IRISA

EWM talk

Spectral theory for dynamical systems arisen by substitutions

Anne Siegel

CNRS

What is a substitution?

A substitution replaces letters with finite words...

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We need to iterate it: extend it as a morphism of a monoid

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$$1$$

$$1 \to 12$$

$$2 \to 3$$

$$3 \to 1$$

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12

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$1\,2$

$12\,3$

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$1231\,12$

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	1
	12
	123
$1 \to 12$	1231
	$1231\,1\,2$
$2 \to 3$	$123112\,12\,3$
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$123112123123112311212311232312311212312 31$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Incidence matrix

We get a fixed point

About symbolic dynamics

Symbolic dynamics consists in studying
the shift map on infinite sequences in $\mathcal{A}^{\mathbb{N}}$

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- CODE the orbits of a system with respect to a cleverly chosen finite partition
- the INTEREST is to replace the initial dynamical system by the shift map on $\mathcal{A}^{\mathbb{N}}$
- it is APPLIED to systems for which past and future are disjoint (toral automorphisms or Pseudo-Anosov diffeo. of surfaces).

Example: the Morse sequence

- Morse studies geodesics (1920)

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Geodesics are curves realizing the minimal distance between two points.

For connected surfaces with constant negative curves, Morse looks at infinite geodesics that stay in a small part of the space.

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Geodesics are coded by infinite sequences of 0 and 1 according to which boundary of the surface they meet

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To advance along a geodesic translates into looking at the next element in the sequence

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A closed geodesic is coded by a periodic sequence

A recurrent geodesic is coded by a minimal sequence

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A minimal an non-periodic sequence is given by the (Prouhet-Thue)-Morse sequence

$$u = 0110100110010110...$$

How is the Morse sequence generated?

$$\sigma : 0 \mapsto 01 \quad 1 \mapsto 10$$

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$$01$$
$$0110$$
$$01101001$$
$$0110100110010110$$
$$0110100110010110100101100110$$

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$X_\sigma \in \{0,1\}^{\mathbb{N}}$ gathers sequences having the same language than u

It is independent from the fixed point that was chosen, provided that the subtitution is primitive

How is the Morse sequence generated?

$$\sigma : 0 \mapsto 01 \quad 1 \mapsto 10$$

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This property is quite general (see Queffelec's book)

Self-similar dynamics and substitutions

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- Self-similar structure: the system defined by the first return map is conjugated to the original system.

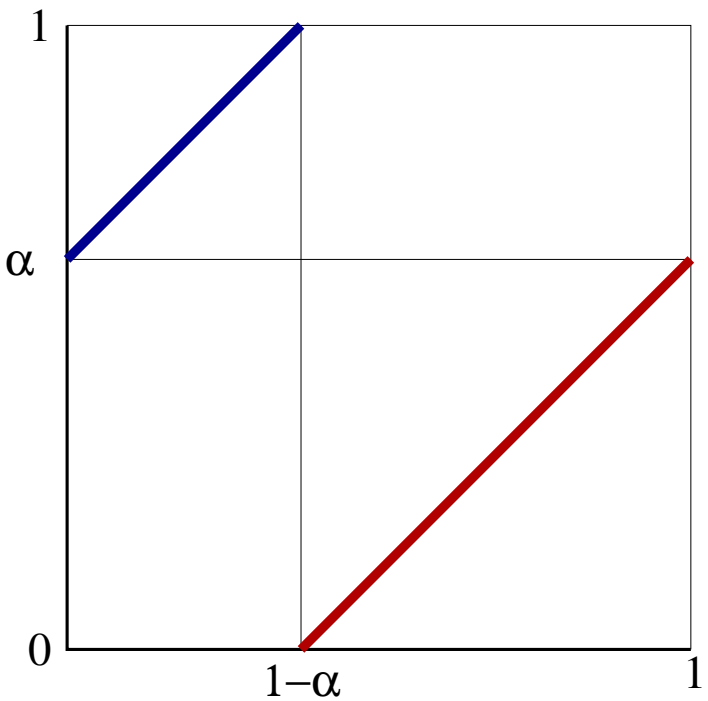
Self-similar dynamics and substitutions

How can one understand the local structure of the orbits of a dynamical system?

- Study the first return map over a neighborhood of a given point (Poincaré)
- Self-similar structure: the system defined by the first return map is conjugated to the original system.
- Basic idea:
  a substitution is hidden behind each self-similar system.

Addition of the golden ratio

- Let φ be the addition of the golden ratio on the 1-torus



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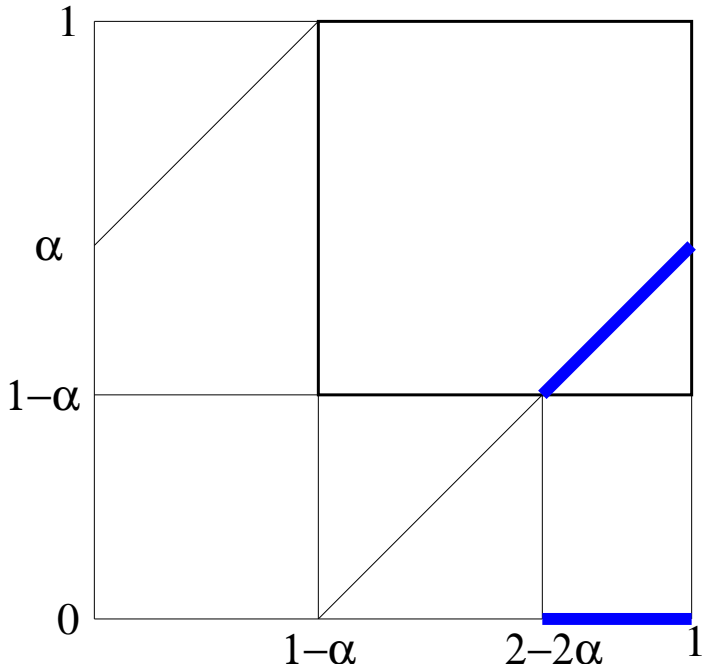

$\varphi(J_2) \subset I$

$(J_2 = [2-2\alpha, 1[)$

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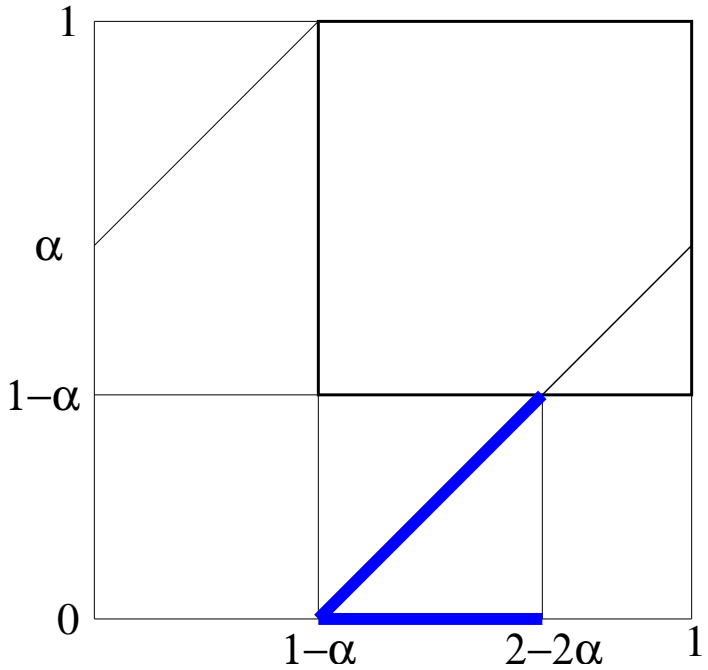

$$\varphi(J_1) = [0, 1-\alpha[\not\subset I$$

$$(J_1 = [1-\alpha, 2-2\alpha[)$$

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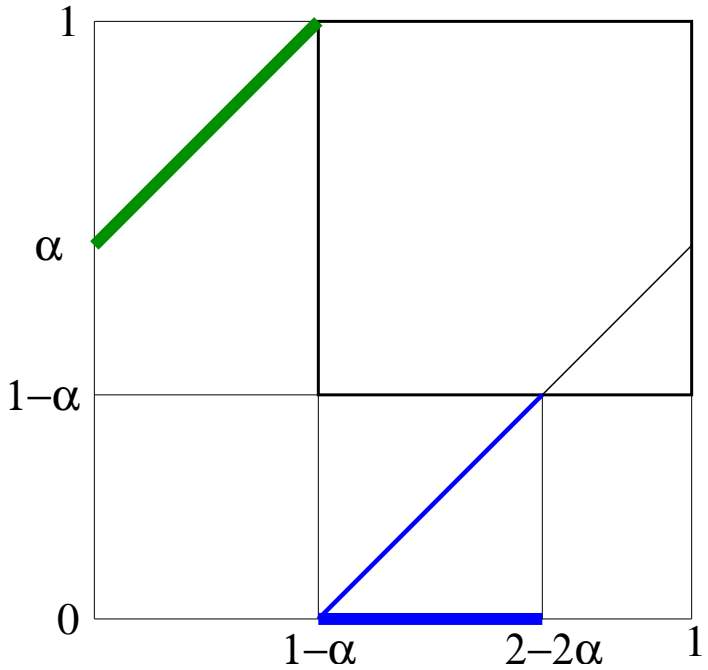

$$\varphi^2(J_1) = [\alpha, 1[\subset I$$

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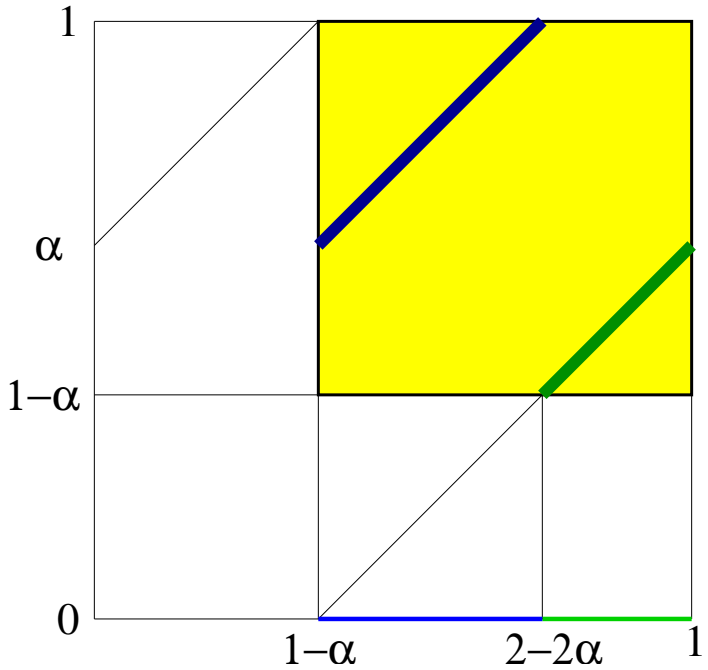

$\varphi^2(J_1) \subset I \qquad \varphi(J_2) \subset I$

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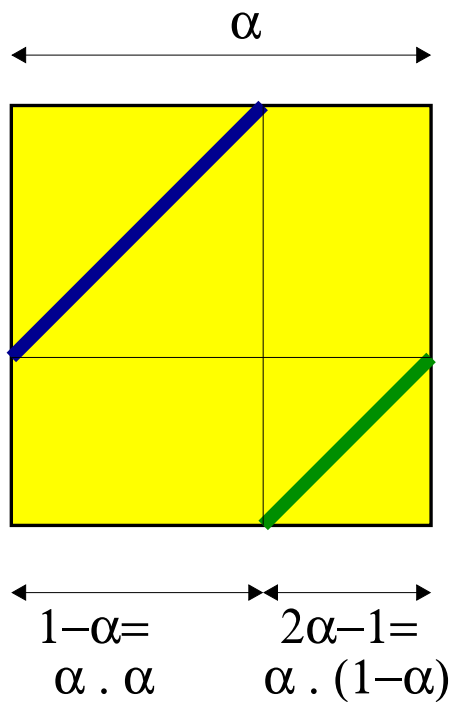

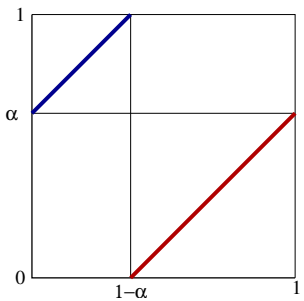

equality up to reversal of the orientation and renormalization

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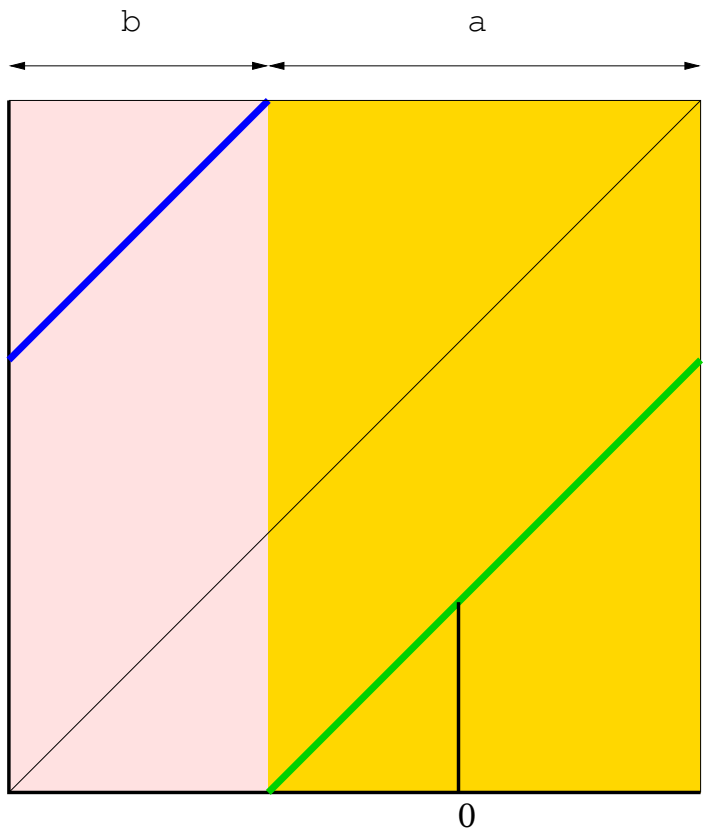

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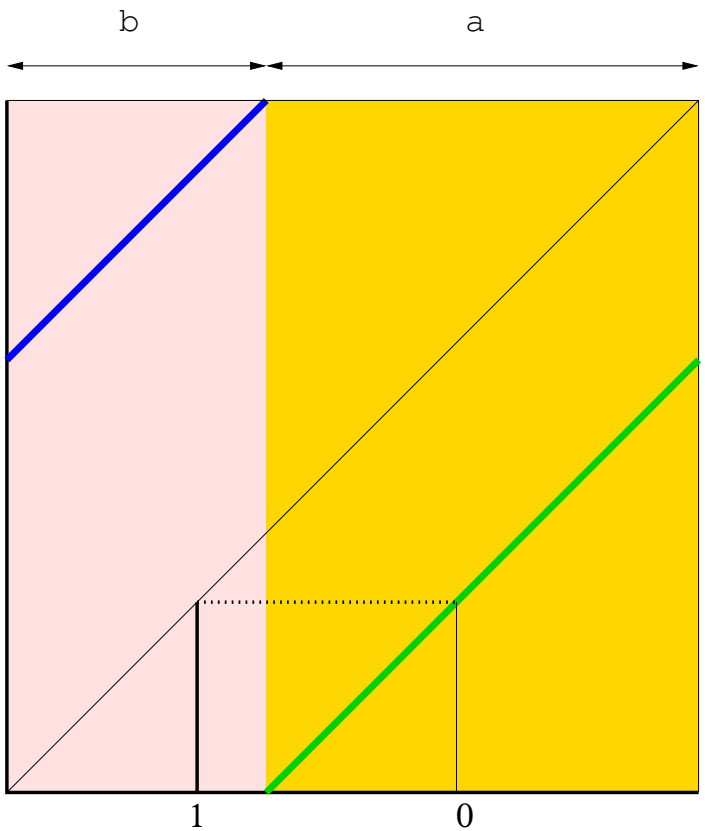

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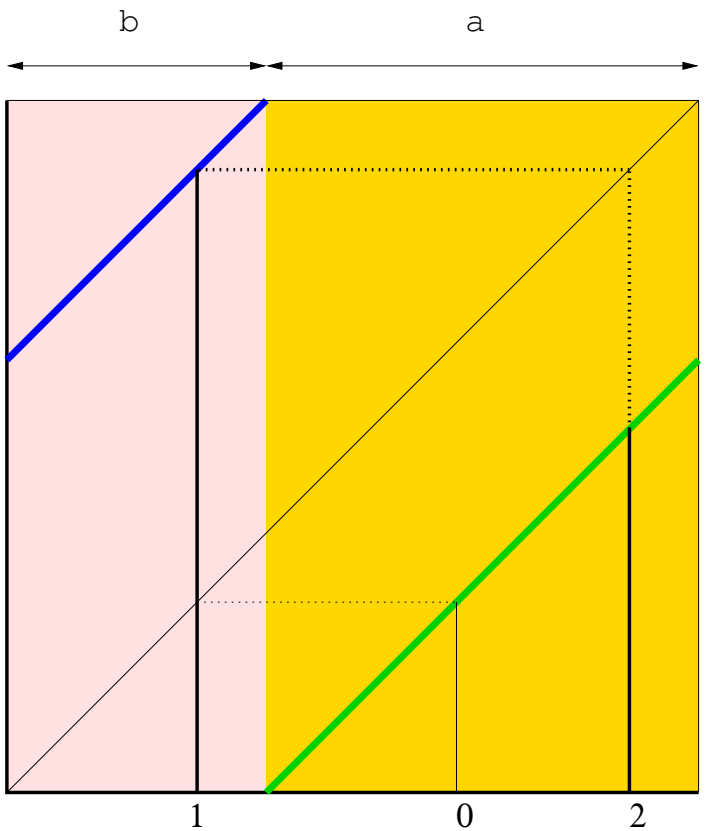

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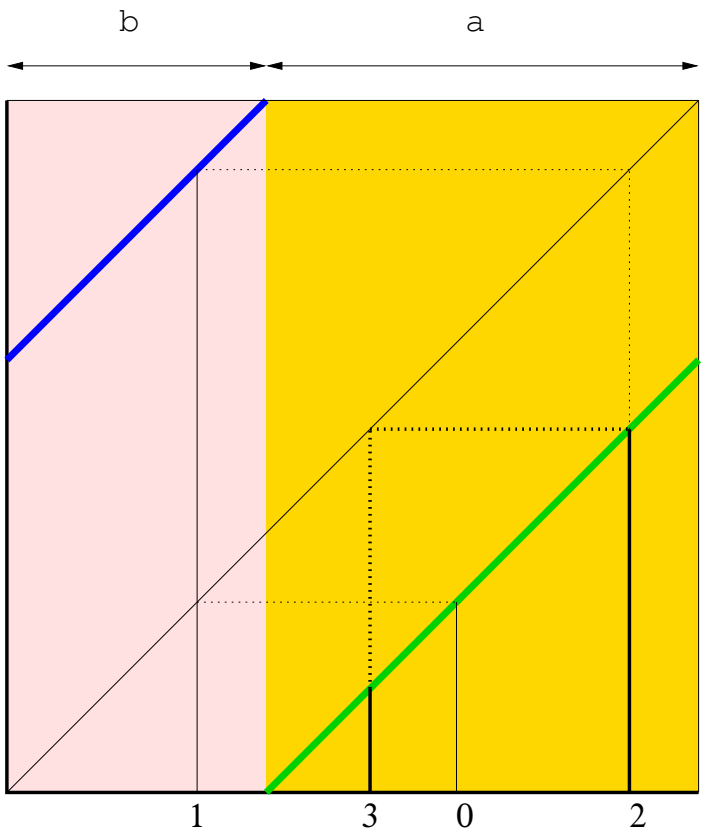

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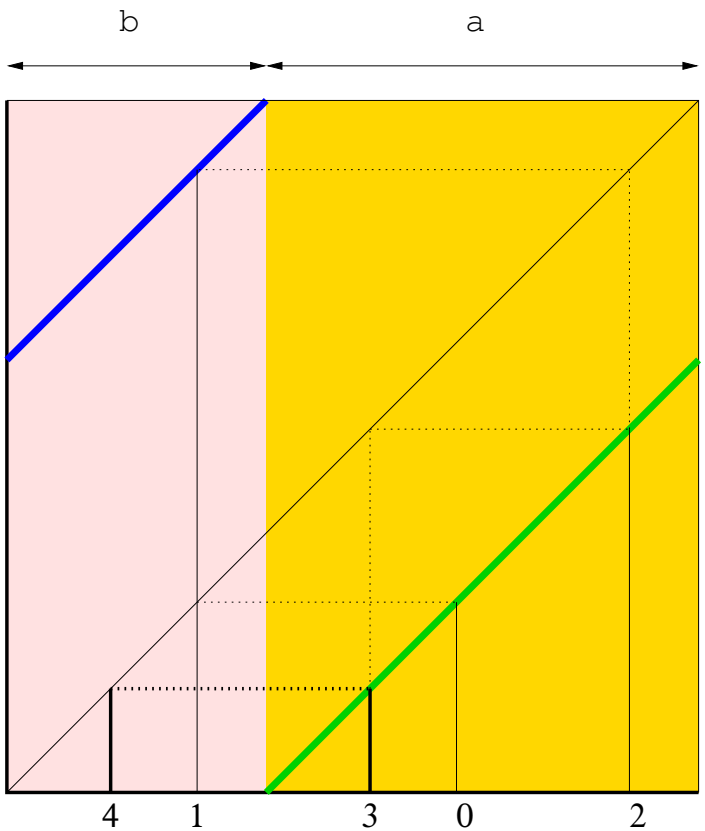

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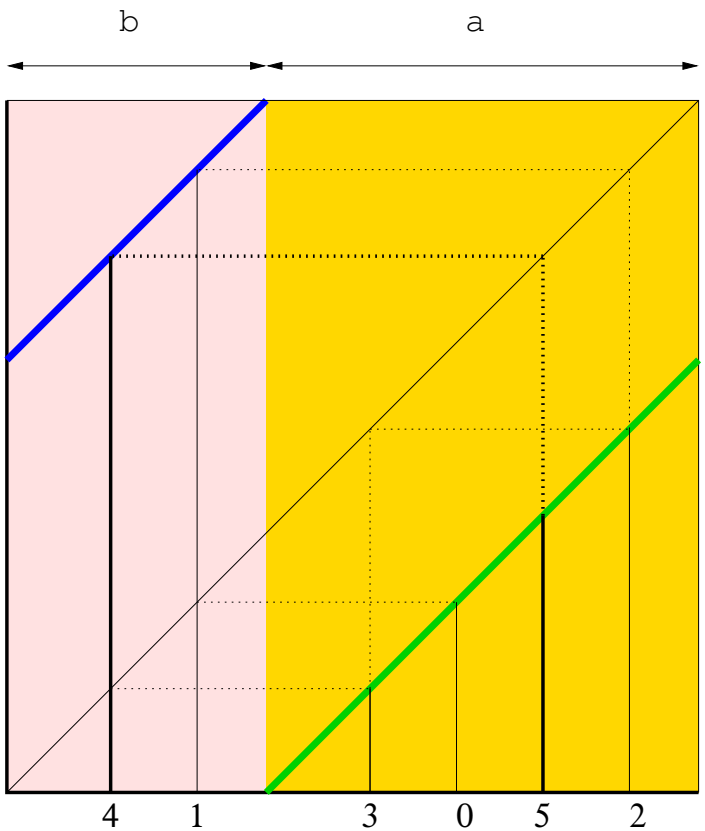

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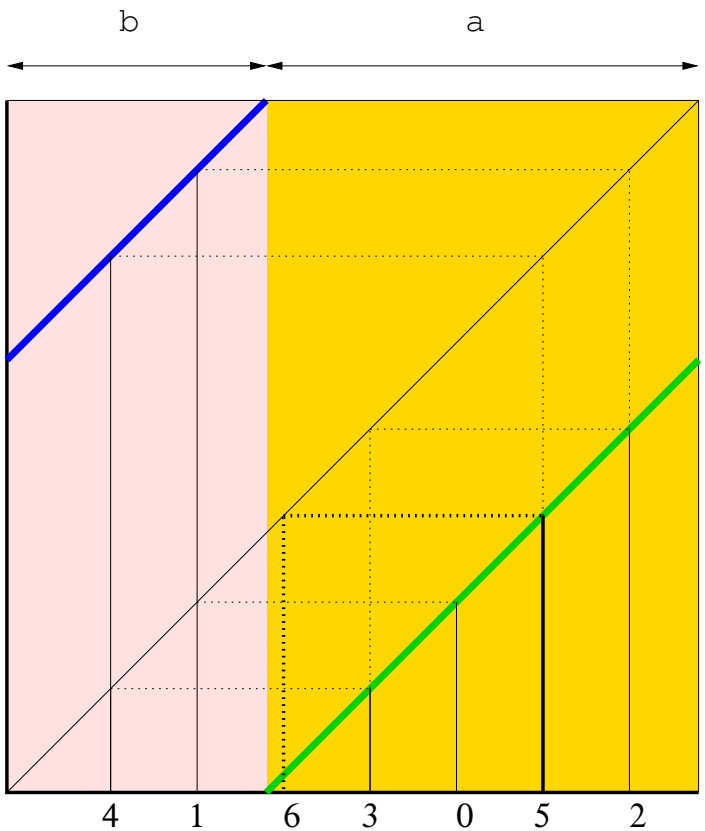

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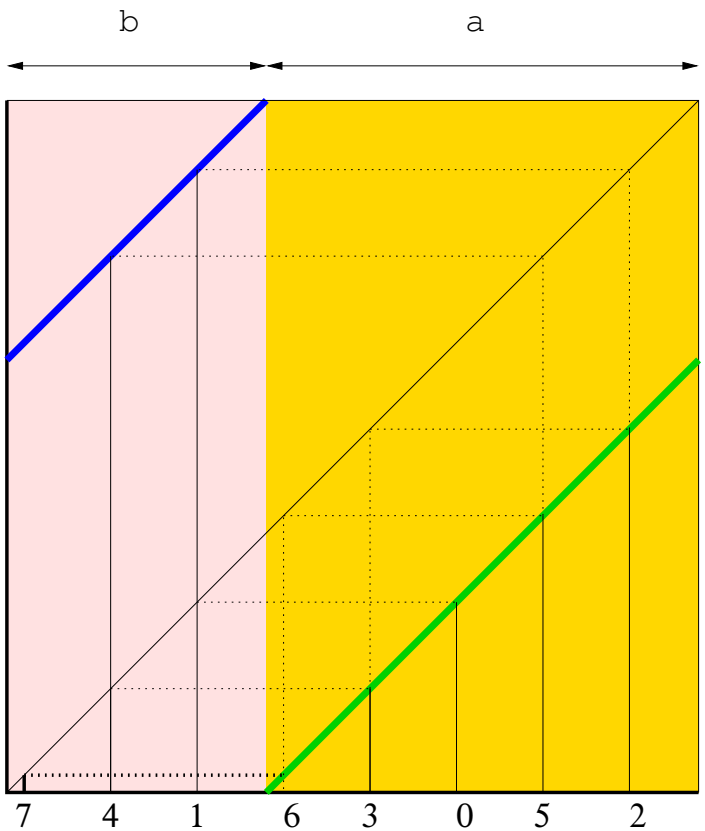

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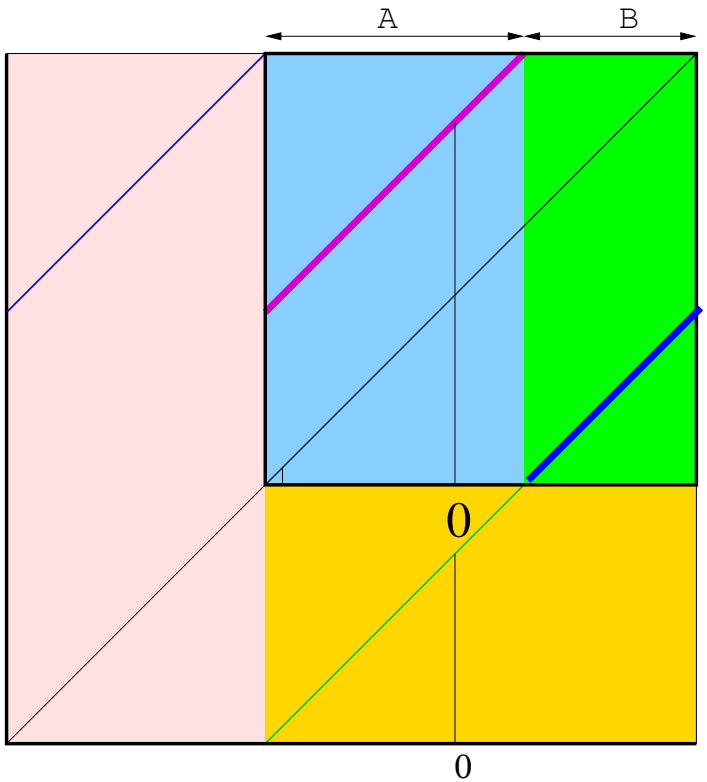

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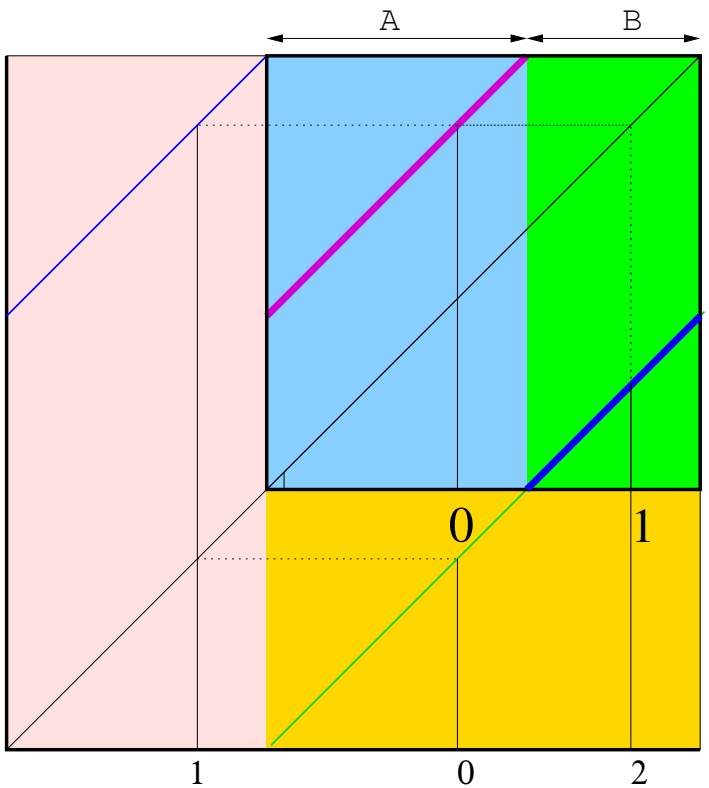

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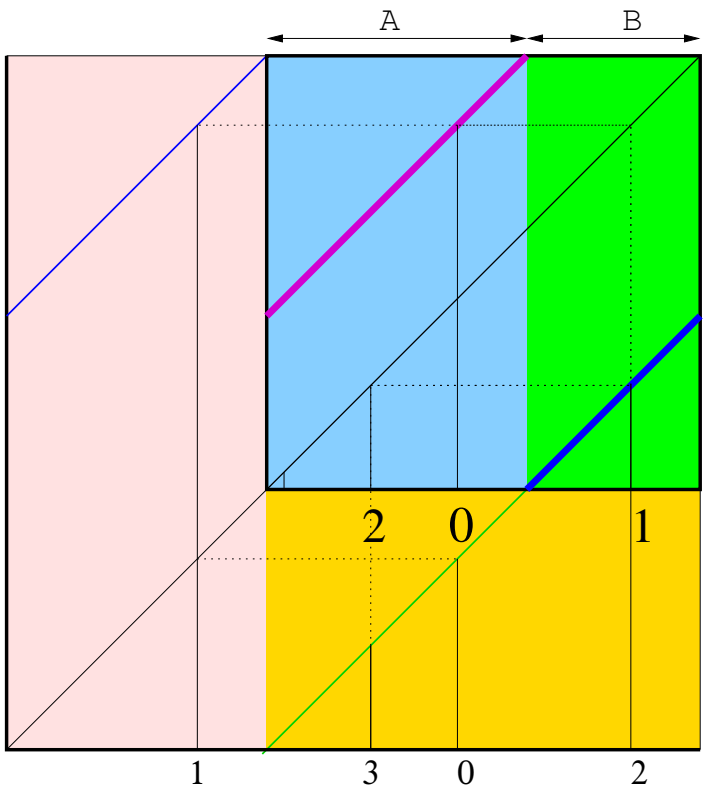

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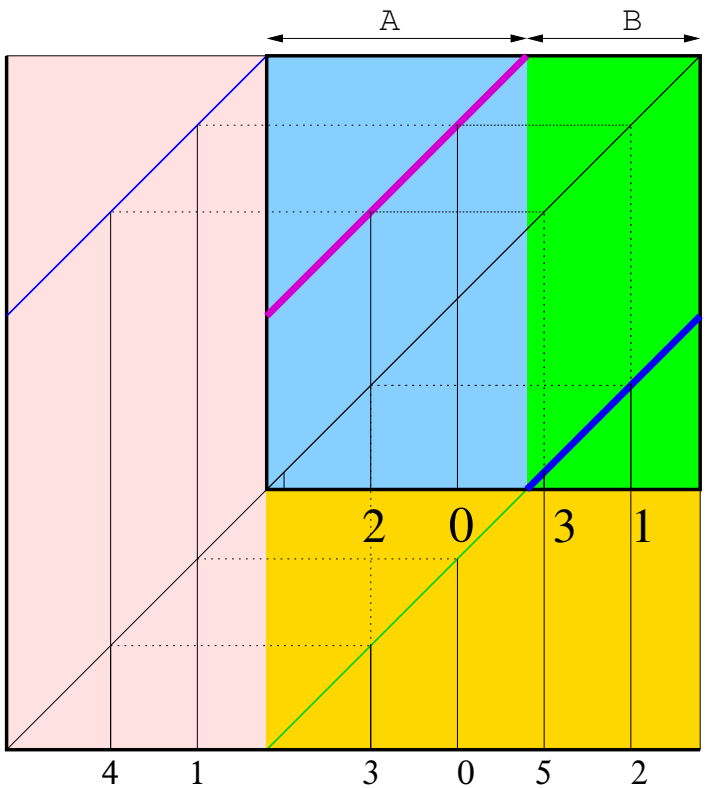

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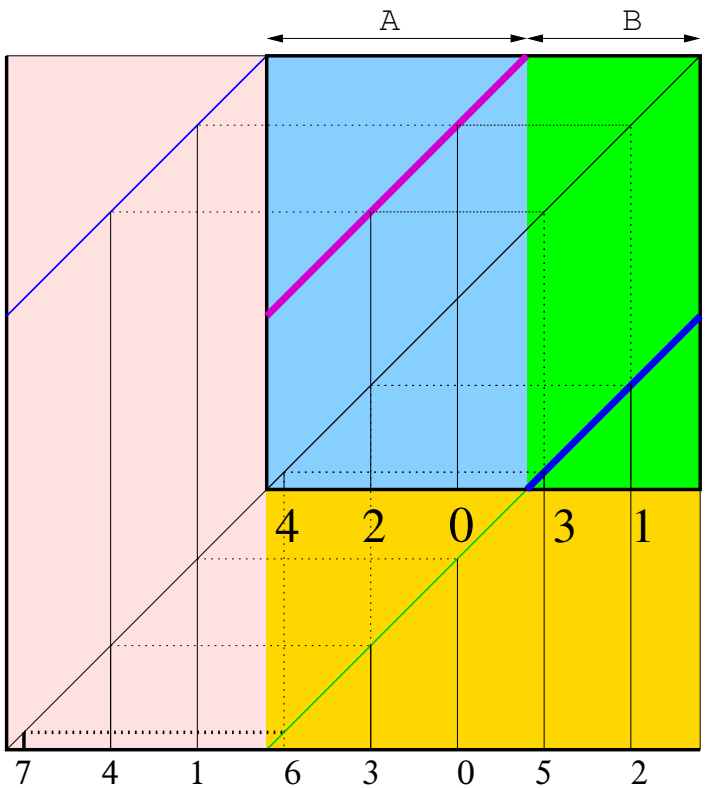
- And obtain a substitution !



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The Fibonacci substitution is

$$a \mapsto ab \qquad b \mapsto a$$

u = abaababaabaababaababaa

Theorem. The addition of the golden ratio is semi-topologically conjugate to the shift map on the symbolic dynamical system associated with the Fibonacci substitution

What can be represented?

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Which are isomorphic to a rotation on a compact group?
What is their maximal equicontinuous factor?

Substitution of constant length

- Caracterization of the maximal equicontinuous factor (Dekking);

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Theorem. *Let σ be non-periodic of constant length n. Let $u = \sigma^k(u)$.*

The height of σ is the greatest m, $(m,n) = 1$, that divides every $i > 0$, $u_i = u_0$.

The maximal equicontinuous factor of (X_σ, S) is the addition of $(1,1)$ on $\mathbb{Z}_n \times \mathbb{Z}/m\mathbb{Z}$.

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$$1 \mapsto 12 \qquad 2 \mapsto 21$$

$$u = 122121122112122121121...$$

$u_0 = 1$ appears at rank 3 and 5

$$\Longrightarrow \quad h = 1.$$

The maximal equicontinuous factor is $\mathbb{Z}_2$.

Substitution of constant length

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$$1 \mapsto 121 \quad 2 \mapsto 312 \quad 3 \mapsto 213$$

$u = 121312121213121312121312$

$u_0 = 1$ appears at rank 2, 4, 6 etc.

$$\Longrightarrow \quad h = 2.$$

The maximal equicontinuous factor is $\mathbb{Z}_3 \times \mathbb{Z}/2\mathbb{Z}$.

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σ satisfies the coincidence condition if the image of each letter under σ^k has the same n-th letter.

$$1 \mapsto 12 \quad 2 \mapsto 23 \quad 3 \mapsto 13$$

$$1223231231312132313121 31$$
$$2313121312231213122323 13$$
$$1223231323131213122323 13$$

Morse has no coincidences: its system is not isomorphic to the dyadic odometer.

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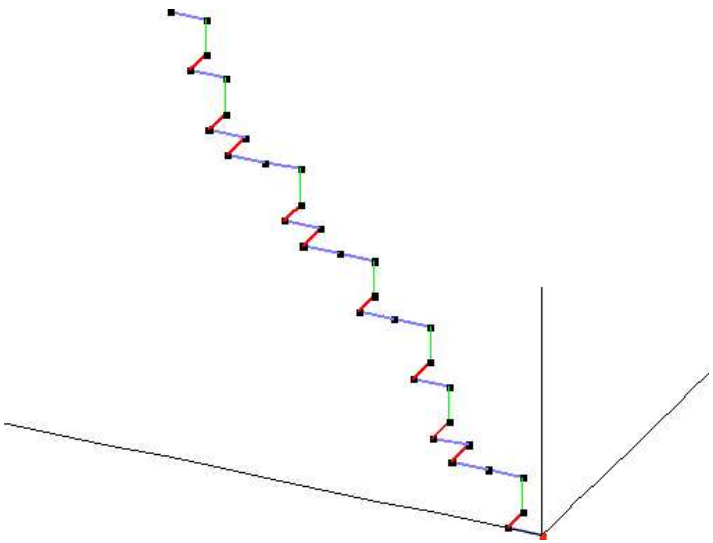

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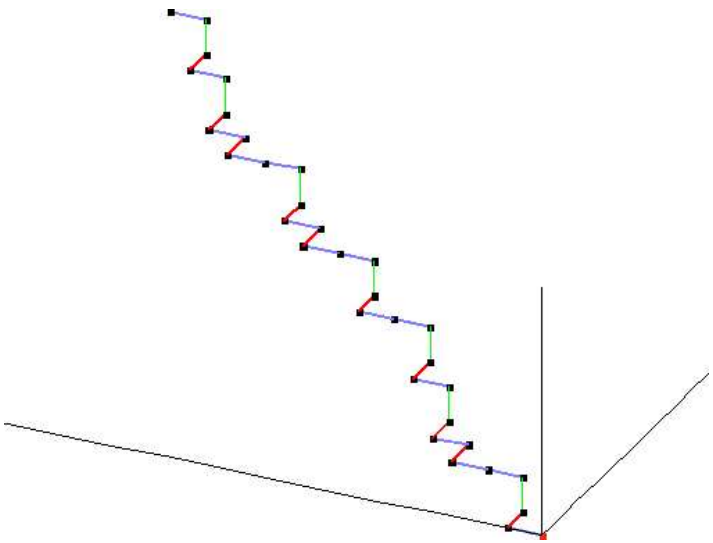

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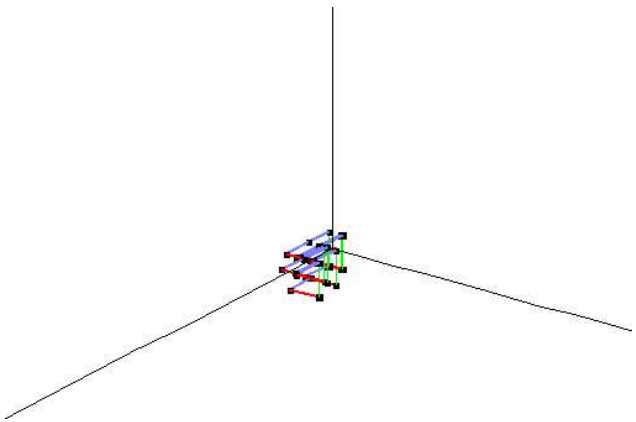

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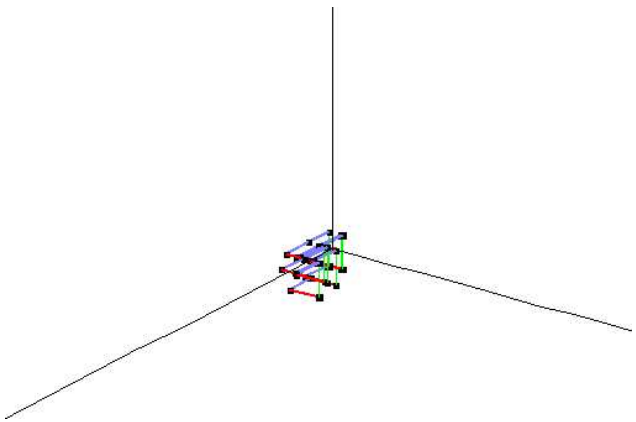

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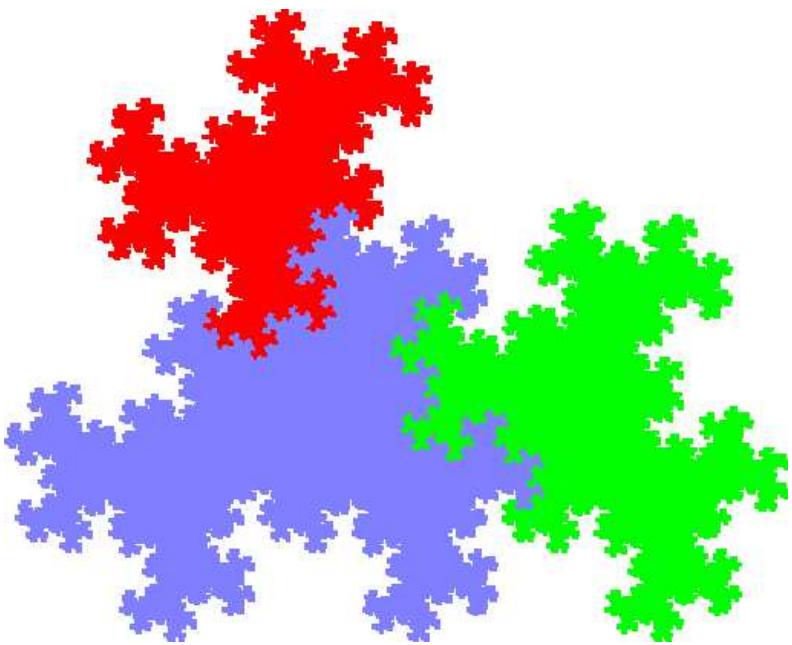

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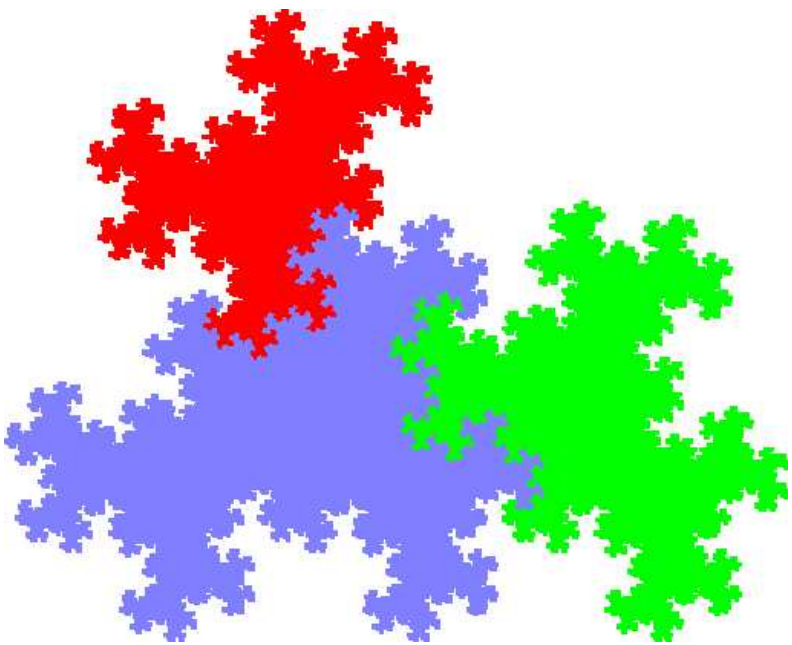

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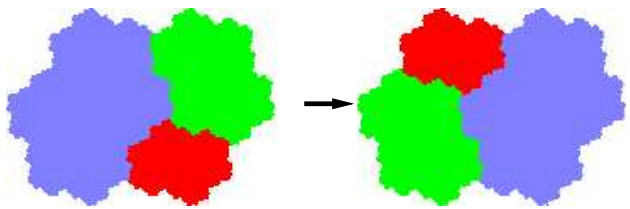

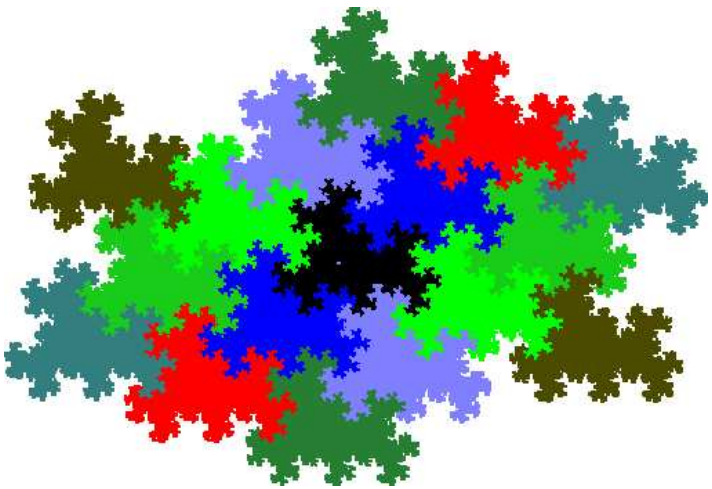

A significant advance (Host)

- Measure-theoretic isomorphism and topological conjugacy are equivalent for primitive systems.

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Eigenvalues and eigenfunctions of (X_σ, S) are those of the unitary operator $f \mapsto f \circ S$ on $\mathcal{L}^2(X_\sigma)$.

Theorem. All eigenfunctions of primitive substitutive systems are continuous

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Coboundary: map $h : \mathcal{A} \to \mathbb{U}$ such that there exists a map $f : \mathcal{A} \to \mathbb{U}$ with $f(b) = f(a)h(a)\ \forall ab$ in the language.

Spectrum: $\lambda \subset \mathbb{U}$ is an eigenvalue of (X_σ, S) iff $\exists\ p > 0$ such that $\forall a \in \mathcal{A}$, the limit

$$h(a) = \lim_{n\to\infty} \lambda^{|\sigma^{pn}(a)|}$$

is a coboundary.

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The eigenvalues for the trivial coboundary are computable and depends only on the incidence matrix of the substitution.

$$\lambda^{|\sigma^{pn}(a)|} \mapsto 1.$$

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- Measure-theoretic isomorphism and topological conjugacy are equivalent for primitive systems.
- Structure of the spectrum: coboundaries
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- Combinatorial spectrum: return words

The eigenvalues for non-trivial coboundaries are purely combinatorial: they depend on return words, playing the role of the height.

Roughly, a return word is a word $W = a_1 \dots a_k$ such that Wa_1 is in the language and $a_i \neq a_1$.

Example: $1 \mapsto 1231$, $2 \mapsto 232$, $3 \mapsto 3123$ has a non-trivial coboundary.

Applications

- Example of weakly mixing substitution

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$1 \mapsto 12121,\ 2 \mapsto 112$
is weakly mixing

(1 is the only eigenvalue)

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Mix of coincidences and return words

$1 \mapsto 23$, $2 \mapsto 12$, $3 \mapsto 23$ as a continuous spectral component but is not weakly mixing (constant length).

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- If the incidence polynomial is irreducible: the existence of discrete spectrum depends on expanding eigenvalues of the matrix (Solomyak).

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- Livshits: conditions for purely discrete spectrum or partially continuous spectrum.
- If the incidence polynomial is irreducible: the existence of discrete spectrum depends on expanding eigenvalues of the matrix (Solomyak).

If $P(\alpha) = C$ for every expanding eigenvalue α of the matrix, then $exp(2\pi i C)$ is an eigenvalue of (X_σ, S).

Partial converse by Ferenczi, Mauduit, Nogueira.

Application: The spectrum of $1 \mapsto 1244$, $2 \mapsto 23$, $3 \mapsto 4$, $4 \mapsto 1$ is $exp(2\pi i \mathbb{Z}\sqrt{2})$.

Substitutions of Pisot type

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Every $P(\alpha)$ is an eigenvalue (Solomyak)

The spectrum contains $\mathbb{Z}[\alpha]$

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In the unimodular case, the arithmetic spectrum is generated by the frequencies of the letters in the fixed point.

In the non-unimodular case, additional rational eigenvalues have to be computed.

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σ satisfies the coincidences conditions if for all b_1, b_2, there exists a such that

$$\sigma^n(b_1) = P_1\, a\, S_1 \quad \sigma^n(b_2) = P_2\, a\, S_2$$

P_1 and P_2 contain the same letters up to the order.

Coincidences $\Longrightarrow$ no combinatorial spectrum

Substitutions of Pisot type

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What about discrete spectrum? They are good candidates...

Spectrum of substitutions of Pisot type

- Pisot type and coincidences on two-letters implies pure discrete spectrum (Host, Hollander)

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Theorem. Every substitutive system of Pisot type on two letters has a pure discrete spectrum.

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- What about three-letters or more?

Spectrum of substitutions of Pisot type

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- What about three-letters or more?

Nothing works...

More intricate results have to be proved in the flavour of Rauzy's work

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Geometric representation of substitutions of Pisot type...

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Coincidences and Pisot type imply that the substitutive system is isomorphic to a domain exchange in a compact set called the Rauzy fractal of the substitution.
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The Rauzy fractal of a substitution with coincidences is a self-similar compact subset of a product of the Euclidean space with finite extensions of p-adic spaces.

It has a non-zero Haar measure.
It is stable under a piecewise translation.

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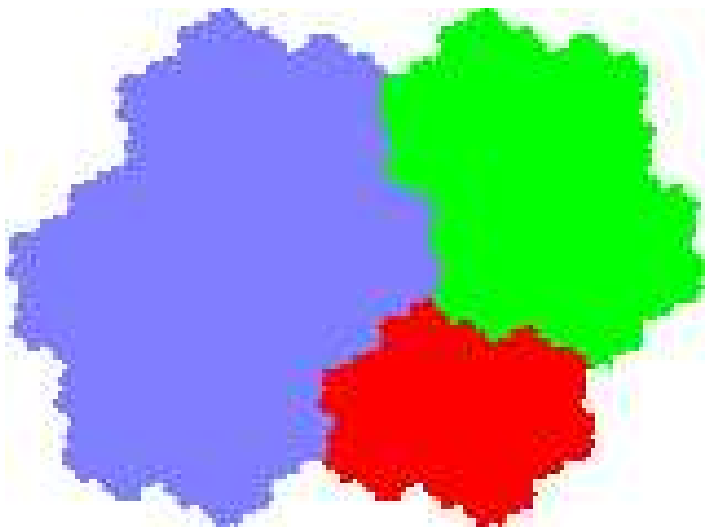

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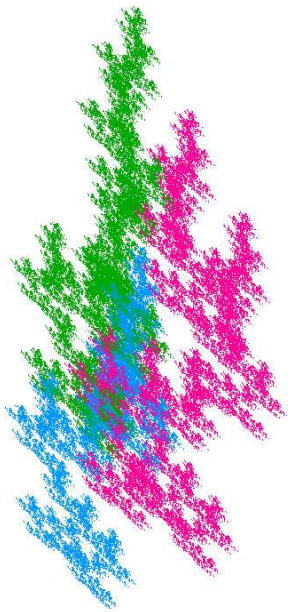

$$1 \mapsto 11223 \; 2 \mapsto 231 \; 3 \mapsto 2$$

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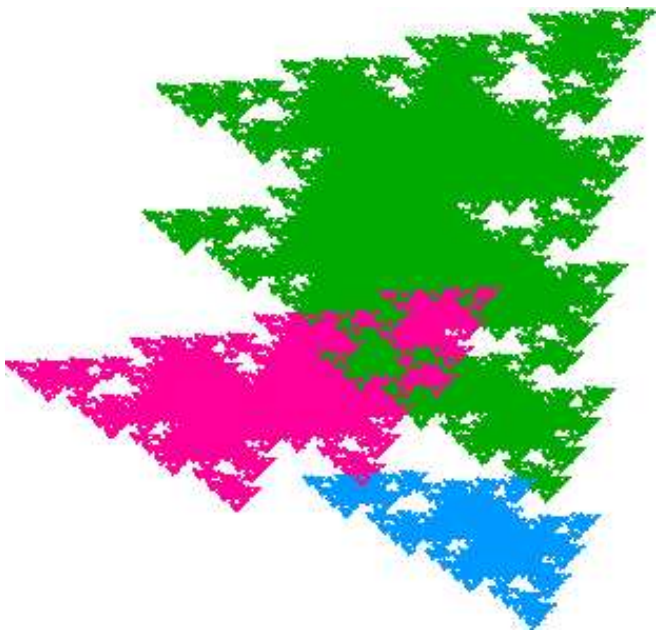

$$1 \mapsto 112 \; 2 \mapsto 31 \; 3 \mapsto 1$$

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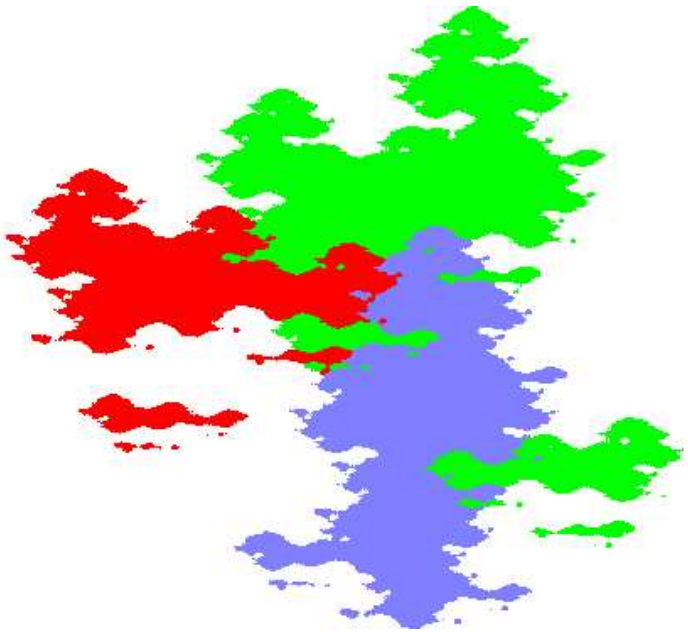

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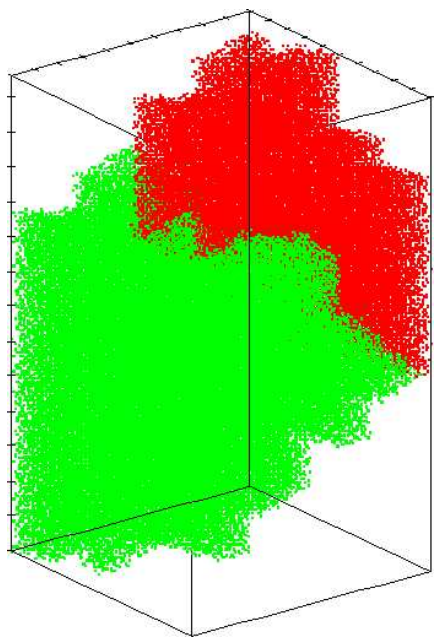

2-adic representation
(in $\mathbb{R} \times \mathbb{Z}_2$)

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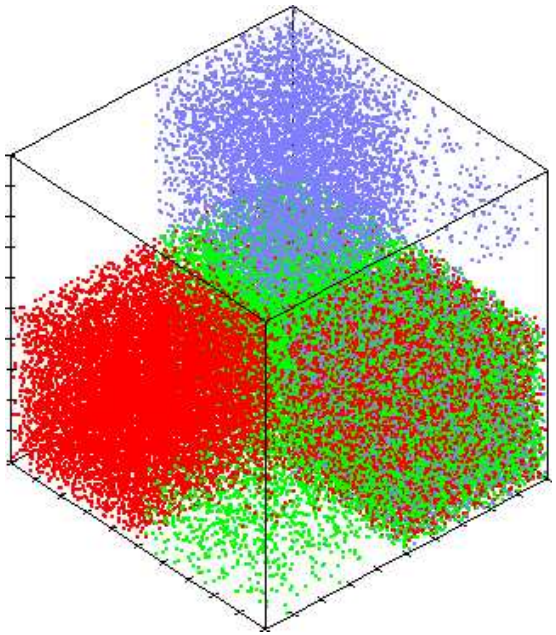

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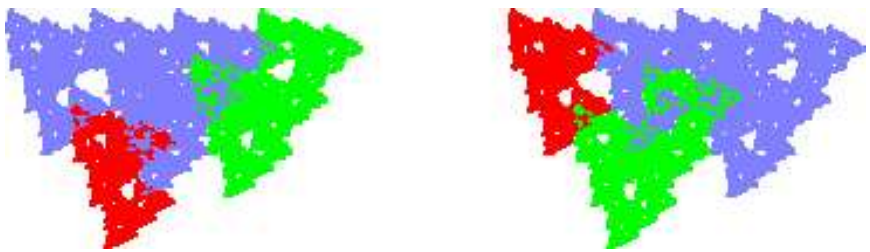

(exchange of domains)

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Notice that we don't know any substitution of Pisot type with no coincidence!

Discrete spectrum...

- The exchange of domains projects onto a translation on an abelian group...

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The question is:

Is this representation one-to-one?

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There is no general result...

Everything has to done by hand (or computer) on each example

Discrete spectrum...

- The exchange of domains projects onto a translation on an abelian group...
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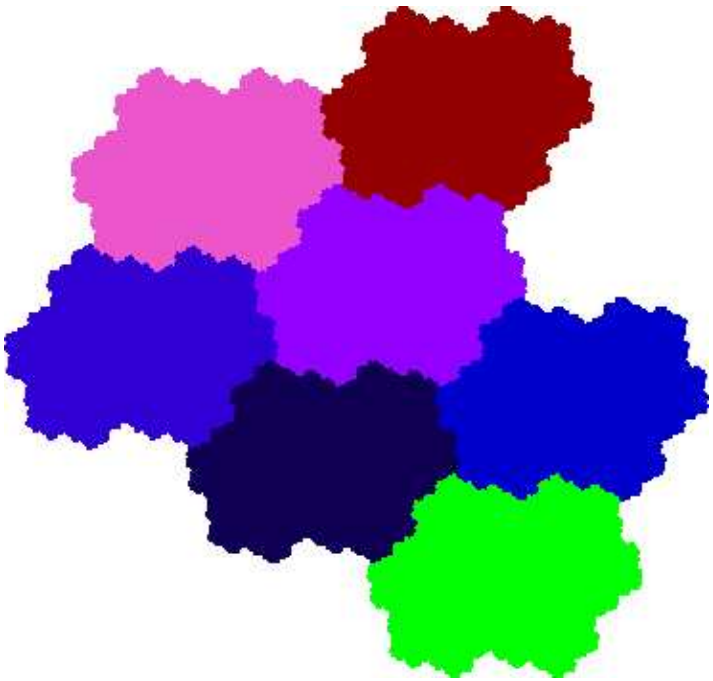
- Geometric interpretation of pure discrete spectrum: tilings.



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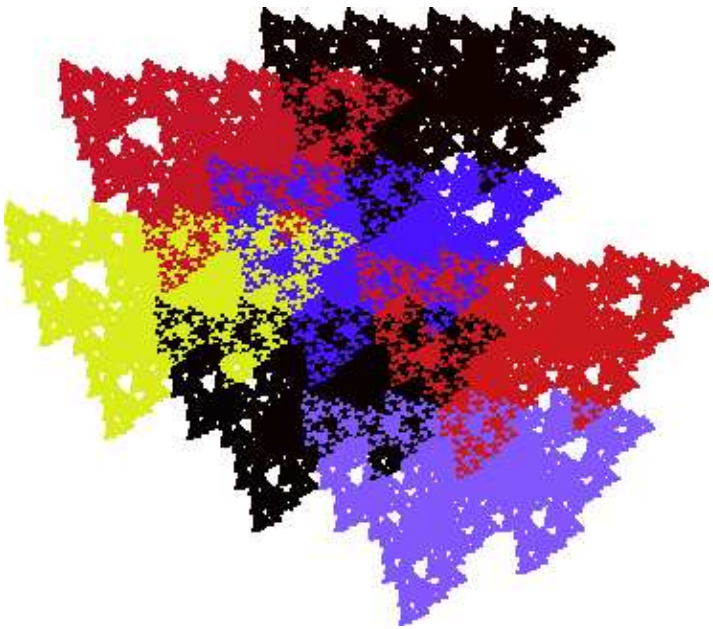

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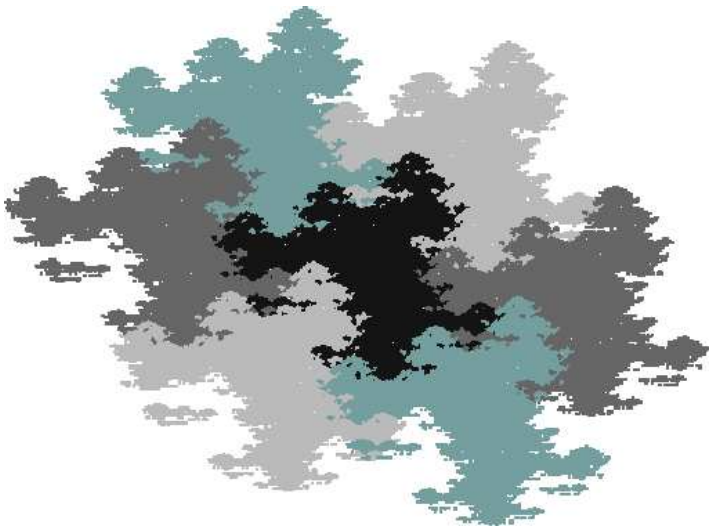

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No counter-example is known!

Each example of a substitution of Pisot type that have been tested has a pure discrete spectrum.

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- Explicit Markov partitions for toral automorphisms.