section III-C, one has

$$\begin{aligned}\left\|\bar{A}_o^+ e\right\|_\infty &= \left\|\left(\bar{A}_o^T\bar{A}_o\right)^{-1}\bar{A}_o^T e\right\|_\infty \\ &\le \epsilon\left\|\left|\left(\bar{A}_o{}^T\bar{A}_o\right)^{-1}\right|\mathbf{1}\right\|_\infty \\ &\le \frac{\epsilon}{1+M-Mp} \\ &\le \frac{\epsilon}{1+M-\beta} \\ \left\|\left(\bar{A}_o^T\bar{A}_o\right)^{-1}\mathrm{sign}(\bar{x}_o)\right\|_\infty &\le \left\|\left|\left(\bar{A}_o{}^T\bar{A}_o\right)^{-1}\right|\mathbf{1}\right\|_\infty \\ &\le \frac{1}{1+M-Mp} \\ &\le \frac{1}{1+M-\beta} \\ \Rightarrow \left\|\bar{A}_o^+ e - h\left(\bar{A}_o^T\bar{A}_o\right)^{-1}\bar{u}\right\|_\infty &\le \frac{\epsilon+h}{1+M-\beta}.\end{aligned}$$

From the last relation and the expression of $\bar{x}(h)$, it follows that $x_m > \frac{\epsilon+h}{1+M-\beta}$ implies $\mathrm{sign}(\bar{x}(h)) = \mathrm{sign}(\bar{x}_o)$. This leads to the announced bound for x_m when

$$h = \left(\frac{1+M-\beta}{1+M-2\beta}\right)\epsilon^+. \qquad \square$$

VI. Conclusion

A. Summary

We have carefully investigated the evolution of the optimum $x(h)$ of (QP) as h varies from zero to infinity. The analysis that we performed could be used to develop a computationally efficient algorithm that solves (QP) starting from $h = \infty$. We used it to investigate the possibility to recover a sparse representation in the presence of bounded noise.

In this context, the idea is to solve (QP) for increasing h. As soon as $\|x(h)\|_0 < \frac{1}{2}(1 + \frac{1}{M})$, the optimum $x(h)$ of the algorithm has a nice rationally expected behavior, i.e., the number of nonzero components in $x(h)$ steadily decreases when h is further increased.

If one assumes to know ϵ, one can decide to increase h until it reaches $\tilde{h}$, the smallest value of h for which it holds that

$$\tilde{\beta} < \frac{1}{2} \quad \text{and} \quad \tilde{h} \ge \left(1 + \frac{\tilde{\beta}}{1+M-2\tilde{\beta}}\right)\epsilon$$

with $\tilde{\beta} = M\|x(\tilde{h})\|_0$. Such an $\tilde{h}$ always exists since $\tilde{\beta}$ and the right-hand side of the second inequality both decrease as h increases. For $h \ge \tilde{h}$, all the retained components belong necessarily to the true sparsest representation.

B. Comparison With Other Methods and Results

While iterative approaches to finding sparse representations have been around for much longer than global approaches, it has only been shown recently that though they are fundamentally suboptimal, they possess quite interesting properties [6]. This is especially true for the approach called Orthogonal Greedy Algorithm (OGA) in [8] or Orthogonal Matching Pursuit in [6] which was introduced in the early 1990s in an effort to improve upon the basic Matching Pursuit (MP) algorithm [18].

In order to describe these algorithms, we denote $r^{(k)}$ the reconstruction error or the residual at step k with initial residual $r^{(0)} = b$. MP is then an iterative method which at step k picks the column-vector of the A matrix with index $j_k = \arg\max_j |a_j^T r^{(k-1)}|$ that has maximal correlation with $r^{(k-1)}$ and assigns to it a weight

$$\chi_{j_k} = a_{j_k}^T r^{(k-1)} = \arg\min_{\chi} \left\| r^{(k-1)} - \chi a_{j_k} \right\|_2^2$$

that minimizes the energy of the new residual vector

$$r^{(k)} = r^{(k-1)} - \chi_{j_k} a_{j_k}.$$

This basic step is repeated until a stopping criterion is satisfied. The following criteria are common: stop after a specified number of atoms have been selected, stop when the maximal correlation is below a threshold, or stop when the residual norm is smaller than some fixed threshold.

In OGA, the difference with MP lies in the re-evaluation of the weights. At the end of step k, when the new entering column a_{j_k} has been selected as in MP, all the weights now denoted $\{\chi_{j_i}^{(k)}, i = 1, \dots, k\}$ are recomputed to minimize the residual norm

$$\min_{\chi_{j_i}, i=1,\dots,k} \left\| b - \sum_{i=1}^{k} \chi_{j_i} a_{j_i} \right\|_2^2.$$

This least-squares fit corresponds to an orthogonal projection of b on the subspace spanned by the k atoms in $\{a_{j_i}, i = 1, \dots, k\}$, hence, the name of the procedure. If we denote $\bar{A}_{(k)}$ the matrix whose columns are the k selected atoms then the optimal weights are given by $\bar{A}_{(k)}^+ b$, with $\bar{A}_{(k)}^+ = \left(\bar{A}_{(k)}^T \bar{A}_{(k)}\right)^{-1} \bar{A}_{(k)}^T$.

It is interesting to notice that (QP) in some sense achieves a goal that is pretty similar to the one of OGA when the stopping rule is set on the maximal acceptable correlation. Indeed, looking at the dual (DQP) of (QP) [20], [5]

$$\min_x \|Ax\|_2^2 \text{ subject to } \|A^T(Ax - b)\|_\infty \leq h \qquad \text{(DQP)}$$

one realizes that the constraint in this problem corresponds precisely with seeking a representation x such that the maximal correlation between the residual or reconstruction error $b - Ax$ and any column in A is smaller than h in absolute value. This means that (QP) performs, according to a minimal energy criterion, in a global way [20] and in a single, possibly computationally intensive, step what OGA achieves iteratively and suboptimally through a much computationally cheaper relaxation procedure.

There is, however, another difference that makes the two approaches distinct. At the optimum x^* of (QP), the correlations of the retained atoms with the final residual are set to $\pm h$ while in OGA they are set to zero. The nonzero components $\bar{x}^*$ of the optimum x^* of (QP) satisfy (6)

$$\bar{x}^* = \bar{A}^+ b - h \left(\bar{A}^T \bar{A}\right)^{-1} \operatorname{sign}(\bar{x}^*)$$

while the weights of the atoms retained by OGA are $\bar{A}_{(k)}^+ b$. There is thus no straight connection between both methods, because even though the second term in $\bar{x}^*$, which can be considered as a bias term induced by the presence of the regularization term in (QP), can be removed *a posteriori* to keep just $\bar{A}^+ b$, one does not know which criterion, if any, this new point optimizes.

To our knowledge, the scenario we consider in this paper has first been proposed in [8], where among other results it is established that if OGA is used with a stopping criterion using the threshold ϵ on the ℓ_2-norm of the residual, and provided the nonzero component of the true representation with the smallest absolute value is greater than a given bound, then OGA recovers the true support. This is also the type of results we propose in the second part of Theorem 4 for (QP). Though it is impossible to compare the two results, since they use different bounds, one can observe that we get a slightly smaller threshold on x_m.

In [8], a further result is proposed that corresponds to the first part of Theorem 4 and holds for (QP) in its equivalent form (3). Again it seems impossible to compare the results since there is no analytical link between B_2 in (3) and h in (QP).

Other major contributions to this domain are to be found in [6], [9], [10], where the problem of the recovery both in the absence and in the presence of bounded noise is addressed. These papers present a nice global view of the existing results as well as innovative contributions. Results that are qualitatively similar to Theorem 4 are presented in [10] as well as some specific quantitative examples for Gaussian noise. Applications to channel coding, linear regression, and numerical analysis are also described in [10].

C. Extensions

In [12], a similar model is considered in conjunction with (QP) but with the assumption that the noise e is Gaussian instead of ℓ_2-bounded. Recovering the exact sparse representation is then a difficult detection problem and only weak asymptotic results (as the signal-to-noise ratio tends to infinity) can be expected. The analysis, which has some similarities with the one where more general scenarios are considered, is extremely technical and of a completely different nature. Indeed, in a detection–estimation context, many practical situations correspond to the case where the true model for the observed vector b is

$$b = \sum_{i=1}^{p} \alpha_i a(\tau_i) + e$$

with $a(\tau)$ a known family of vectors parameterized by a scalar τ, α_i the scalar weights, and e some additive noise. In order to apply the previous setting to this scenario where one wants to detect the number p of components and estimate theirs weights α_i and parameters τ_i, a natural way to proceed is to uniformly discretize the values of τ over its compact domain to get the m columns $a_j = a(\tau)|_{\tau=t_j}$ of the A matrix and to apply for instance (QP). Since the true τ_i will generically not belong to the sampling points used to build the columns in A, there is no exact sparse representation of b even if the true number of components, p, is small. Hence, it is necessary to tune the scalar h in (QP) not only in order to erase the noise e but also to take care of the approximation errors that depend upon the discretization step. In the case of a single noiseless component, when $b = \alpha_1 a(\tau_1)$, the weights of the optimum of (QP), x^*, can be shown to be close to samples of a minimum L_1-norm interpolating function [26]. A more general scenario is considered in [20] where the issue is super-resolution. Heuristic rules are presented that indicate how both the discretization step and the parameter h in (QP) should be chosen in terms of the signal-to-noise ratio and the expected performance evaluated by means of the Cramér–Rao bounds.

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Gaussian Class Multivariate Weibull Distributions: Theory and Applications in Fading Channels

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***Abstract*—Ascertaining on the suitability of the Weibull distribution to model fading channels, a theoretical framework for a class of multivariate Weibull distributions, originated from Gaussian random processes, is introduced and analyzed. Novel analytical expressions for the joint probability density function (pdf), moment-generating function (mgf), and cumulative distribution function (cdf) are derived for the bivariate distribution of this class with not necessarily identical fading parameters and average powers. Two specific distributions with arbitrary number of correlated variates are considered and studied: with exponential and with constant correlation where their pdfs are introduced. Both cases assume equal average fading powers, but not necessarily identical fading parameters. For the multivariate Weibull distribution with exponential correlation, useful corresponding formulas, as for the bivariate case, are derived. The presented theoretical results are applied to analyze the performance of several diversity receivers employed with selection, equal-gain, and maximal-ratio combining (MRC) techniques operating over correlated Weibull fading channels. For these diversity receivers, several useful performance criteria such as the moments of the output signal-to-noise ratio (SNR) (including average output SNR and amount of fading) and outage probability are analytically derived. Moreover, the average symbol error probability for several coherent and noncoherent modulation schemes is studied using the mgf approach. The proposed mathematical analysis is complemented by various evaluation results, showing the effects of the fading severity as well as the fading correlation on the diversity receivers performance.**

***Index Terms*—Bit-error rate (BER), correlated fading, diversity, equal-gain combining (EGC), maximal-ratio combining (MRC), multichannel reception, multivariate analysis, outage probability, selection combining (SC), Weibull fading.**

I. Introduction

Multivariate statistics is a useful mathematical tool for modeling and analyzing realistic wireless channels with correlated fading. Such fading channels are usually met in digital contemporary communications systems employed with diversity receivers with not sufficiently separated antennas where space or polarization diversity is applied (e.g., hand-held mobile terminals and indoor base stations). In these applications, the correlation among the channels results in a degradation of the diversity gain obtained [1]–[3].

Reviewing the open technical literature, one can encounter several papers applying multivariate statistics for fading channel modeling, most of them concerning the Rayleigh and Nakagami-m distributions. In an early work, Nakagami has defined the m-bivariate probability density function (pdf) [4, p. 31], while many years later, an infinite series representation for the bivariate Rayleigh and Nakagami-m cumulative distribution functions (cdf)s have been presented by Tan and

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