Minimal L_1-Norm Reconstruction Function for Oversampled Signals: Applications to Time-Delay Estimation

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***Abstract*—We consider the problem of the reconstruction of an oversampled band-limited signal and obtain an explicit expression for the reconstruction function with minimal L_1-norm. It has good sparseness and localization properties that recommends its use in time-delay estimation but the result may be of interest in other domains. Compared to the standard sine cardinal reconstruction function, its rate of decrease is almost one order of magnitude higher.**

***Index Terms*—Minimal L_1-norm, oversampling, signal reconstruction, time-delay estimation.**

I. INTRODUCTION

The well-known Whittaker–Shannon interpolation theorem [1]–[3] states that if a signal $s(t)$ is band-limited, i.e., if its Fourier transform $\hat{s}(f)$ satisfies $\hat{s}(f) = 0$ for $|f| > 1/2$, it can be reconstructed from its samples $s(nh)$ provided the sampling period h satisfies $h \leq 1$

$$s(t) = \sum_{n=-\infty}^{+\infty} s(nh) \frac{\sin\left(\frac{\pi}{h}(t-nh)\right)}{\frac{\pi}{h}(t-nh)} = \sum_{n=-\infty}^{+\infty} s(nh)\,\mathrm{sinc}\left(\frac{1}{h}(t-nh)\right). \tag{1}$$

This relation can be interpreted in two different ways. It says how, for a fixed value of t, $s(t)$ can be reconstructed as a weighted sum of the samples $s(nh)$ and it indicates how the whole function $s(t)$ can be reconstructed as a sum of scaled, shifted, and weighted sine cardinal functions.

While, for $h = 1$, the expansion (1) is unique, if the signal is oversampled $(h < 1)$, other similar expansions of the form

$$s(t) = \sum_{n=-\infty}^{+\infty} s(nh)\psi(t-nh) \tag{2}$$

exist, with the reconstruction function $\psi(t)$ independent of the signal $s(t)$. For $h = 1/2$, for instance, one can obviously reconstruct $s(t)$ using just the even samples or the odd samples or using any convex combinations of these two expansions. Reconstruction functions different from the sinc function can be obtained using wavelet techniques [4], [5].

In the sequel, we will restrict ourselves to sampling periods of the form $h = 1/\ell$ with the integer ℓ defined as the oversampling factor. For ℓ large, one then expects the existence of *localized* reconstruction functions that decrease rapidly to zero. Though an infinite number of terms will always be required for exact reconstruction, approximate reconstruction should be feasible with a small number of samples when ℓ is large. We will present an analytic expression of the minimal L_1-norm reconstruction function. It is such quite *localized* reconstruction function.

There are actually two different optimization problems each associated with one of the two different interpretations of ((1), (2)). One can either consider the minimization of the L_1-norm of the reconstruction function: $\min \int |\psi(t)|dt$ or the minimization of the ℓ_1-norm of the sequence of weights: $\min \sum_n |\psi(t-nh)|$ the solution of which depends upon $t \in [0, h[$. There is a strong link between both problems. Since, by Fubini's theorem

$$\int_0^h \sum_n |\psi(t-nh)|dt = \int_{-\infty}^{\infty} |\psi(t)|dt$$

one can argue that, as t varies in $[0, h[$, the solutions of the ℓ_1 problem define a function ψ which is a solution of the L_1 problem.

We primarily sought this solution in order to apply it to a time-delay estimation problem [6] that is commonly encountered in radar, sonar, or geophysics. In this context, one expects this criterion to yield a sparse solution to an underdetermined system of linear equations. As such, it has also been applied to the general class of linear inverse problems [7].

The specific time-delay estimation problem we consider is the following. A known bandlimited signal $s(t)$ with maximal sampling period $h = 1$ is observed through more than one path

$$y(t) = \sum_{p=1}^{P} \alpha_p s(t-\tau_p). \tag{3}$$

Observing $\{y(t)\}$, one wants to detect the number of paths or replicas P and to estimate their characteristics $\{\alpha_p, \tau_p\}$. We ignore here the fact that noise may be present and to get a clear link with the reconstruction problem we consider the case of a single replica with unit amplitude. In this case, the problem reduces to observing $y(t) = s(t-\tau)$ and knowing $s(t)$, get an estimate of the delay τ. One possibility is to reconstruct the observation $y(t)$ using shifted versions of the known signal $s(t)$, i.e., to seek the weights $\{x_n\}$ in

$$y(t) = s(t-\tau) = \sum_n s(t-nh)x_n. \tag{4}$$

Comparing this relation with (2) yields $x_n = \psi(nh-\tau)$. The weights $\{x_n\}$ that allow for perfect reconstruction are thus samples from a reconstruction function $\psi(t)$. An estimate of τ can then be deduced from the location of the maximum of these weights or better from the location of the maximum of the underlying interpolating function $\psi(t)$. To get a *precise* estimate it will be helpful to have the most *localized* and rapidly decreasing reconstruction function ψ. We thus decide to take $h = 1/\ell$ small and try to estimate samples from a *localized* interpolating function $\psi(t)$. Since we will show later that the minimal L_1-norm reconstruction function is quite *localized*, we propose to solve the following minimal ℓ_1-norm problem:

$$\min \sum_n |x_n| \text{ subject to } y(t) = \sum_n s(t-n/\ell)x_n \quad \text{for } \ell > 1$$

and to get an estimate of τ from the so-obtained samples $\{x_n\}$ of $\psi(t)$.

In the next section, we present the main result of this correspondence in a more abstract setting with the more technical part presented in the Appendix. In Section III, we come back to the time-delay estimation context and more generally to the sparseness issues.

II. A RECONSTRUCTION THEOREM FOR OVERSAMPLED SIGNALS

A. The Properties of the Interpolating Function

We now investigate the properties that characterize the reconstruction functions $\psi(t)$ in (2) for oversampled signals. For a sampling pe-

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riod $h = 1/\ell$, the discrete-time Fourier transform $\hat{s}_d(f)$ of the sequence $\{s(nh)\}$

$$\hat{s}_d(f) = \sum_{n=-\infty}^{+\infty} s(nh)e^{-2i\pi nfh} \tag{5}$$

is an even periodic function with period equal to ℓ, that is identical to $\ell\hat{s}(f)$ for $|f| \leq 1/2$ [2], [3]. This implies that it vanishes outside the following intervals:

$$\hat{s}_d(f) = 0, \qquad \text{for } f \notin \bigcup_k \left]\ell k - \frac{1}{2}, \ell k + \frac{1}{2}\right[. \tag{6}$$

Since the reconstruction formula (2) can be seen as a discrete convolution product, the following relation holds:

$$\hat{s}(f) = \hat{s}_d(f)\hat{\psi}(f)$$

where $\hat{\psi}(f)$, the Fourier transform of $\psi(t)$, is an even function for symmetry reasons. Since $\hat{s}_d(f)$ satisfies (6) and $\hat{s}(f)$ vanishes for $|f| > \frac{1}{2}$, for this relation to hold, $\hat{\psi}(f)$ must satisfy

$$\begin{aligned} \hat{\psi}(f) &= 1/\ell: \quad f \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\ \hat{\psi}(f) &= 0: \qquad f \in \bigcup_{k\neq 0} \left[\ell k - \frac{1}{2}, \ell k + \frac{1}{2}\right]. \end{aligned} \tag{7}$$

These support conditions characterize the set of all interpolating functions $\psi(t)$. See the upper-right part of Fig. 6. We now seek, among these functions, the one with minimal L_1 norm, i.e., the solution of the following functional optimization problem:

$$\begin{aligned} &\text{Min} \int |\psi(t)|dt \quad \text{s.t. the constraints (7)} \\ &\text{where } \hat{\psi}(f) \text{ is the Fourier transform of } \psi(t). \end{aligned} \tag{8}$$

Remember that we expect the solution to be a *localized* function that decreases rapidly to zero.

Note that the minimal L_2 norm solution under the same constraints is easy to find. By Parseval's equality, it is the function whose Fourier transform is equal to $1/\ell$ on $[-1/2, 1/2]$ and zero elsewhere. It is thus $(1/\ell)\,\text{sinc}\,(t)$, the *standard* sine cardinal function divided by ℓ, and is probably the least *localized* interpolating function. It should not be confused with $\psi(t) = \text{sinc}\,(\ell t)$ which also satisfies (7) and is the *scaled* sine cardinal function that takes the oversampling into account. Note, however, that the rate of decrease of both functions is $1/t$ while it is of order $t^{-2+1/l}$ for the optimum of (8), as we shall see.

B. The Minimal L_1 Norm Solution

To get some insight on the optimum, we consider the easier situation where the function $s(t)$ to be interpolated is periodic with integer period N. Exact reconstruction of $s(t-\tau)$ for $\tau \in [0, 1/\ell]$ is then possible with (ℓN) samples $s(k/\ell)$ and the problem becomes finite-dimensional. For fixed τ, the minimization of the ℓ_1 norm of the corresponding sequence $x_j = \{\psi(j/\ell - \tau)\}$

$$\min \sum_{j=1}^{\ell N} |x_j| \text{ under } s(t-\tau) = \sum_{j=1}^{\ell N} s(t - j/\ell)x_j, \qquad \text{for } t \in (0, N)$$

is further equivalent, since $s(t)$ has then only N degrees of freedom, to

$$\begin{aligned} &\min \quad \sum_{j=1}^{\ell N} |x_j| \\ &s(k-\tau) = \sum_{j=1}^{\ell N} s(k + j/\ell)x_j, \qquad k = 1, \cdots, N \end{aligned}$$

that can be transformed in a linear program.

In this easier situation, using a specific band-limited and periodic function $s(t)$, it is possible, using a computer, to build the function $\psi(t)$ pointwise and to find out that, for fixed τ, among the ℓN optimal weights only $N+1$ are nonzero. This means that to reconstruct $s(t-\tau)$, except for the two neighboring sample points, only one out of every ℓ sample points is used. The function $\psi(t)$ thus actually vanishes on large intervals. This property, as we shall prove, is actually preserved for aperiodic $s(t)$ functions

$$\psi(t) = 0 \text{ for } |t| \in \bigcup_{k\geq 0} \left]k + \frac{1}{\ell}, k+1\right[. \tag{9}$$

See the upper-left part of Fig. 6.

Indeed, it may be true that both support conditions (condition (7) in the Fourier domain and condition (9) in the time domain) completely define the optimum of (8), since they seem to fix all the available degrees of freedom. However, finding the (or a) function satisfying these conditions is again difficult to achieve.

In order to find the potential solution, one should first observe that the well-known infinite product expression for $\sin \pi x$

$$\sin \pi x = \pi x \prod_{n\neq 0}\left(1 - \frac{x}{n}\right)$$

says that $\sin \pi x$ is a "polynomial" that vanishes on $\mathbb{Z}$. The sine cardinal function is then the "polynomial" that vanishes at these same points except for the origin were it is equal to one. This means that the shifted sine cardinal functions can be seen as the Lagrangian polynomials on $\mathbb{Z}$ and the Whittaker-Shannon interpolation formula (1) as Lagrange's interpolation formula [2], [8]. Knowing from the computer experimentations exactly which points have to be used, it is possible to write a similar Lagrange's interpolation formula. For instance, one can observe that for $\tau \in (0, 1/\ell)$, the set of interpolation points is $\mathbb{Z}_- \bigcup (\mathbb{Z}_+ + \frac{1}{\ell})$. The function which cancels exactly on this set is

$$\phi(x) = \frac{1}{\Gamma(x)\Gamma(\frac{1}{\ell} - x)}$$

and from Lagrange's formula one gets

$$s(\tau) = \phi(\tau)\left(\sum_{k\geq 0} s\left(k + \frac{1}{\ell}\right) \frac{\alpha_k}{\tau - k - \frac{1}{\ell}} + \sum_{k\leq 0} s(k)\,\frac{\alpha_k}{\tau - k}\right)$$

where α_k is the value of $1/\phi'$ at the interpolation point [2]. With a little algebra, this can be rewritten as in (2). The difficulty however is that for $\tau \in (1/\ell, 2/\ell)$, a different set of interpolation points will be chosen (obviously the previous one translated by $1/\ell$). A precise statement of this result is given in the following theorem whose proof is given in the Appendix.

A Reconstruction Theorem for Oversampled Signals*:*

Let $s(t)$ be a deterministic signal whose Fourier transform vanishes for $|f| > \frac{1}{2}$. For an integer oversampling ratio ℓ, i.e., at a rate of ℓ samples per second, a reconstruction function $\psi(t)$, which is such that

$$s(t) = \sum_{j=-\infty}^{\infty} s\left(\frac{j}{l}\right)\,\psi\left(t - \frac{j}{\ell}\right)$$

that has minimal L_1 norm is given by

$$\begin{aligned} \psi(t) &= \sum_{k\geq 0} \beta_k\,\frac{\phi(|t| - k)}{|t|}\,1_{|t|\in[k, k+\frac{1}{\ell}]} \\ \phi(x) &= \frac{1}{\Gamma(x)\Gamma(\frac{1}{\ell} - x)}, \qquad x \in \left[0, \frac{1}{\ell}\right] \\ \beta_k &= (-1)^k = \frac{\Gamma(k + \frac{1}{\ell})}{\Gamma(k+1)}. \end{aligned} \tag{10}$$

Where $\Gamma(\cdot)$ is the standard gamma-function

$$\Gamma(t) = \int_0^{+\infty} e^{-x}x^{t-1}\,dx. \qquad \square$$

As expected (9), the function $\psi(t)$ is nonzero only for $|t| \in [k, k + 1/\ell]$ with $k \geq 0$. This means that to reconstruct the signal at any point, except for the two neighboring sample points, only one every ℓ sample points is used in the reconstruction procedure leading to quite a parsimonious representation. Moreover, the weights to be given to these

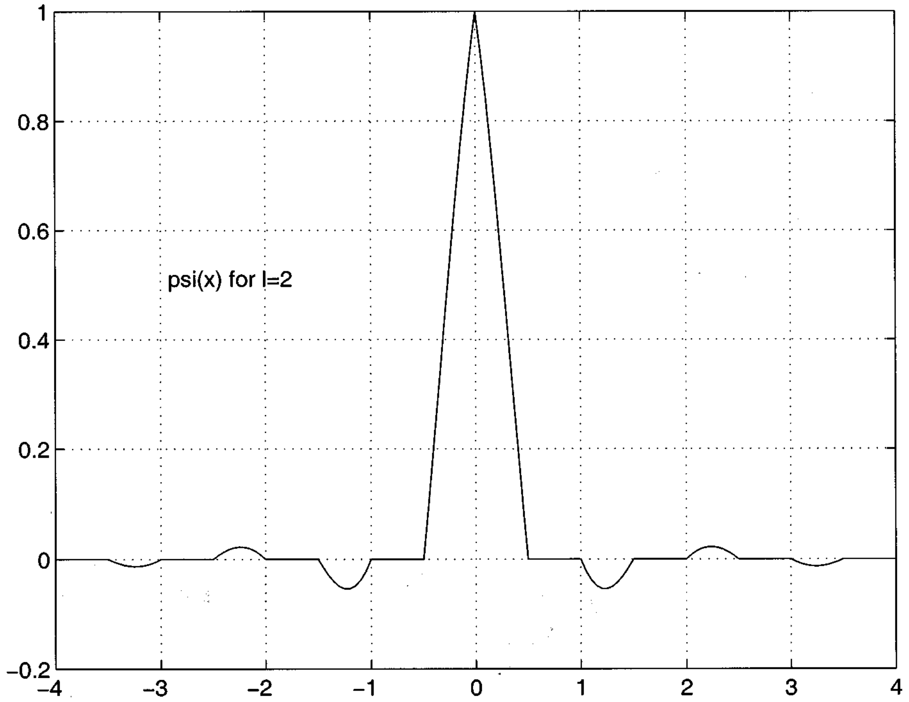

Fig. 1. The central part of the minimal L_1 norm interpolating function in case of an oversampling ratio of two. It uses one sample out of two.

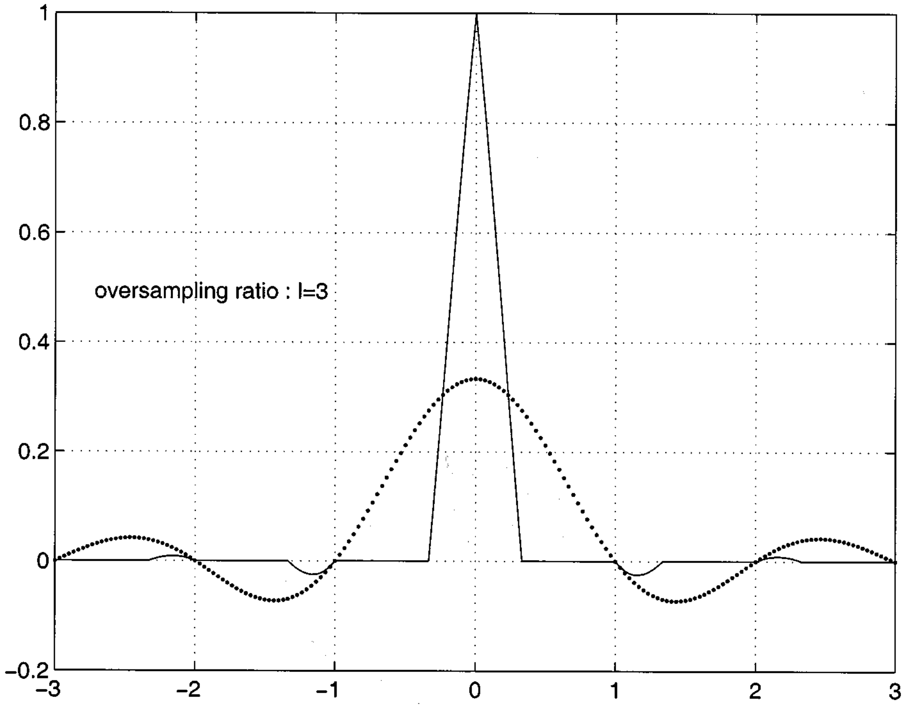

Fig. 2. The central part of the minimal L_1 norm interpolating function in case of an oversampling ratio of three. It uses one sample out of three. The dotted curve is the minimal L_2 norm interpolating function: the sine cardinal function.

samples decrease rapidly, they are of order $t^{-2+1/l}$, about one order of magnitude faster than the sine cardinal which decreases as $1/t$.

The central part of interpolating function $\psi(t)$ is shown in Figs. 1 and 2. For oversampling factors ℓ equal 2 and 3, respectively. In Fig. 2, we have added the interpolating function with minimal L_2 norm, i.e., the standard sine cardinal function divided by ℓ, to highlight the difference. One can check that for $\ell = 1$ the function $\psi(t)$ is nothing but the standard sine cardinal function.

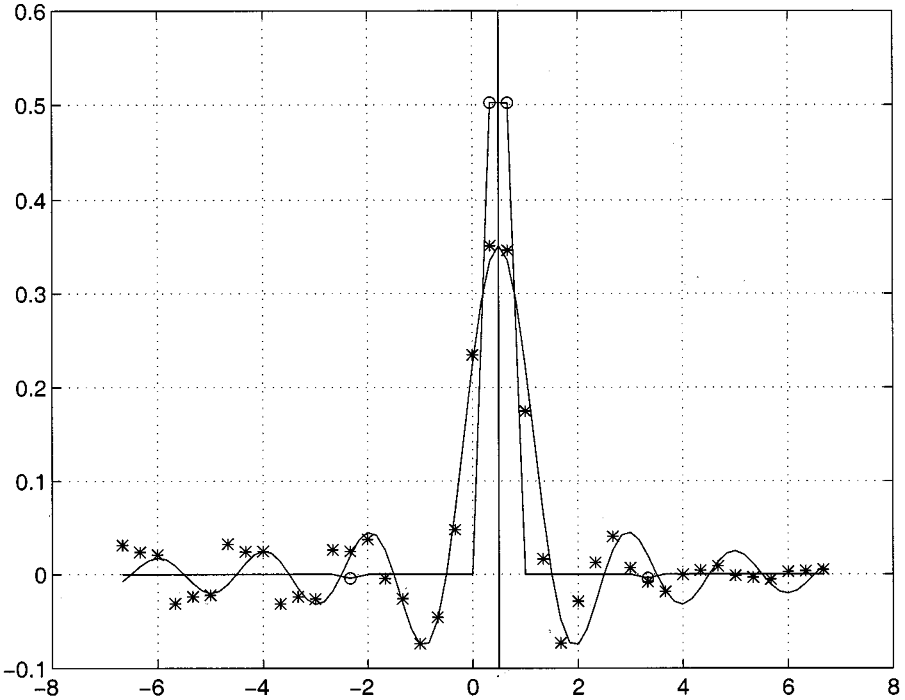

Fig. 3. Single replica scenario. The central part of the weights estimated for an oversampling factor of three. The minimal ℓ_2-norm weights, marked by "*," are close to the sine cardinal function. The minimal ℓ_1-norm weights, marked by "o," are very sparse. The true delay is given by the vertical solid line.

III. APPLICATION TO TIME-DELAY ESTIMATION

A. The Model

Let us apply this result to the time-delay estimation problem (3). Even in the simplest case of a single replica (4), many restrictions and approximations enter the ideal reconstruction scheme described above. The signal $s(t)$ is never, strictly speaking, bandlimited. Only a finite number, say m, of samples y_k of the signal $y(t)$ are available. It is thus realistic to plan to estimate a finite number n of weights x_j solving

$$\min \quad \sum_{j=1}^{n} |x_j|$$

$$y_k = \sum_{j=1}^{n} s(k - j/\ell)x_j, \qquad k = 1, \cdots, m. \tag{11}$$

This means that at best only biased estimates of samples of the central part of the reconstruction function ψ will be obtained. It appears that the analysis presented in Section II is nevertheless meaningful and the minimum ℓ_1 norm solution to the finite-dimensional underdetermined linear system (11) will indeed both lead to a sparse solution and allow for high resolution. We do intentionally severely truncate the observations to demonstrate these features.

We take for $s(t)$ a compressed real chirp signal [9], [6]. Consider the real linear frequency-modulated signal

$$h_k = u_k \cdot \sin\left(2\pi(\alpha k^2 + \beta k)\right)$$

with u_k a window function, $k \in [0, \cdots, 749]$, $\beta = 0.1$ and $\alpha = 1/3 \cdot 10^{-4}$ and take $s(t)$ in (3) and (4) as its autocorrelation.

B. Simulation of a Single Replica

We simulate the simplest case of a single replica (4) with delay $\tau = 0.5$ and unit amplitude, i.e., $y_k = s(k - 0.5)$. We keep $m = 49$ observations y_k for $k = -24$ to 24. The same time span is taken for the potential delays. For an oversampling ratio $\ell = 3$, there are then $n = 48 \times 3 + 1$ weights x_j in (11). We compute both the minimal ℓ_2 norm and ℓ_1 norm solution to the underdetermined linear system (11). The minimal ℓ_1 norm solution is obtained by transforming the problem into a linear program and using the simplex algorithm [10]. The results are presented in Fig. 3. Only the 41 central values out of the 145 components of the optimum X^* are represented. The 41 values of the minimal ℓ_2 norm solution are represented by $*$'s, the curve that is close to these stars is the standard sine cardinal function. The other curve is identically zero except for four points marked by o's and represents the weights of the minimal ℓ_1 norm solution. From both solutions one can deduce that the delay is close to 0.5, the true value represented by a vertical solid line, though this is clearly easier to do from the minimal ℓ_1 norm solution. Fig. 4 corresponds to the same scenario with an oversampling rate of 6. X has now 289 components and the true delay is among the potential values j/ℓ. The minimal ℓ_2 norm solution follows closely the scaled sine cardinal function and the minimal ℓ_1 norm solution has just one nonzero component at the exact position.

While the minimal ℓ_2-norm solution has all its components nonzero and looses in resolution when ℓ the oversampling ratio increases, the opposite is true for the minimal ℓ_1-norm solution. As the oversampling ratio increases it gets sparser, more *localized*, and gains in resolution.

Note that there appear indeed quite significant discrepancies in Figs. 3 and 4 between the theoretical weights (the sine cardinal for the ℓ_2-norm or the ψ function for the minimal ℓ_1 norm) and those obtained by solving the optimization problem. These are due to the fact that both m and n are finite. Remember that we take as observations $\{y_k\}$ the very central part (49 samples) of $s(t - \tau_1)$ the compressed chirp which has about 300 significant samples, introducing an important truncation effect.

C. Two Close Replicas

We simulate the case of two close replicas with unit amplitude and delays $\tau_1 = -0.5$ and $\tau_2 = 1$ in (3). We keep an oversampling rate of six so that both delays correspond to the potential values j/ℓ.

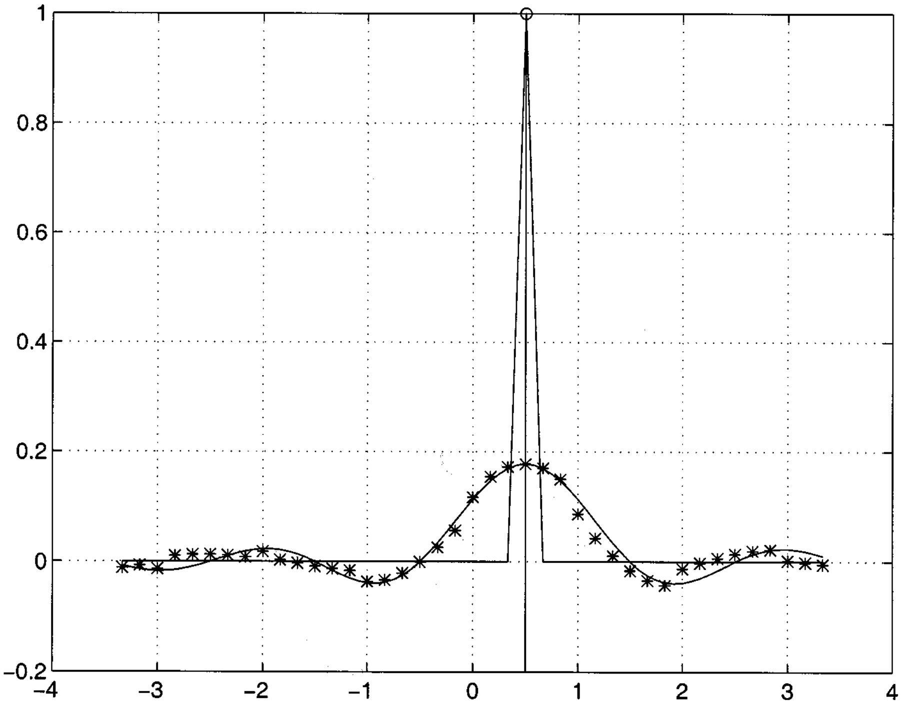

Fig. 4. Single replica scenario. The central part of the weights estimated for an oversampling factor of six. The minimal ℓ_2-norm weights, marked by "*," are close to the sine cardinal function that further spreads out. The minimal ℓ_1-norm solution has just one nonzero weight marked by a "o," exactly located at the true delay.

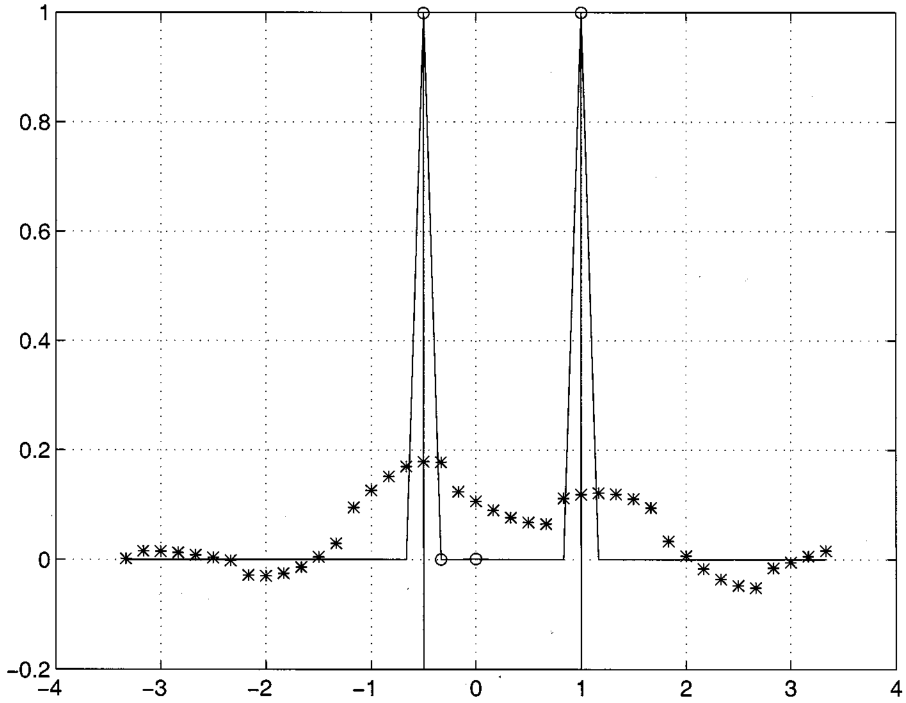

Fig. 5. Two-replicas scenario. The central part of the weights estimated for an oversampling factor of six. The minimal ℓ_2-norm weights, marked by "*," are quite low and flat. The minimal ℓ_1-norm solution has just four nonzero weights marked by a "o," with two of them exactly located at the true delays, the vertical lines.

The results are given in Fig. 5. The minimal ℓ_2 norm solution is represented by $*$'s and it would be difficult to guess that there are two replicas. The minimal ℓ_1 norm solution is represented by o's, there are just four nonzero components, two of them are at the true locations with *exact* amplitudes. The remaining two nonzero coefficients are extremely small and due to truncation errors.

IV. CONCLUSIONS

We have obtained the explicit expression of the minimum L_1 norm reconstruction function for oversampled bandlimited signals with integer oversampling ratios denoted ℓ. As compared to other possible reconstruction functions such as the *standard* sine cardinal $(1/\ell)\,\mathrm{sinc}\,(t)$ or the *scaled* sine cardinal $\mathrm{sinc}\,(\ell t)$ whose rate of decrease is of order $1/t$, its rate of decrease is of order $t^{-2+1/\ell}$ about one order of magnitude faster, a feature that should be of interest in many domains.

We have highlighted the importance of this function in the context of a specific time-delay estimation problem. The sinusoids in additive noise signal is another example where the same technique can be applied to obtain estimates of the characteristics of the signal. It suffices to observe that the periodogram of the observations is then a sum of shifted Fejer kernels [11], [12] to realize that it is similar to the time-delay estimation problem.

It appears that the applicability of this approach goes beyond the specific applications we have considered. Sparseness has become an important issue in signal processing where in many problems one has to select a sparse solution to an underdetermined system of linear equations. This is the case if one wants to represent a signal as a linear combination of components belonging to redundant or overcomplete bases [7]. The present analysis reveals, more generally, why seeking the minimal ℓ_1 norm solution is of interest in this context.

We have considered noiseless signals. If noise is present, it can either be considered as a perturbation and one has to rely on the robustness of the approach or can be modeled and taken care of by changing the criterion [11], [12], [6], [13] and allowing for reconstruction errors.

APPENDIX
PROOF OF THE THEOREM

Let us prove the thorem presented in Section II-B. For completness we restate it.

Theorem 1: The solution of the problem

$$\min\ \|\psi\|_1$$

under the constraints

$$\hat{\psi}(f) = \frac{1}{\ell}, \qquad f \in \left[-\frac{1}{2}, +\frac{1}{2}\right]$$
$$\hat{\psi}(f) = 0, \qquad f \in \cup_{k\neq 0}\left[\ell k - \frac{1}{2}, \ell k + \frac{1}{2}\right]$$

is

$$\psi(x) = \sum_{k\geq 0} \beta_k\ \frac{\phi(|x|-k)}{|x|}\ 1_{|x|\in[k,k+\frac{1}{\ell}]}$$
$$\phi(t) = \frac{1}{\Gamma(t)\Gamma(\frac{1}{\ell}-t)}, \qquad t\in\left[0, \frac{1}{\ell}\right]$$
$$\beta_k = (-1)^k\ \frac{\Gamma(k+\frac{1}{\ell})}{\Gamma(k+1)}.$$

Proof: Let us first show that ψ satisfies the constraints. We have to prove that for $|\tau|\leq 1/2$, $\hat{\psi}(\tau+n\ell)=\frac{1}{\ell}$, if $n=0$ and $\hat{\psi}(\tau+n\ell)=0$ if n is another integer. This is the same as proving that

$$\sum_n \hat{\psi}(\tau+n\ell)e^{2i\pi yn\ell} = \frac{1}{\ell} \tag{12}$$

for any $y\in[0,\frac{1}{\ell}]$ and $|\tau|\leq 1/2$. One has, using the Poisson formula (see Lemma 4)

$$\begin{aligned}\ell\sum_n \hat{\psi}(\tau+n\ell)e^{2i\pi y(n\ell+\tau)} &= \sum_n e^{-2i\pi n\tau/\ell}\psi\left(\frac{n}{\ell}+y\right)\\ &= \sum_k\sum_n \beta_k e^{-2i\pi n\tau/\ell}\frac{\phi(|\frac{n}{\ell}+y|-k)}{|\frac{n}{\ell}+y|}1_{|\frac{n}{\ell}+y|\in[k,k+\frac{1}{\ell}]}.\end{aligned}$$

Since $y\in[0,\frac{1}{\ell}]$, the sum with respect to (w.r.t.) n reduces to $n=\ell k$ or $n=-\ell k-1$, and

$$\begin{aligned}\ell\sum_n \hat{\psi}(\tau+n\ell)e^{2i\pi y(n\ell+\tau)} &= \sum_k e^{-2i\pi\tau k}\beta_k\frac{\phi(y)}{k+y} + \sum_k e^{2i\pi\tau(k+\frac{1}{\ell})}\beta_k\frac{\phi(\frac{1}{\ell}-y)}{k+\frac{1}{\ell}-y}\\ &= \phi(y)\sum_k\beta_k\frac{e^{-2i\pi\tau k}}{k+y} + \phi(y)\sum_k\beta_k\frac{e^{2i\pi\tau(k+\frac{1}{\ell})}}{k+\frac{1}{\ell}-y}\\ &= e^{2i\pi y\tau}\end{aligned}$$

the last equality is established in Lemma 2 below, the previous uses the fact that $\phi(\frac{1}{\ell}-y)=\phi(y)$; (12) is thus proved and function $\hat{\psi}$ satisfies the constraints.

We shall prove now that ψ is a solution to the minimization problem. Let us introduce the piecewise-constant function

$$\lambda(x) = \frac{1}{\cos(\pi/2\ell)}\ \cos(\pi([\ell x]+1/2)/\ell)$$

where $[x]$ stands for the integer part of x. In Fig. 6, we present for an oversampling factor $\ell=3$, the function $\psi(\cdot)$ and the function $\lambda(\cdot)$ together with their Fourier transforms.

Notice that $|\lambda(x)|\leq 1$ and $\lambda(x)\psi(x)=|\psi(x)|$ (because ψ is nonzero only on $|x|\in[k,k+1/\ell]$ with $\mathrm{sign}\,(-1)^k$). It is proved in Lemma 3 that λ is the Fourier transform of

$$\hat{\lambda}(\nu) = \frac{\ell}{2\pi}\tan(\pi/2\ell)\sum_k\frac{\delta_{k\ell+1/2}+\delta_{-k\ell-1/2}}{k\ell+1/2}.$$

Let $\theta(x)$ be another function satisfying the constraints, then

$$\begin{aligned}\|\theta\|_1 \geq \int\theta(x)\lambda(x)\,dx &= \int\hat{\theta}(\nu)\hat{\lambda}(\nu)\,d\nu\\ &= \frac{\ell}{4\pi}\ \tan(\pi/2\ell)(\hat{\theta}(-1/2)+\hat{\theta}(1/2)) = \frac{1}{2\pi}\ \tan(\pi/2\ell)\end{aligned}$$

which is $\|\psi\|_1$ since for $\theta=\psi$, the first inequality is an equality. □

Lemma 1 is used in the proof of Lemma 2.

Lemma 1: For $t\in[0,1/\ell]$, the following is true:

$$\phi(t)^{-1} = \Gamma(1/\ell)\int_0^\infty\ u^{t-1}(1+u)^{-1/\ell}\,du.$$

Proof: We recall that

$$\Gamma(x) = \int_0^\infty\ e^{-t}t^{x-1}\,dt, \qquad \mathrm{Re}\,(x)>0.$$

We have,

$$\begin{aligned}\phi(x)^{-1} &= \Gamma(x)\Gamma(1/\ell-x)\\ &= \int_0^\infty\int_0^\infty e^{-t-s}t^{x-1}s^{1/\ell-x-1}\,dt\,ds\\ &= \int_0^\infty\int_0^\infty e^{-s(1+u)}u^{x-1}s^{1/\ell-1}\,ds\,du, \qquad t=su\\ &= \int_0^\infty\int_0^\infty e^{-v}u^{x-1}v^{1/\ell-1}(1+u)^{-1/\ell}\,dv\,du,\\ &\qquad\qquad s(1+u)=v\\ &= \Gamma(1/\ell)\int_0^\infty u^{x-1}(1+u)^{-1/\ell}\,du.\end{aligned}$$ □

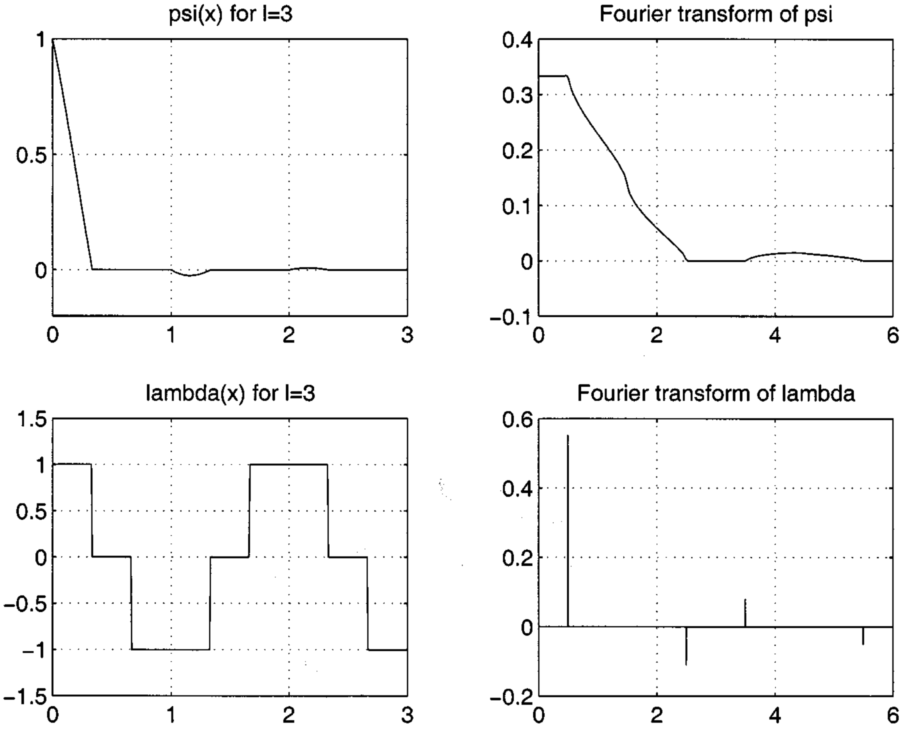

Fig. 6. For an oversampling ratio $\ell = 3$, the reconstruction function $\psi(x)$, the associated $\lambda(x)$ function, and their Fourier transforms.

Lemma 2: For $z = e^{i\theta}$, $\theta \in (-\pi, \pi)$, and $x > 0$, the following is true:

$$\sum_k \beta_k \left(\frac{z^k}{x+k} + \frac{z^{-k-1/\ell}}{\frac{1}{\ell} - x + k} \right) = \phi(x)^{-1} e^{-i\theta x}.$$

For $z \in \Omega = \boldsymbol{C} \backslash \boldsymbol{R}_-$, we define z^x by $e^{x \log(z)}$, where the principal determination of the logarithm is used. z^x is an entire function on this open domain.

Convergence of the Taylor expansion gives for $|z| < 1$

$$\sum_k \beta_k z^k = \Gamma(1/\ell)(1+z)^{-1/\ell}.$$

Hence

$$\begin{aligned}\sum_k \frac{\beta_k}{x+k} z^k &= \sum_k \beta_k z^k \int_0^1 u^{x+k-1}\, du \\ &= \Gamma(1/\ell) \int_0^1 (1+uz)^{-1/\ell} u^{x-1}\, du.\end{aligned}$$

Since $\beta_k \simeq k^{1/\ell - 1}$, this equality remains true for $|z| = 1$.

Note that for z as in the lemma and $v > 0$, $1 + zv$ has also its argument in $(-\pi, \pi)$ and

$$(1 + z^{-1}v)^{1/2} z^{1/2} = (z+v)^{1/2}$$

because both functions are entire on Ω, and coincide on $\boldsymbol{R}_+$. Finally,

$$\begin{aligned}\Gamma(1/\ell)^{-1} &\sum_k \beta_k \left(\frac{z^k}{x+k} + \frac{z^{-k-1/\ell}}{1/\ell - x + k} \right) \\ &= \int_0^1 (1+zu)^{-1/\ell} u^{x-1}\, du \\ &\quad + \int_0^1 (1+vz^{-1})^{-1/\ell} (vz^{-1})^{1/\ell} v^{-x-1}\, dv \\ &= \int_0^1 (1+zu)^{-1/\ell} u^{x-1}\, du \\ &\quad + \int_0^1 (1+zv^{-1})^{-1/\ell} v^{-x-1}\, dv \\ &= \int_0^1 (1+zu)^{-1/\ell} u^{x-1}\, du \\ &\quad + \int_1^\infty (1+zu)^{-1/\ell} u^{x-1}\, du \\ &= \int_0^\infty (1+zu)^{-1/\ell} u^{x-1}\, du.\end{aligned}$$

Note that for $z \in \boldsymbol{R}_+$, one has

$$\int_0^\infty (1+zu)^{-1/\ell} u^{x-1}\, du = z^{-x} \int_0^\infty (1+u)^{-1/\ell} u^{x-1}\, du.$$

The identity between those two entire functions extends to Ω, and using Lemma 1

$$\begin{aligned}\sum_k \beta_k &\left(\frac{z^k}{x+k} + \frac{z^{-k-1/\ell}}{1/\ell - x + k} \right) \\ &= \Gamma(1/\ell) z^{-x} \int_0^\infty (1+u^2)^{-1/2} u^{x-1}\, du \\ &= z^{-x} \phi(x)^{-1}.\end{aligned}$$

□

Lemma 3: For any $a > b > 0$, the Fourier transform of

$$\frac{a \sin(\pi b/a)}{2\pi} \sum_n \frac{\delta_{an+b} + \delta_{-an-b}}{an+b}$$

is

$$\cos(2\pi b([ax] + 1/2)/a).$$

Proof: We shall prove

$$\frac{a \sin(\pi b/a)}{\pi} \sum_n \frac{e^{-2i\pi(an+b)x}}{an+b} = e^{-2i\pi b([ax]+1/2)/a} \tag{13}$$

whose real part is exactly what is needed. By a simple normalization argument (consider the variables $b' = b/a$ and $x' = ax$) we see that it suffices to prove this identity in the case $a = 1$. We have, denoting by $\chi(x)$ the indicator of $[0, 1]$, and using the Poisson formula (Lemma 4)

$$\begin{aligned} e^{-2i\pi b[x]} &= \sum_n \chi(x-n)e^{-2i\pi bn} \\ &= -\,e^{-2i\pi bx} \sum_n \hat{\chi}(-b-n)e^{2i\pi xn} \text{(Poisson)} \\ &= -\,e^{-2i\pi bx} \sum_n \frac{e^{i\pi(b+n)}\sin(\pi(b+n))}{\pi(b+n)}\, e^{2i\pi xn} \\ &= -\,e^{i\pi b}\, \frac{\sin(\pi b)}{\pi} \sum_n \frac{e^{2i\pi x(n-b)}}{n-b} \\ &= e^{i\pi b}\, \frac{\sin(\pi b)}{\pi} \sum_n \frac{e^{-2i\pi x(n+b)}}{n+b} \end{aligned}$$

which is (13) with $a = 1$. □

Lemma 4 (Poisson Summation Formula [3]): For any square integrable function f and any real numbers x, ν, T, one has

$$T \sum_n f(nT+x)e^{-2i\pi n\nu T} = e^{2i\pi x\nu} \sum_n \hat{f}(n/T+\nu)e^{-2i\pi nx/T}$$

where the Fourier transorm of f is

$$\hat{f}(\nu) = \int f(x)e^{-2i\pi x\nu}\, dx.$$

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A Note on the Discrete Wavelet Transform of Second-Order Processes

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***Abstract*—Some properties of second-order random processes such as, stationarity and scaling, are characterized via corresponding (multiscale) properties of their discrete wavelet transform. The single scale characterizations valid in the continuous setting do not extend to the discrete setting. Some extensions of these results are also briefly indicated.**

***Index Terms*—Random fields, random processes, self-similarity, stationarity, wavelet transform.**

I. INTRODUCTION

The purpose of this note is to present some complements to the study of the discrete wavelet transform of second-order random processes [1], [2], [4]–[6], [8]–[10].

A main result of [2] was the characterization of some second-order properties of a process via corresponding properties of its continuous wavelet transform. Another main feature of [2] was the identification of a tradeoff between the regularity of the wavelet and characterizations valid at a single scale or at every scale. These results were extended to random processes (and fields) without finite moments in [1]. Here the role of oversampling (inherent to the continuous setting) is filtered, and characterizing properties of second-order random processes are given via corresponding properties of the discrete wavelet transform. In the discretized framework the above conclusions will only persist as long as the multiscale analysis is concerned, and we show that the single scale, i.e., the linear, analysis no longer provides such characterizations.

Throughout the correspondence let $(\Omega, \mathcal{B}, P)$ be a probability space, let $X = \{X_t\}_{t\in\mathbb{R}}$ be a second-order process, i.e., X is jointly measurable and X_t is square integrable for all $t \in \mathbb{R}$. Let $\psi : \mathbb{R} \to \mathbb{C}$ be an analyzing wavelet. The *discrete wavelet transform* of X is the (discrete) random field

$$\{W(j,k)\}_{j,\,k\in\mathbb{Z}} = \left\{2^{-j/2} \int_{\mathbb{R}} X(u)\psi(2^{-j}u-k)\, du\right\}_{k,\,j\in\mathbb{Z}} \quad (1)$$

provided the path integral in (1) is defined with probability one. We often write W^ψ to emphasize the fact that the wavelet transform is taken with respect to ψ.

The discrete wavelet transform of X is the discretization (at times $k2^j$ and scales 2^j) of the continuous wavelet transform of X. Since X has finite second moments, with $R_X(t,s) = EX_t\overline{X_s}$, $t, s \in \mathbb{R}$, a generic condition which ensures that (1) is well defined and is also a second-order random sequence is

$$\int_{\mathbb{R}} \sqrt{R_X(u,u)}\,|\psi(2^{-j}u-k)|\, du < +\infty \quad (2)$$

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