

Finiteness properties for substitution dynamical systems

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Grenoble, March 05

Overview

- Substitution: **expand a real number in an exotic basis**
- Define a multiple tiling and a generalized (F)-property
- **Interpret the (F)-property in discrete geometry**
- Application in beta-numeration: new class of sufficient conditions for (F)
- Application in dynamical systems: new condition for 0 to be an inner point

β -expansion

$\beta > 1$ unit Pisot number of degree d

- **β -transformation** $T_\beta : x \mapsto \beta x \pmod{1}$
- **β -expansion** $d_\beta : x \in [0, 1[\mapsto (u_i)_{i \geq 1} \in \{0 \dots [\beta]\}^{\mathbb{N}}$

$$x = \sum_{i \geq 1} u_i \beta^{-i}, \quad u_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$$
- **Pseudo-expansion of 1** $d_\beta^*(1) = (t_i)_{i \geq 1}$ ultimately periodic

$$1 = \sum_{i \geq 1} t_i \beta^{-i}$$

Theorem

$(u_i)_{i \geq 1}$ is a β -expansion iff $\forall k \geq 1, (u_i)_{i \geq k} <_{\text{lex}} d_\beta^*(1)$.

β -shift

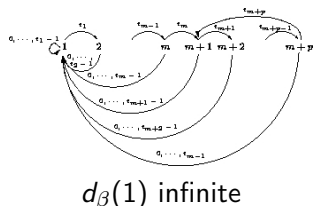
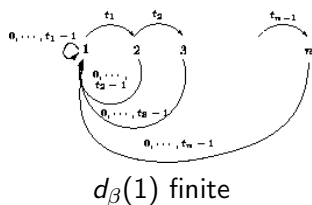
The condition of admissibility is extended to two-sided sequences

$$y := \dots \omega_3 \omega_2 \omega_1 \omega_0 u_1 u_2 u_3 \dots$$

$$X_\beta := \{(y_i)_{i \in \mathbb{Z}} \text{ in } \mathcal{A}_\beta^{\mathbb{Z}}; \quad \forall k \in \mathbb{Z}, (y_i)_{i \geq k} \leq_{\text{lex}} d_\beta^*(1)\}$$

Theorem (Bertrand)

(X_β, S) is a *sofic dynamical system*: the set of finite factors F_β is recognized by a finite graph.



Expansion on the real line

$$x = \underbrace{\omega_M \beta^M + \dots + \omega_0}_{\text{integer part}} + \underbrace{u_1 \beta^{-1} + \dots + u_L \beta^{-L} + \dots}_{\text{fractional part}}$$

- Reals with **finite fractional part**: $Fin(\beta)$.

$$Fin(\beta) = \{ \omega_M \beta^M + \dots + \omega_0 + u_1 \beta^{-1} + \dots + u_L \beta^{-L}; \\ \omega_M \dots \omega_0 u_1 \dots u_L \in F_\beta \}$$

- Integer** in base β : no fractional part

$$\mathbb{Z}_\beta^+ = \{ \omega_M \beta^M + \dots + \omega_0; \omega_M \dots \omega_0 \in F_\beta \}$$

- Successor map on \mathbb{Z}_β^+** : $Succ(x)$ is the smallest element of \mathbb{Z}_β^+ strictly larger than x .

$$(Succ(x) - x \text{ takes } n \text{ values: } T_\beta^{i-1}(1), 1 \leq i \leq n)$$

Central tile

- **Geometric representation**

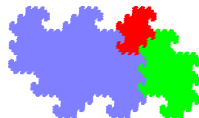
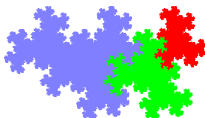
- β -conjugates: $\beta_2, \dots, \beta_r, \beta_{r+1}, \overline{\beta_{r+1}}, \dots, \beta_{r+s}, \overline{\beta_{r+s}}$
- **Canonical embedding:**

$$\Phi_\beta : P(\beta) \in \mathbb{Z}[\beta^{-1}] \mapsto (P(\beta_2), \dots, P(\beta_{r+s})) \\ \in \mathbb{K}_\sigma = \mathbb{R}^{r-1} \times \mathbb{C}^s \simeq \mathbb{R}^{d-1}$$

- $\Phi_\beta(\beta^i)$ tends to 0; $\Phi_\beta(\beta^{-i})$ is expanding.

- **Central tile:** representation of integers in base β :

$$\begin{aligned} \mathcal{T}_\beta &= \overline{\Phi_\beta(\mathbb{Z}_\beta^+)} \\ &= \{\omega_0 + \omega_1 \Phi_\beta(\beta) + \dots + \omega_M \Phi_\beta(\beta^M) + \dots; \\ &\quad \omega_0 \dots \omega_M \dots \text{path in the reverse graph}\} \end{aligned}$$



Basic tiles and self-replicating central tile

- **Basic tiles** : subdivision into pieces depending on the successor map.

$$\mathcal{T}_\beta(i) = \{\omega_0 + \omega_1\Phi_\beta(\beta) + \dots + \omega_M\Phi_\beta(\beta^M) + \dots; \\ \omega_0 \dots \omega_M \dots \text{path starting from the node } i\}$$

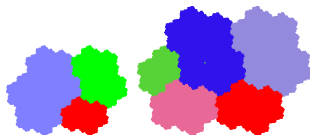
- **Homothety** h_β : multiplication by each conjugate of β on \mathbb{K}_σ .

Theorem (Akiyama, Sirvent-Wang, Ito-Rao)

The basic tiles of the central tile \mathcal{T}_β satisfy the following graph IFS:

$$\forall a, \mathcal{T}_\beta(a) = \bigcup_{b \mapsto_i a} h_\beta(\mathcal{T}(b)) + \Phi_\beta(i).$$

The basic tiles have disjoint interiors and they are the closure of their interior.



Fractional parts: self-replicating multiple tiling

- Set of **fractional parts**: $\text{Frac}(\beta) = \{d_\beta(x), x \in \mathbb{Z}[\beta] \cap [0, 1)\}$
- **Tile associated with a fractional part**
representation of reals having u as β -fractional part

$$\mathcal{T}_u = \underbrace{\left\{ \sum_{i \geq 1} w_i \phi_\beta(\beta^i); (w, u) \in X_\beta \right\}}_{\substack{\text{variable integer part} \\ \text{union of basic tiles}}} + \underbrace{\phi_\beta\left(\sum_{i \geq 1} u_i \beta^{-i}\right)}_{\substack{\text{fixed fractional part} \\ x = \sum_{i \geq 1} u_i \beta^{-i} \in \mathbb{Z}[\beta]}}$$

Theorem (Akiyama)

The set of tiles \mathcal{T}_u , for $u \in \text{Frac}(\beta)$, is finite up to translation.

The set of tiles \mathcal{T}_u covers \mathbb{K}_β .

The covering is a repetitive self-replicating multiple tiling with constant degree almost everywhere.

$$\mathbb{K}_\beta = \bigcup_{u \in \text{Frac}(\beta)} \mathcal{T}_u = \bigcup_{\substack{u \in \text{Frac}(\beta), \\ (u_j) \text{ read from } i}} \mathcal{T}(i) + \Phi_\beta\left(\sum_{j \geq 1} u_j \beta^{-j}\right).$$

(F) property and condition for tiling

Theorem (Akiyama)

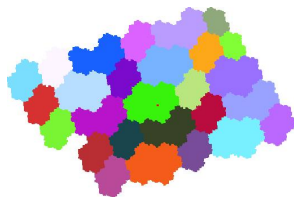
If every polynomial has a finite β -expansion:

$$(F) : \text{Fin}(\beta) = \mathbb{Z}[\frac{1}{\beta}] \cap [0, +\infty]$$

then 0 is an exclusive inner point of the central tile and the self-replicating multiple tiling is a tiling.

- Algorithm condition to check (F)
- Frougny-Solomyak sufficient condition

$$\beta^d = a_{d-1}\beta^{d-1} + \dots + a_1\beta + 1$$
 with $a_{d-1} \geq a_{d-2} \geq \dots \geq a_1 \geq 1$.
- Other conditions: Hollander, Akiyama-Rao-Steiner



Substitution

- A substitution **replaces letters by finite words**
- Iterations provide periodic or **fixed points**

$$\sigma(1) = 12, \sigma(2) = 13, \sigma(3) = 1.$$

$$121312112131212131211213$$

- Algebraic properties are contained in the incidence matrix

- primitivity: $\exists n, \mathbf{M}^n > 0$

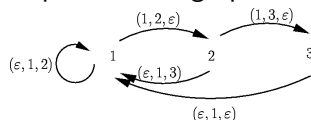
$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

- Pisot type

- **Combinatorial structure** given by the prefix-suffix graph

$$b \xrightarrow{(\rho, a, s)} a \text{ iff } \sigma(b) = pas$$

$F_{\mathcal{P}}$: finite walks in the graph

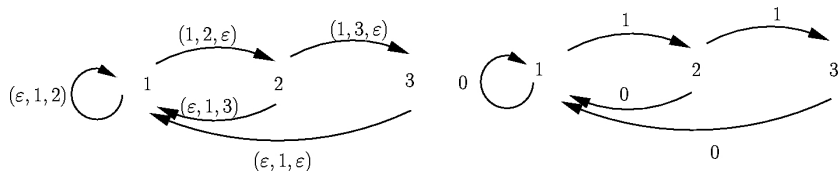


- **Strong coincidence property**

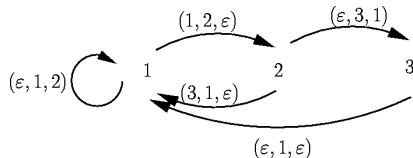
$\forall b_1, b_2, \exists a, \sigma^n(b_1) = P_1 a S_1, \sigma^n(b_2) = P_2 a S_2, P_1$ and P_2 contain the same letters.

Examples

- beta-substitutions** : The β -graph is the prefix-suffix graph of the β -substitution where prefixes 1^a are replaced by a .



- Flipped Tribonacci substitution** $1 \rightarrow 12, 2 \rightarrow 31, 3 \rightarrow 1$



Dumont-Thomas numeration

β dominant eigenvalue (unit Pisot), v_β expanding left eigenvector.

- **Numerical digit:** $\delta_\sigma(\sigma(P)) = \beta\delta_\sigma(P)$
 $(\delta_\sigma : P \in \mathcal{A}^* \rightarrow \langle \mathbf{l}(P), \mathbf{v}_\beta \rangle \in \mathbb{Q}(\beta))$

- **Polynomial representation of a prefix-suffix path**

$$(p_n, a_n, s_n) \dots (p_0, a_0, s_0) \in F_{\mathcal{P}} \rightarrow \delta_\sigma(p_n)\beta^n + \dots + \delta_\sigma(p_0) \in \mathbb{Z}[\beta^{-1}]$$

Theorem ((σ, a_0)-expansion, Dumont, Thomas)

Let $a_0 \in \mathcal{A}$. Every $x \in [0, \delta_\sigma(a_0))$ can be uniquely expanded as

$$x = \sum_{i \geq 1} \delta_\sigma(p_i)\beta^{-i}, \quad (p_i, a_i, s_i)_{i \geq 1} := d_{(\sigma, a_0)}(x)$$

- $(p_i, a_i, s_i)_{i \geq 1}$ in the prefix-suffix graph is issued from the letter a_0 : $\forall i \geq 0, \sigma(a_i) = p_{i+1}a_{i+1}s_{i+1}$
- infinitely non-empty suffixes in $(s_i)_{i \geq 1}$ (*Parry condition*).

Underlying dynamics and exotic numeration

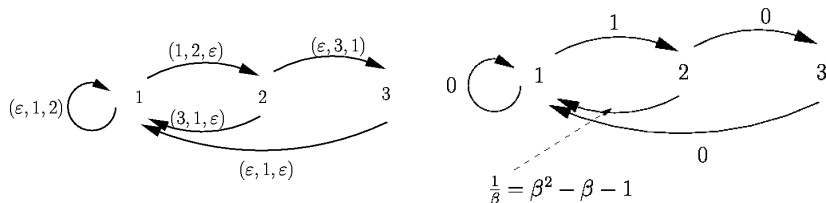
$$T_\sigma : \bigcup_{a \in \{1 \dots d\}} [0, \delta_\sigma(a)) \times \{a\} \quad \text{to} \quad \bigcup_{a \in \{1 \dots d\}} [0, \delta_\sigma(a)) \times \{a\}$$

$$(x, a) \mapsto (\beta x - \delta_\sigma(p), b)$$

$$\text{with } \begin{cases} \sigma(a) = pbs \\ \beta x - \delta_\sigma(p) \in [0, \delta_\sigma(b)). \end{cases}$$

- The map T_σ is **well defined**: for every (x, a) there exists a unique (y, b) satisfying the conditions.
- A **different numeration for each letter**
- beta-substitution: recover beta-numeration
- Rigo-Steiner: ultimately periodic expansions for abstract numeration systems

Example: flipped Tribonacci substitution



Flipped Tribonacci substitution:

$d_{(\sigma,1)}(1/\beta^2) = d_{(\sigma,3)}(1/\beta^2) = 0/\beta + 1/\beta^2 + 0/\beta^3 + \dots$
 (two walks are labelled by $(0, 1, 0, 0, \dots)$, starting from 1 and 3)

$d_{(\sigma,2)}(1/\beta^2) = (1/\beta)/\beta + 0/\beta^2 + 0/\beta^3 + \dots$
 (a walk is labelled by $(1/\beta, 0, 0, 0, \dots)$, starting from 2)

Substitution expansion of reals

We have a dynamics that describes an exotic expansion.

- expansion on the real line \mathbb{R}^+ ,
- reals with finite expansions

$$\begin{aligned} \text{Fin}(\sigma) &= \{ \delta_\sigma(q_M)\beta^M + \dots + \delta_\sigma(p_0) + \delta_\sigma(p_1)\beta^{-1} + \dots + \delta_\sigma(p_L)\beta^{-L}; \\ &M, L \in \mathbb{N}, (q_M, b_M, r_M) \dots (p_0, a_0, s_0)(p_1, a_1, s_1) \dots (p_L, a_L, s_L) \in F_{\mathcal{P}} \}, \end{aligned}$$

- integers for the substitution numeration system

$$\begin{aligned} \mathbb{Z}_\sigma^+ &= \{ \delta_\sigma(q_M)\beta^M + \dots + \delta_\sigma(q_0); M \in \mathbb{N}, \\ &(q_M, b_M, s_M) \dots (q_0, b_0, s_0) \in F_{\mathcal{P}} \} \subset \text{Fin}(\sigma). \end{aligned}$$

Substitution central tile

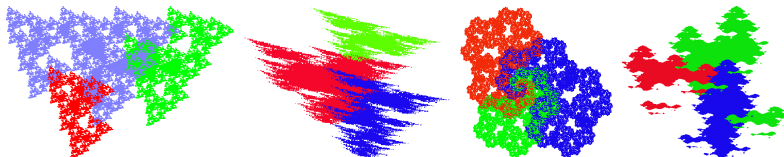
- **Representation space**: representation space of β .

$$\Phi_\sigma : P(\beta) \in \mathbb{Z}[\beta^{-1}] \mapsto (P(\beta_2), \dots, P(\beta_{r+s})) \\ \in \mathbb{K}_\sigma = \mathbb{R}^{r-1} \times \mathbb{C}^s \simeq \mathbb{R}^{d-1}$$

- **Central tile** : no fractional part

$$\mathcal{T}_\sigma(a) = \overline{\Phi_\sigma(\mathbb{Z}^+)} \\ = \{ \Phi_\sigma(\delta_\sigma(q_0)) + \dots + \Phi_\sigma(\delta_\sigma(q_M)\beta^M) + \dots, \\ (q_i, a_i, s_i)_{i \geq 0} \text{ reversed prefix-suffix path} \}$$

- **Basic tiles** $\mathcal{T}_\sigma(i)$: paths starting from node i .
- Recover **Rauzy fractals**



Tiling: suitable fractional set

$$\mathbb{Z}[\mathbf{v}_\beta] := \mathbb{Z}[\delta_\sigma(1), \dots, \delta_\sigma(n)]$$

- Main property: $\mathbb{Z}[\mathbf{v}_\beta]$ is **uniformly discrete** and $\Phi_\sigma \mathbb{Z}[\mathbf{v}_\beta]$ is **dense** in \mathbb{K}_σ .
 - **Fractional parts** $\text{Frac}(\sigma) = \cup_{a \in \{1, \dots, n\}} d_{(\sigma, a_0)}(\mathbb{Z}[\mathbf{v}_\beta] \cap [0, \delta_\sigma(a_0)))$
 - $\mathbb{Z}[\mathbf{v}_\beta] = \mathbb{Z}[1/\beta]$ for **beta-substitution**
 - **Tiles for any element $u \in \text{Frac}(\sigma)$** : representation of reals whose fractional part is u .
- (nonempty interior; closure of their interior; finite union of basic tiles)

Substitution tiling

The proofs for beta-numeration are generalizable.

Theorem

Let σ be a primitive substitution whose dominant eigenvalue is a unit Pisot number.

The finite up to translation set of tiles \mathcal{T}_u , for $u \in \text{Frac}(\sigma)$, covers \mathbb{K}_σ .

The covering is a self-replicating repetitive multiple tiling.

The basic tiles satisfy the IFS:

$$\forall a \in \mathcal{A}, \mathcal{T}_\sigma(a) = \bigcup_{b \mapsto (p_i, a_j, s_j) a} h_\beta(\mathcal{T}(b)) + \Phi_\sigma(\delta_\sigma(p_i)).$$

Provided that the strong coincidence condition holds, the basic tiles have disjoint interiors.

Example: flipped Tribonacci substitution



Substitutive (F)-property

Every fractional part in $\text{Frac}(\sigma)$ is finite:

$$(F) \forall a, \forall z \in \mathbb{Z}[\mathbf{v}_\beta] \cap [0, \delta_\sigma(a)), d_{(\sigma,a)}(z) \text{ is finite.}$$

Theorem

*The substitutive (F)-property provided with the strong coincidence condition implies that **the multiple tiling is a tiling** and **0 is an exclusive inner point of the central tile**.*

How to check the substitutive (F)-property ?

Discrete geometry

Restriction: σ is **of Pisot type** (all the eigenvalues are conjugate)

- **Pisot type**: the contracting plane is generated by the conjugates of the dominant eigenvector.
- **Unit faces**: $(\mathbf{x}, i) := \{\mathbf{x} + \sum_{j \neq i} \lambda_j \mathbf{e}_j; 0 \leq \lambda_j \leq 1\}$.
- **Discrete plane**: the contracting plane is best-approximated by

$$\mathcal{S}(\mathbf{v}_\beta) = \bigcup_{0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle} (\mathbf{x}, i).$$

- **Discrete geometry versus numeration**: the numeration multiple tiling is equal to the self-similar discrete multiple tiling (Ei-Rao-Ito)

$((\mathbf{x}, i) \in \mathcal{S}(\mathbf{v}_\beta)$ corresponds to $\Phi_\sigma(z) + \mathcal{T}(i)$, $z = \langle \mathbf{x}, \mathbf{v}_\beta \rangle \in \mathbb{Z}[\mathbf{v}_\beta] \cap [0, \delta_\sigma(i))$.)

Projection of paths on the discrete plane

Lemma

A finite *path* in the prefix-suffix graph corresponds to a *face* in the discrete surface:

If $a_{-k} \mapsto_{(p_{-k}, a_{-k}, s_{-k})} \cdots \mapsto_{(p_{-1}, a_{-1}, s_{-1})} a_0$ is a prefix-suffix path then $(M^{k-1}\mathbf{l}(p_{-1}) + \cdots + p_{-k}, a_0) \in E_1^*(\sigma)^{k-1}(\mathbf{0}, a_{-k})$.

Proof

- Substitution rule on faces (Arnoux-Ito)

$$E_1^*(\sigma)(\mathbf{x}, a) = \bigcup_{\sigma(b)=pas} (\mathbf{M}_\sigma^{-1}(\mathbf{x} + \mathbf{l}(p)), b).$$

- Stability of the discrete plane

$$(\mathbf{0}, 1), \dots, (\mathbf{0}, d) \cup E_1^*(\sigma)\mathcal{S}(\mathbf{v}_\beta) \subset \mathcal{S}(\mathbf{v}_\beta)$$

- Stability of expansion of integer points of $\mathcal{S}(\mathbf{v}_\beta)$

$$(\mathbf{x}, a) \in \mathcal{S}(\mathbf{v}_\beta), b \rightarrow_{(p, a, s)} a \implies (M^{-1}\mathbf{x} + M^{-1}\mathbf{l}(p), b) \in \mathcal{S}(\mathbf{v}_\beta)$$

(F) combinatorial condition

Theorem

A unit substitution of Pisot type satisfies the *substitution (F) property* iff the images of the unit cube located at the origin under the iterated action of $E_1^*(\sigma)$ *cover the full discrete surface*:

$$\mathcal{S}_\sigma = \bigcup_{n \in \mathbb{N}} E_1^*(\sigma)^n((\mathbf{0}, 1) \cup \dots \cup (\mathbf{0}, d)).$$

Application

- (F) implies that **0 is an inner point** of the Central tile/Rauzy fractal
- (F) implies that the multiple tiling is a **tiling**
- **Flipped Tribonacci substitution**: The combinatorial (F) property holds (ring condition: check a finite number of possibilities)
0 is an inner point of the central tile and the multiple tiling is a tiling.
- **Discussion**: Reducible case ? (see Ito-Rao, reducible case)