Combinatorial Optimization 4

Rumen Andonov

Irisa/Symbiose and University of Rennes 1

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Modelling

CPU Scheduling

10 tasks must be run on 3 CPUs at 1.33, 2 and 2.66 GHz (each processor can run only one task at a time). The number of elementary operations of the tasks (expressed in billions of instructions (BI)) is as in the below table. Schedule tasks to processors so that the completion time of the last task is minimized. *[Leo Liberti, Ecole Polytechnique]*

process	1	2	3	4	5	6	7
BI	1.1	2.1	3	1	0.7	5	3

- Parameters
  - b_i : number of BI (billion instructions) in task i ;
  - s_k : speed of CPU k in GHz ;
  - W_{max} : upper bound for completion time of all tasks.
  - set P of tasks ;
  - set C of CPUs.

CPU Scheduling : Solution

- Variables
  - $x_i \geq 0$: starting time of task i ;
  - $y_i \in Z_+$: CPU ID to which task i is assigned ;
  - $z_{ik} = 1$ if task i is assigned to CPU k, 0 otherwise ;
  - $\sigma_{ij} = 1$ if task i ends before task j starts, 0 otherwise ;
  - $\varepsilon_{ij} = 1$ if task i is executed on a CPU having lower ID than task j ;
  - $L_i \geq 0$: length of task i ;
  - $W \geq 0$: completion time of all tasks.
- Objective function : $\min W$

CPU Scheduling : Solution

- Constraints :
  - (lengths) $\forall i \in P (L_i = \sum_{k \in C} \frac{b_i}{s_k} z_{ik})$;
  - (times) $\forall i \in P (t_i + L_i \leq W)$;
  - (assignment) $\forall i \in P (\sum_{k \in C} z_{ik} = 1)$;
  - (cpudef) $\forall i \in P (y_i = \sum_{k \in C} k z_{ik})$;
  - (horizontal non-overlapping) $\forall i \neq j \in P (x_j - x_i - L_i - (\sigma_{ij} - 1) W_{max} \geq 0)$;
  - (vertical non-overlapping) $\forall i \neq j \in P (y_j - y_i - 1 - (\varepsilon_{ij} - 1) \mid P \mid \geq 0)$;
  - (at least one position) $\forall i \neq j \in P (\sigma_{ij} + \sigma_{ji} + \varepsilon_{ij} + \varepsilon_{ji} \geq 1)$;
  - (horizontal : at most one) $\forall i \neq j \in P (\sigma_{ij} + \sigma_{ji} \leq 1)$;
  - (vertical : at most one) $\forall i \neq j \in P (\varepsilon_{ij} + \varepsilon_{ji} \leq 1)$;

Network Design

Orange is the unique owner and handler of the telecom network in the figure below. The costs on the links are proportional to the distances d(i, j) between the nodes, expressed in units of 10km. Because of anti-trust regulations, Orange must delegate to SFR and Bouygtel two subnetworks each having at least two nodes (with Orange handling the third part). Orange therefore needs to design a backbone network to connect the three subnetworks. Transforming an existing link into a backbone link costs c = 25 euros/km. Formulate a mathematical program to minimize the cost of implementing a backbone connecting the three subnetworks. [*Leo Liberti, Ecole Polytechnique*]

Network Design : solution

Let $G = (V, E)$ be the graph of the network. The problem can be formalized as looking for the partition of V in three disjoint subsets V_1, V_2, V_3 such that the sum of the backbone update cost are minimum on the edges having one adjacent vertex in a set of the partition, and the other adjacent vertex in another set of the partition. This problem is often called Graph Partitioning or Min-k-Cut problem.

- Parameters :
  - $K = \{1, 2, 3\}$: set of disjoint subsets ;
  - for each $i, j \in E, d_{ij}$ is the edge weight (distance between i and j) ;
  - c : backbone updating cost ;
  - m : minimum cardinality of the subnetworks.
- Variables
  - for each $i \in V, h \in K$, let $x_{ih} = 1$ if vertex i is in V_h, and 0 otherwise.
- Objective function : $\min \frac{1}{2} \sum_{h \neq k \in K} \sum_{i,j \in E} c d_{ij} x_{ih} x_{jk}$

Network Design : solution

- Constraints :
  - $\forall i \in V \sum_{k \in K} x_{ik} = 1$; (assignment)
  - $\forall h \in K \sum_{i \in V} x_{ih} \geq m$; (subnetwork cardinality)
  - $\forall i \in V, h \in K\ x_{ih} \in \{0,1\}$

  Linearization : each quadratic product $x_{ih}x_{jk}$ is replaced by a the variable w_{ij}^{hk} s.t. $0 \leq w_{ij}^{hk} \leq 1$ and the below constraints are added :
- $\forall i,j \in E, h \neq k \in K\ w_{ij}^{hk} \geq x_{ih} + x_{jk} - 1$ (if $x_{ih} = x_{jk} = 1, w_{ij}^{hk} = 1)$
- $\forall i,j \in E, h \neq k \in K\ w_{ij}^{hk} \leq x_{ih}$ (if $x_{ih} = 0, w_{ij}^{hk} = 0)$
- $\forall i,j \in E, h \neq k \in K\ w_{ij}^{hk} \leq x_{jk}$ (if $x_{jk} = 0, w_{ij}^{hk} = 0)$.

2-player zero-sum strategic games

2-player zero-sum strategic games (Illustration)

Two players pick a move from the set $\{e,s,m,t\}$. Player1 picks a move from $\{e,s\}$, while Player2 picks a move from $\{m,t\}$. They then look up the entry corresponding to their moves in the matrix U called *payoff matrix*, and Player2 pays Player1 this amount. Thus, U gives Player1's gain and Player2's loss.

U=

	m	t
e	3	-1
s	-2	1

Player1$(x_1) \Longleftrightarrow$ Row(x)
Player2$(x_2) \Longleftrightarrow$ Column(y)

Consider the two scenarios ;

- First Row announces her strategy, then Column picks his.
- First Column announces his strategy, then Row chooses her.

We expect the first option to favor Column, and the second to favor Row.
Is this true ?

2-player zero-sum strategic games (Illustration)

U=

	m	t
e	3	-1
s	-2	1

Player1$(x_1) \Longleftrightarrow$ Row(x)
Player2$(x_2) \Longleftrightarrow$ Column(y)

Amazingly, this is not the case : if both play optimally, then it doesn't hurt a player to announce his or her strategy in advance ! This remarkable property is a consequence of–and in fact equivalent to–linear programming duality ! Let us check it.
Suppose Row announces that she will play the mixed strategy $x = (\frac{1}{2}, \frac{1}{2})$. What should Column do ? Column's best response is the *pure strategy* $y = (0,1)$.
More generally, once Row's strategy $x = (x1, x2)$ is fixed, there is always a pure strategy that is optimal for Column : either move m, with payoff $3x1 - 2x2$, or t, with payoff $-x1 + x2$,whichever is smaller.

2-player zero-sum strategic games (Illustration)

Hence if Row is forced to announce x before Column plays, she knows that his best response will achieve an expected payoff of $\min\{3x1-2x2,-x1+x2\}$. She should choose x defensively to maximize her payoff against this best response :

$$\text{Pick}(x_1,x_2) \text{ that maximizes } \min\{3x_1-2x_2,-x_1+x_2\}$$

This choice of xi's gives Row the best possible guarantee about her expected payoff. And it can be found by an LP !
Simple because for *fixed* $x1$ and $x2$ the following are equivalent :

$$\max\ \min\{3x_1-2x_2,-x_1+x_2\} \qquad \begin{array}{ll} & \max u \\ u & \leq 3x_1-2x_2 \\ u & \leq -x_1+x_2 \end{array}$$

2-player zero-sum strategic games (Illustration)

And Row needs to choose $x1$ and $x2$ to solve the below LP.

$$\begin{array}{ll} \max u & \\ -3x_1+2x_2+u & \leq 0 \\ x_1-x_2+u & \leq 0 \\ x_1+x_2 & = 1 \\ x_1,x_2 \geq 0 & \end{array}$$

2-player zero-sum strategic games (Illustration)

Symmetrically, if Column has to announce his strategy first, his best bet is to choose the mixed strategy y that minimizes his loss under Row's best response, in other words :

$$\text{Pick}(y_1,y_2) \text{ that minimizes } \max\{3y_1-y_2,-2y_1+y_2\}$$

which is equivalent to :

$$\begin{array}{ll} & \min v \\ v & \geq 3y_1-y_2 \\ v & \geq -2y_1+y_2 \end{array}$$

2-player zero-sum strategic games (Illustration)

In LP form, this is :

$$\begin{array}{ll} \max u & \\ -3x_1+2x_2+u & \leq 0 \\ x_1-x_2+u & \leq 0 \\ x_1+x_2 & = 1 \\ x_1,x_2 \geq 0 & \end{array} \qquad \begin{array}{ll} \min v & \\ -3y_1+y_2+v & \geq 0 \\ 2y_1-y_2+v & \geq 0 \\ y_1+y_2 & = 1 \\ y_1,y_2 \geq 0 & \end{array}$$

where (x_1,x_2) and (y_1,y_2) are mixed strategies.

These two LPs are dual to each other and, hence, they have the same optimum. I.e., by solving an LP, Row can determine a strategy that guarantees an expected outcome of at least V no matter what Column does. And by solving the dual LP, Column can guarantee an expected outcome of at most V, no matter what Row does. It follows that this is the uniquely defined optimal play !
In our example, it is $\frac{1}{7}$ and is realized when Row plays her optimum mixed strategy $(\frac{3}{7},\frac{4}{7})$ and Column plays his optimum mixed strategy $(\frac{2}{7},\frac{5}{7})$.

2-player zero-sum strategic games

This is easily generalized to arbitrary games and shows the existence of mixed strategies that are optimal for both players and achieve the same value–a fundamental result of game theory called the *Minmax Theorem*. In equation form

$$\max_x \min_y \sum_{i,j} U_{i,j} x_i y_j = \min_y \max_x \sum_{i,j} U_{i,j} x_i y_j$$

We will prove it formally.

2-player zero-sum strategic games

Given a 2-player zero-sum strategic game. Let $U^{m\times n}$ be the matrix payoff (for Player1). We denote by X_1 (resp. X_2) the n-dimensional (resp. m-dimensional) set of mixed strategies of Player1 (resp. Player2).

Question (2)

Assuming some mixed strategy $\overline{x_2} \in X_2$ for Player2 is given, define an optimization problem to characterize strategies of Player1 which are best responses to $\overline{x_2}$.

Solution 1. Problem (1) is given by :

$$\textbf{maximize} \quad x_1^t U \overline{x_2} \quad (1)$$

$$\textbf{subject to} \quad \sum_{j=1}^{n} x_1(j) = 1 \quad (2)$$

$$x_1(j) \geq 0, \quad \forall j = 1, \ldots, n \quad (3)$$

Problem (1) can be written as $\max\limits_{x_1 \in X_1} x_1^t U \overline{x_2}$, since (2)-(3) express $x_1 \in X_1$.

2-player zero-sum strategic games II

Question (3)

Justify why the objective function of Problem (1) *can be written as follows :*

$$\max_{j=1,\ldots,n} e_j^t U \overline{x_2}$$

where e_j is the n(resp. m)-dimensional vector whose components all equal 0 *but the j-th equals* 1.

Hint : *Apply the fundamental theorem of linear programming to relate the basic feasible solutions of Problem* (1) *to the pure strategies of Player1.*

Solution 2. From the fundamental theorem of linear programming, we know that if Problem (1) has a finite optimum, then it has an optimal basic feasible solution which is an extreme point of the set X_1. Notice that the extreme points of X_1 are all vectors whose components equal zero but one equals to one. Consequently, the basic optimal solutions of Problem (1) correspond to pure strategies, and we have $\max\limits_{x_1 \in X_1} x_1^t U \overline{x_2} = \max\limits_{j=1,\ldots,n} e_j^t U \overline{x_2}$.

2-player zero-sum strategic games III

Question (4)

Keeping in mind that Player1 is rational (she wants to maximize her payoff), characterize the strategy Player2 should follow provided he is rational too.

Solution 3. Player2 should adopt an optimal strategy x_2^* which satisfies the following equation

$$x_2^* = \operatorname*{argmin}_{x_2 \in X_2} \max_{x_1 \in X_1} x_1^t U x_2$$

Similarly, denote by x_1^* the optimal strategy of Player 1.

2-player zero-sum strategic games IV

Question (5)

By combining the answers of questions (2) *and* (3), *describe the entire optimization problem corresponding to an optimal strategy for Player2.*

Solution 4. Problem (4) is given by :

$$\begin{array}{lll} \textbf{minimize} & \max_{j=1,\dots,n} e_j^t U x_2 & \\ \textbf{subject to} & \sum_{i=1}^{m} x_2(i) = 1 & \\ & x_2(i) \geq 0, & \forall i = 1,\dots,m \end{array}$$

2-player zero-sum strategic games V

Question (6)

By introducing a new variable v representing an upper bound on the terms $e_j^t U x_2$ ($j = 1,\dots,n$), recast Problem (4) *as a linear program (LP). The optimal value of this LP hence denotes the optimal payoff of Player2.*

Solution 5. Problem (5) consists in the following :

$$\begin{array}{lll} \textbf{minimize} & v & \\ \textbf{subject to} & v \geq e_j^t U x_2,\ j = 1,\dots,n & \\ & \sum_{i=1}^{m} x_2(i) = 1 & \\ & x_2(i) \geq 0, & \forall i = 1,\dots,m \end{array}$$

2-player zero-sum strategic games VI

Question (7)

Write the dual of Problem (5).

Solution 6. Problem (6), the dual of Problem (5), is

$$\begin{array}{lll} \textbf{maximize} & u & \\ \textbf{subject to} & u \leq e_i^t U^t x_1, i = 1,\dots,m & \\ & \sum_{j=1}^{n} x_1(j) = 1 & \\ & x_1(j) \geq 0, & \forall j = 1,\dots,n \end{array}$$

2-player zero-sum strategic games VII

Question (8)

Apply the LP Duality to Problem (5) *and Problem* (6) *in order to prove the Minmax Theorem stated as follows : There exist stochastic vectors x_1^* and x_2^* such that*

$$\max_{x_1 \in X_1} x_1^t U x_2^* = \min_{x_2 \in X_2} x_1^{*t} U x_2$$

Solution 7. By the LP Duality Theorem, we have $u^* = v^*$. By definition of v^* and u^* we have

$$\begin{cases} v^* = \max_{j=1,\dots,n} e_j^t U x_2^* = \max_{x_1 \in X_1} x_1^t U x_2^* \\ u^* = \min_{i=1,\dots,m} e_i^t U^t x_1^* = \min_{x_2 \in X_2} x_2^t U^t x_1^* = \min_{x_2 \in X_2} x_1^{*t} U x_2 \end{cases}$$

which concludes.