Master2 RI **January 2008**

OPT : Introduction to Numerical and Combinatorial Optimization

(2 hours - with lecture notes - Deliver you work on separate sheets for each section)

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1 Numerical optimization

1. We consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x) = 2\,{x_1}^2 + {x_2}^2 + 20\,x_1$.

   Is it a positive quadratic form ? Put it under the form $f(x) = x^t Ax + B^t x + c$ where A is a symmetric matrix. Where are the minima of f ? Express $f(x)$ as $(x - x^0)^t A(x - x^0) + d$ for a suitable point $x^0 \in \mathbb{R}^2$ and $d \in \mathbb{R}$.
2. Consider the equality constraint $\theta_1(x) = {x_1}^2 + {x_2}^2 - 5 = 0$. Does it define a domain of regular points ? Solve the problem

   $$\arg\min_x f(x) \quad \text{s.t.} \quad \theta_1(x) = 0$$

   explaining all steps of your method. Is there a unique local minimum ?
3. We now consider the domain defined by $\theta_1(x) \leq 0$ and $\theta_2(x) = 3\,x_1 + x_2 - 5 \leq 0$. Is it a regular domain ? Solve the problem

   $$\arg\min_x f(x) \quad \text{s.t.} \quad \theta_1(x) \leq 0, \;\; \theta_2(x) \leq 0$$

   explaining all steps of your method. Which constraints are active at the optimum ?

2 Combinatorial optimization

1. **Branch-&-Bound :** You are in the process of solving an integer linear maximization problem, in variables x_1, x_2, x_3, by Branch-&-Bound. The current value of the lower bound LB is $-\infty$. The list of active subproblems is

subproblem	z^*	x_1^*	x_2^*	x_3^*
P^1 with $x_1 \geq 6, x_2 \leq 3$	90.50	6.00	3.00	0.50
P^2 with $x_1 \leq 5, x_2 \leq 13$	165.25	5.00	13.00	5.75
P^3 with $x_1 \leq 5, x_2 \geq 14, x_3 \geq 1$	138.00	4.25	16.00	1.00
P^4 with $x_1 \leq 5, x_2 \geq 14, x_3 \leq 0$	121.25	3.75	15.25	0.00

   where x^* is the optimal solution for the linear-programming relaxation of a subproblem and z^* is the objective value of x^*.

   Question 1. *What is the current value of the upper bound UB ? Explain.*

   Solution 1. UB=165.25

   Question 2. *Have we fathomed any subproblem by integrality yet ? Explain.*

   Solution 2. No subproblem has been solved since LB=$-\infty$. Hence, no subproblem is fathomed by integrality.

Question 3. *Have we fathomed any subproblem by bounds or by unfeasibility yet ? Explain.*

Solution 3. We observe that subproblem $x_1 \geq 6, x_2 \geq 4$ is not in the list and has therefore been fathomed by bounds or by unfeasibility.

Assume now that you have selected P^2 to solve, and branching on x^3 you have obtained two new subproblems P^5 with $x_1 = 5, x_2 = 13, x_3 = 5, z^* = 120$ and P^6 with $x_1 = 5, x_2 = 13, x_3 = 6, z^* = 130$.

Question 4. *What are now the current values of the upper and lower bound ?*

Solution 4. LB=130 since P^5 and P^6 have been solved but $130 > 120$. Upper bound does not change (i.e. UB=138).

Question 5. *Is the problem solved ? Argue why.*

Solution 5. The problem is not solved since since $UB > LB$ (i.e. P^3 has been not explored).

Question 6. *If the problem is not solved, what will be the list of active subproblems ?*

Solution 6. The list of active subproblems contains P^3 (P^1 and P^4 have been fathomed).

2. **Modeling :** Given a rectangular grid V of n rows and m columns. To any point (i, j), where i corresponds to the row, while j corresponds to the column number, is a associated a weight $w(i, j)$. A set of points, such that for any couple (i_1, j_1) and (i_2, j_2) we have the relation $j_1 \neq j_2$, and $j_1 < j_2$ implies $i_1 \leq i_2$, will be called *non-decreasing* path. We look for the non-decreasing path passing through any column in V and with maximum total weight.

Question 7. *Give the MIP formulation of the corresponding optimization problem without introducing edges in V.*

Solution 7. To any point (i, j) we associate a binary variable $x(i, j)$ such that $x(i, j) = 1$ if the path passes through (i, j) (otherwise $x(i, j) = 0$). The problem to solve can be formulated as follows :

$$\begin{aligned} \textbf{maximize} \quad & \sum_{i=1}^{n} \sum_{j=1}^{m} w(i,j)x(i,j) \\ \textbf{subject to} \quad & \sum_{i=1}^{n} x(i,j) = 1, \quad j = 1, \dots, m \\ & \sum_{k=1}^{i} x(k,j) \geq \sum_{k=1}^{i} x(k,j+1), \quad i = 1, \dots, n, \; j = 1, \dots, m-1, \end{aligned}$$

3 Theory of linear programming and 2-player zero-sum strategic games

The present exercise aims at characterizing optimal strategies of players as solutions of optimization problems. In the following, we take the convention that optimization problems or equations are numbered according to the question they come from : for example, the optimization problem of Question 8 should be called Problem (8) and the objective function in Question 9 should be called Equation (9). You are welcome to use this jargon in your answers.

Given a 2-player zero-sum strategic game, we denote by U the matrix payoff (for Player1) and we assume its dimension is $n \times m$; hencefoth, n (resp. m) is the dimension of the pure strategy space S_1 of Player1 (resp. S_2 of Player2). As in the lecture notes, we denote by X_1 (resp. X_2) the n-dimensional (resp. m-dimensional) set of mixed strategies of Player1 (resp. Player2).

Question 8. *Assuming some mixed strategy $\bar{x}_2 \in X_2$ for Player2 is given, define an optimization problem to characterize strategies of Player1 which are best responses to $\bar{x}_2$. Be accurate when you write the objective function and the constraints of this problem.*
Solution 8. Problem (8) is given by :

$$\begin{aligned} \textbf{maximize} \quad & x_1^t U \bar{x}_2 & (1)\\ \textbf{subject to} \quad & \sum_{j=1}^{n} x_1(j) = 1 & (2)\\ & x_1(j) \geq 0, \quad \forall j = 1, \ldots, n & (3) \end{aligned}$$

Problem (8) can be written as $\max_{x_1 \in X_1} x_1^t U \bar{x}_2$, since (2)-(3) express $x_1 \in X_1$.

Question 9. *Justify why the objective function of Problem* (8) *can be written as follows :*

$$\max_{j=1,\ldots,n} e_j^t U \bar{x}_2$$

where e_j is the n(resp. m)-dimensional vector whose components all equal 0 *but the j-th equals* 1*.*
Hint : *Apply the fundamental theorem of linear programming to relate the basic feasible solutions of Problem* (8) *to the pure strategies of Player1.*
Solution 9. From the fundamental theorem of linear programming, we know that if Problem (8) has a finite optimum, then it has an optimal basic feasible solution which is an extreme point of the set X_1. Notice that the extreme points of X_1 are all vectors whose components equal zero but one equals to one. Consequently, the basic optimal solutions of Problem (8) correspond to pure strategies, and we have $\max_{x_1 \in X_1} x_1^t U \bar{x}_2 = \max_{j=1,\ldots,n} e_j^t U \bar{x}_2$.

Question 10. *Keeping in mind that Player1 is* rational *(she wants to maximize her payoff), characterize the strategy Player2 should follow provided he is rational too.*
Solution 10. Player2 should adopt an optimal strategy x_2^* which satisfies the following equation

$$x_2^* = \operatorname*{argmin}_{x_2 \in X_2} \max_{x_1 \in X_1} x_1^t U x_2$$

Similarly, denote by x_1^* the optimal strategy of Player 1.

Question 11. *By combining Equations* (9) *and* (10)*, describe the entire optimization problem corresponding to an optimal strategy for Player2.*
Solution 11. Problem (11) is given by :

$$\begin{aligned} \textbf{minimize} \quad & \max_{j=1,\ldots,n} e_j^t U x_2 \\ \textbf{subject to} \quad & \textstyle\sum_{i=1}^m x_2(i) = 1 \\ & x_2(i) \geq 0, \qquad \forall i = 1, \ldots, m \end{aligned}$$

Question 12. *By introducing a new variable v representing an upper bound on the terms $e_j^t U x_2$ $(j = 1, \ldots, n)$, recast Problem* (11) *as a linear program (LP). The optimal value of this LP hence denotes the optimal payoff of Player2.*
Solution 12. Problem (12) consists in the following :

$$\begin{aligned} \textbf{minimize} \quad & v \\ \textbf{subject to} \quad & v \geq e_j^t U x_2, j = 1, \ldots, n \\ & \textstyle\sum_{i=1}^m x_2(i) = 1 \\ & x_2(i) \geq 0, \qquad \forall i = 1, \ldots, m \end{aligned}$$

Question 13. *Write the dual of Problem* (12)*.*
Solution 13. Problem (13), the dual of Problem (12), is

$$\begin{array}{ll} \textbf{maximize} & u \\ \textbf{subject to} & u \leq e_i^t U^t x_1, i = 1, \ldots, m \\ & \sum_{j=1}^n x_1(j) = 1 \\ & x_1(j) \geq 0, \qquad\qquad \forall j = 1, \ldots, n \end{array}$$

Question 14. *Apply the LP Duality to Problem* (12) *and Problem* (13) *in order to prove the Minmax Theorem stated as follows :*

There exist stochastic vectors x_1^* *and* x_2^* *such that*

$$\max_{x_1 \in X_1} x_1^t U x_2^* = \min_{x_2 \in X_2} {x_1^*}^t U x_2$$

Solution 14. By the LP Duality Theorem, we have $u^* = v^*$. By definition of v^* and u^* we have

$$\begin{cases} v^* = & \max_{j=1,\ldots,n} e_j^t U x_2^* = \max_{x_1 \in X_1} x_1^t U x_2^* \\ u^* = & \min_{i=1,\ldots,m} e_i^t U^t x_1^* = \min_{x_2 \in X_2} x_2^t U^t x_1^* = \min_{x_2 \in X_2} {x_1^*}^t U x_2 \end{cases}$$

which concludes.