Spectral theory of Substitutive systems : Combinatorial conditions for Pure spectrum and tilings

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Representation of substitutive dynamical systems

Let σ be a of Pisot type and (X_σ, S) be its symbolic system.

- The Fibonacci substitution ($1 \mapsto 12$, $2 \mapsto 1$) provides the best representation for the addition of the golden ratio.
- The Morse substitution ($1 \mapsto 12$, $2 \mapsto 21$) is a coding of a two-point extension of the dyadic odometer.
- The Tribonacci substitution ($1 \mapsto 12$, $2 \mapsto 13$, $3 \mapsto 1$) codes a domain exchange in $\mathbb{R}^2$ as well as a two-dimensional toral rotation.

Which substitive systems are isomorphic to a rotation on a compact group?

What is their maximal equicontinuous factor?

Substitution of constant length

- Caracterization of the maximal equicontinuous factor.
- the Morse substitution has height 1.
- Example with height 3.
- Pure discrete spectrum if $h = 1$: the coincidence condition.
- Pure discrete spectrum if $h \neq 1$: recoding in a pure base.

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Theorem (Dekking)

Let σ be non-periodic of constant length n. Let $u = \sigma^k(u)$.

The height of σ is the greatest m, $(m, n) = 1$, that divides every $i > 0$, $u_i = u_0$.

The maximal equicontinuous factor of (X_σ, S) is the addition of $(1, 1)$ on $\mathbb{Z}_n \times \mathbb{Z}/m\mathbb{Z}$.

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Morse substitution

$$1 \mapsto 12 \qquad 2 \mapsto 21$$

$u = 1221211221121221211 21 \ldots$
$u_0 = 1$ appears at rank 3 and 5 $\Longrightarrow \quad h = 1$.

The maximal equicontinuous factor is $\mathbb{Z}_2$.

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$$1 \mapsto 121 \quad 2 \mapsto 312 \quad 3 \mapsto 213$$

$$u = 121312121213121312121312$$

$u_0 = 1$ appears at rank 2, 4, 6 etc.

$$\implies \quad h = 2.$$

The maximal equicontinuous factor is $\mathbb{Z}_3 \times \mathbb{Z}/2\mathbb{Z}$.

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coincidence condition

All $\sigma^k(a)$'s have the same n-th letter.

$1 \mapsto 12 \quad 2 \mapsto 23 \quad 3 \mapsto 13$

122323123131213231312131
231312131223121312232313
122323132313121312232313

Morse has no coincidences : it is a dyadic odometer.

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Non trivial height

$$1 \mapsto 121 \quad 2 \mapsto 312 \quad 3 \mapsto 213$$

has a pure discrete spectrum.

Non constant-length : Host

- If σ is primitive, measure-theoretic isomorphism = topological conjugacy.
- Structure of the spectrum: coboundaries.
- Arithmetic spectrum: incidence matrix.
- Combinatorial spectrum: return words.

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Theorem (Host). All eigen-functions of primitive substitutive systems are continuous.

Non constant-length : Host

- If σ is primitive, measure-theoretic isomorphism = topological conjugacy.
- Structure of the spectrum: coboundaries.
- Arithmetic spectrum: incidence matrix.
- Combinatorial spectrum: return words.

Coboundary: map $h : \mathcal{A} \to \mathbb{U}$ such that $f : \mathcal{A} \to \mathbb{U}$ with $f(b) = f(a)h(a)\ \forall ab \in \mathcal{L}$.

Spectrum (Host): $\lambda \subset \mathbb{U}$ is an eigenvalue of (X_σ, S) iff $\exists\ p > 0$ such that $\forall a \in \mathcal{A}$, the limit

$$h(a) = \lim_{n\to\infty} \lambda^{|\sigma^{pn}(a)|}$$

is a coboundary.

Non constant-length : Host

- If σ is primitive, measure-theoretic isomorphism = topological conjugacy.
- Structure of the spectrum: coboundaries.
- Arithmetic spectrum: incidence matrix.
- Combinatorial spectrum: return words.

Arithmetic spectrum

The eigenvalues for the trivial coboundary are computable and depends only on the incidence matrix of the substitution.

$$\lambda^{|\sigma^{pn}(a)|} \mapsto 1.$$

Non constant-length : Host

- If σ is primitive, measure-theoretic isomorphism = topological conjugacy.
- Structure of the spectrum: coboundaries.
- Arithmetic spectrum: incidence matrix.
- Combinatorial spectrum: return words.

Combinatorial spectrum

The eigenvalues for non-trivial coboundaries depend on return words, playing the role of the height.

A return word is a word $W = a_1 \ldots a_k$ such that $W a_1$ is in the language and $a_i \neq a_1$ (Livshits, Durand...).

$1 \mapsto 1231,\ 2 \mapsto 232,\ 3 \mapsto 3123$ has a non-trivial coboundary.

Applications

- Example of weakly mixing substitution
- Livshits: conditions for purely discrete spectrum or partially continuous spectrum.
- If the incidence polynomial is irreducible: the existence of discrete spectrum depends on expanding eigenvalues of the matrix (Solomyak).

$1 \mapsto 12121,\ 2 \mapsto 112$
is weakly mixing.

(1 is the only eigenvalue)

Applications

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Mix of coincidences and return words

$1 \mapsto 23,\ 2 \mapsto 12,\ 3 \mapsto 23$ as a continuous spectral component but is not weakly mixing (constant length).

Applications

- Example of weakly mixing substitution
- Livshits: conditions for purely discrete spectrum or partially continuous spectrum.
- If the incidence polynomial is irreducible: the existence of discrete spectrum depends on expanding eigenvalues of the matrix (Solomyak).

If $P(\alpha) = C$ for every expanding eigenvalue α of the matrix, then $exp(2\pi iC)$ is an eigenvalue of (X_σ, S) (Solomyak).

Partial converse by Ferenczi, Mauduit, Nogueira.

Application: The spectrum of $1 \mapsto 1244$, $2 \mapsto 23$, $3 \mapsto 4$, $4 \mapsto 1$ is $exp(2\pi i\mathbb{Z}\sqrt{2})$.

Substitutions of Pisot type

Pisot type: *the incidence matrix has only one expanding eigenvalue* α*.*

- They are never weakly mixing
- Their arithmetical spectrum can be computed
- The combinatorial spectrum is empty

Non empty spectrum

Every $P(\alpha)$ is an eigenvalue (Solomyak)

The spectrum contains $\mathbb{Z}[\alpha]$

Substitutions of Pisot type

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Arithmetical spectrum

In the unimodular case, the arithmetic spectrum is generated by the frequencies of the letters in the fixed point.

In the non-unimodular case, additional rational eigenvalues have to be computed.

Substitutions of Pisot type

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- Their arithmetical spectrum can be computed
- The combinatorial spectrum is empty

Combinatorial spectrum

Theorem (Barge, Kwapisz). Pisot type and unimodular

$$\Longrightarrow$$

no combinatorial spectrum.

Pure discrete spectrum $\Longleftrightarrow$ there exists a metric conjugacy with an abelian translation defined by the arithmetic spectrum.

Pisot on two letters

Strong coincidences condition : $\forall\, b_1, b_2, \exists\, a\ \sigma^n(b_1) = P_1\, a\, S_1$, $\sigma^n(b_2) = P_2\, a\, S_2$, P_1 and P_2 contain the same letters.

- Pisot type and strong coincidences on two-letters implies pure discrete spectrum (Host, Hollander-Solomyak)
- Pisot type on two letters implies strong coincidences (Barge-Diamond).

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- Pisot type and strong coincidences on two-letters implies pure discrete spectrum (Host, Hollander-Solomyak)
- Pisot type on two letters implies strong coincidences (Barge-Diamond).

Theorem. Every substitutive system of Pisot type on two letters has a pure discrete spectrum.

Super/Geometrical coincidences

Combinatorial condition for tilings in the unimodular case.

Def : Two unit integral segment that cross the contracting plane have to share a common third segment after a finite iteration of the substitution.

Theorem (Barge-Kwapisz, Ito-Rao) : In the Pisot type and unimodular case, pure discrete spectrum $\Longleftrightarrow$ Super coincidences.

Effectivity: Balanced pair algorithm checks super coincidences (but does not check the converse).

The Tribonacci substitution

$$1 \mapsto 12 \quad 2 \mapsto 13 \quad 3 \mapsto 1$$

- Studied by Rauzy in 1981.
- The fixed point provides a $\mathbb{R}^3$ stair.
- A projection: the Rauzy fractal.
- Ergodic properties.
- Periodic tiling.

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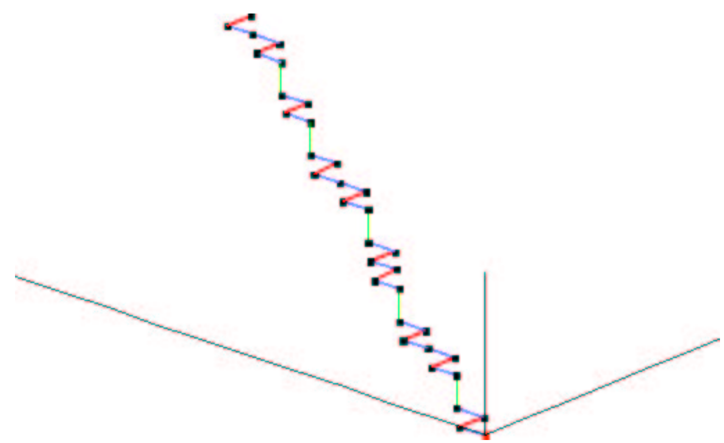

$121312121312131211213121121312$

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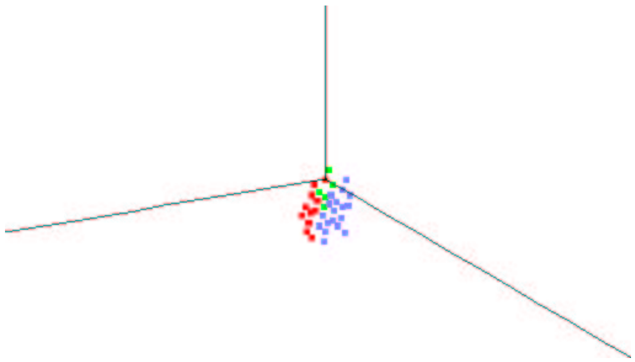

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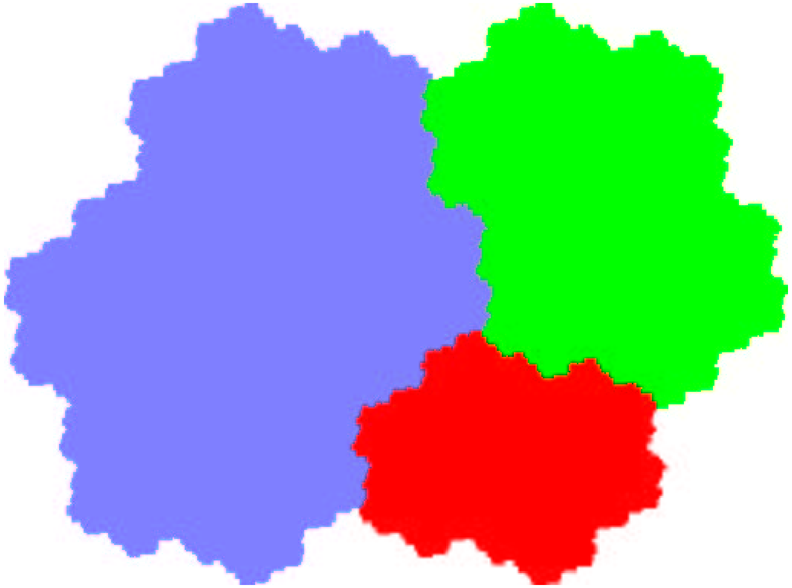

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Theorem (Rauzy)

The substitutive system is measure-theoretically isomorphic with a domain exchange on the Rauzy fractal.

It is isomorphic with a two-dimensional toral translation.

It has an explicit pure discrete spectrum.

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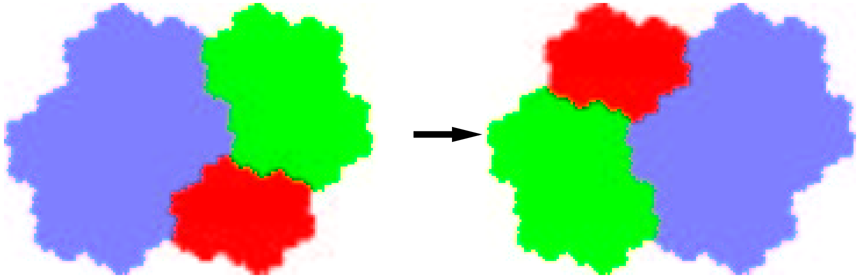
Geometric representation

Theorem (Rauzy, Arnoux-Ito, Canterini, S.)

Strong coincidences and Pisot type

$\Longrightarrow$

(X_σ, S) is isomorphic with a domain exchange in a compact set.



(exchange of domains)

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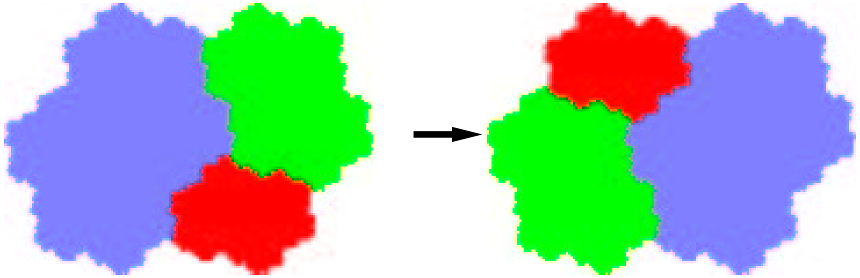

(exchange of domains)

Rauzy fractals

- Pisot substitution with coincidences
- self-similar compact
- Belong to an Euclidean space cross finite extensions of p-adic spaces.
- Non-zero Haar measure.

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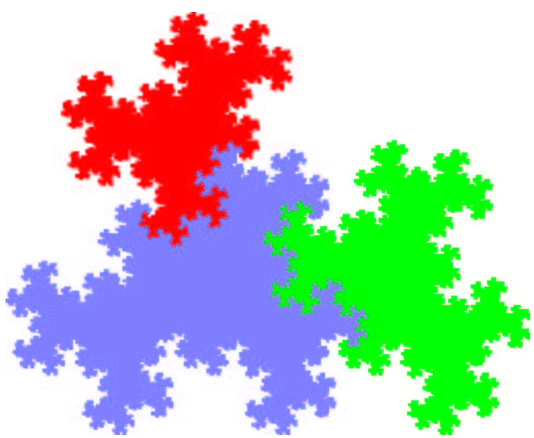

$1 \mapsto 12 \; 2 \mapsto 3 \; 3 \mapsto 1$

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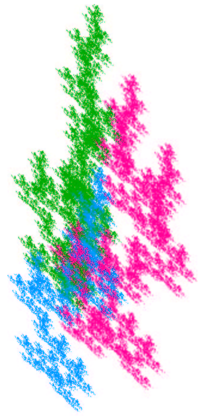

$1 \mapsto 11223 \; 2 \mapsto 231 \; 3 \mapsto 2$

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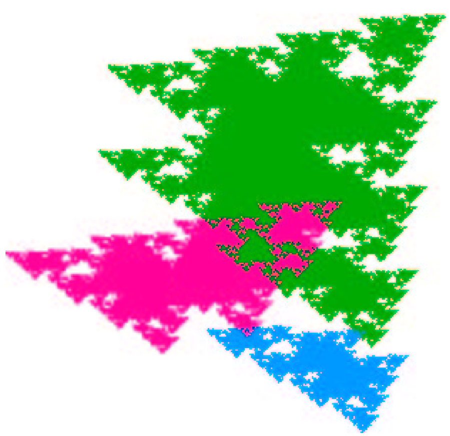

$1 \mapsto 112 \; 2 \mapsto 31 \; 3 \mapsto 1$

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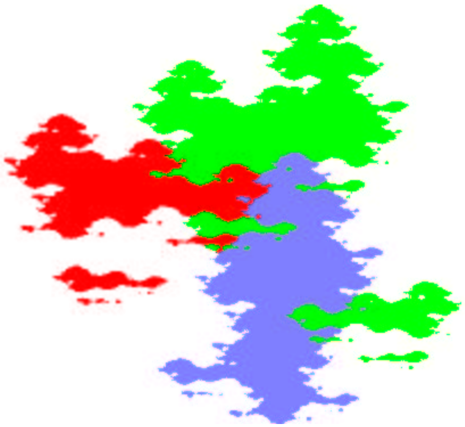

non connected example

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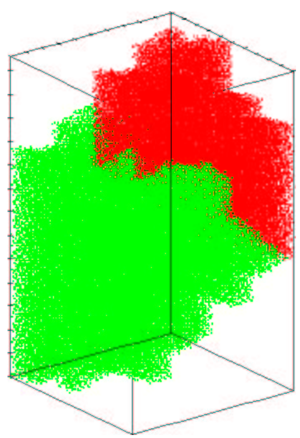

$1 \mapsto 1112$ $2 \mapsto 12$ (in $\mathbb{R}^2 \times \mathbb{Z}_2$)

Rauzy fractals

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Rauzy fractals

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All known substitutions of Pisot type have strong coincidences !

Pure discrete spectrum from numeration systems

Quotient by the translations vectors implied in the exchange of domains

$\Longrightarrow$ translation on a compact abelian group

The quotient map is finite to one (Host). Is it one-to-one ?

Pure discrete spectrum from numeration systems

Quotient by the translations vectors implied in the exchange of domains

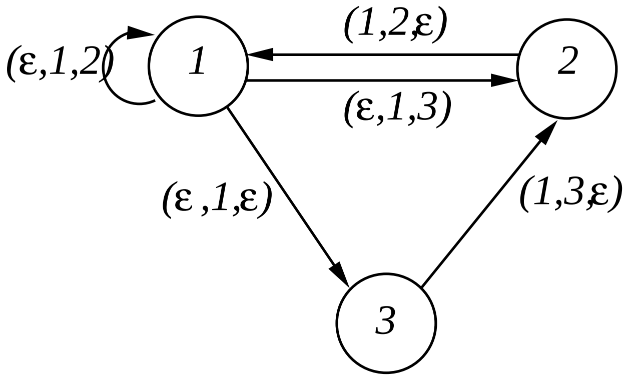
$\Longrightarrow$ translation on a compact abelian group

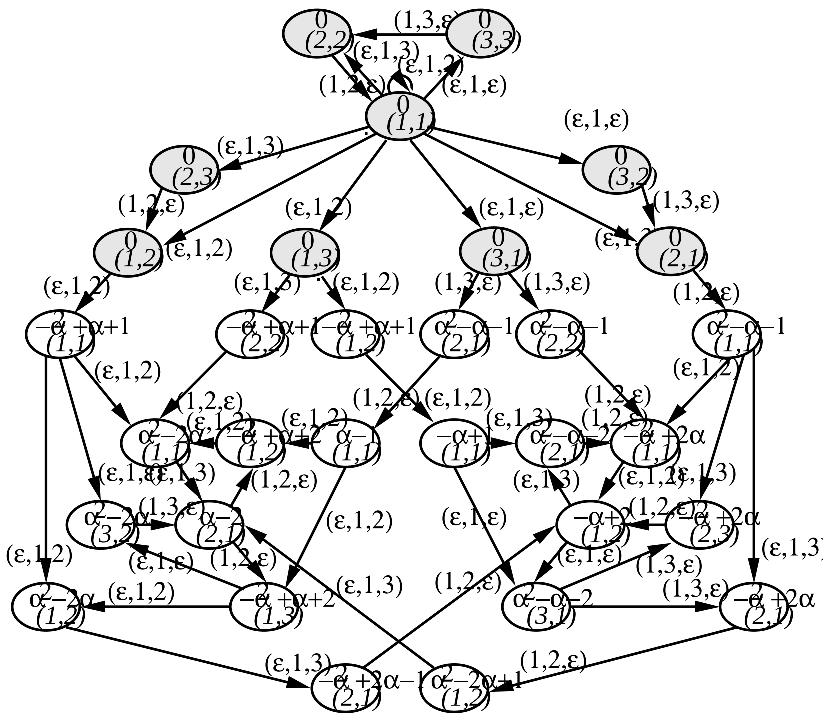
The quotient map is finite to one (Host). Is it one-to-one ?

- Desubstitution = combinatorial “division” by σ over ${}_{\sigma}$.
- Combinatorial expansions are recognized by an automaton or a Brattelli diagram.
- Non simple fibers of the quotient map correspond to non-proper expansions.
- An automaton recognize non-proper expansions.

Pure discrete spectrum from numeration systems

Theorem (S., Thuswaldner): An explicit algorithm checks wether a Pisot substitutive system has a pure discrete spectrum.
The condition is a CNS if σ is unimodular.





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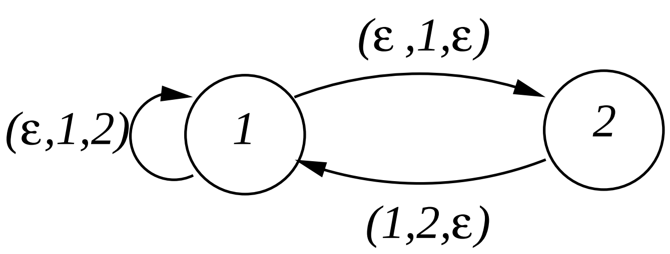
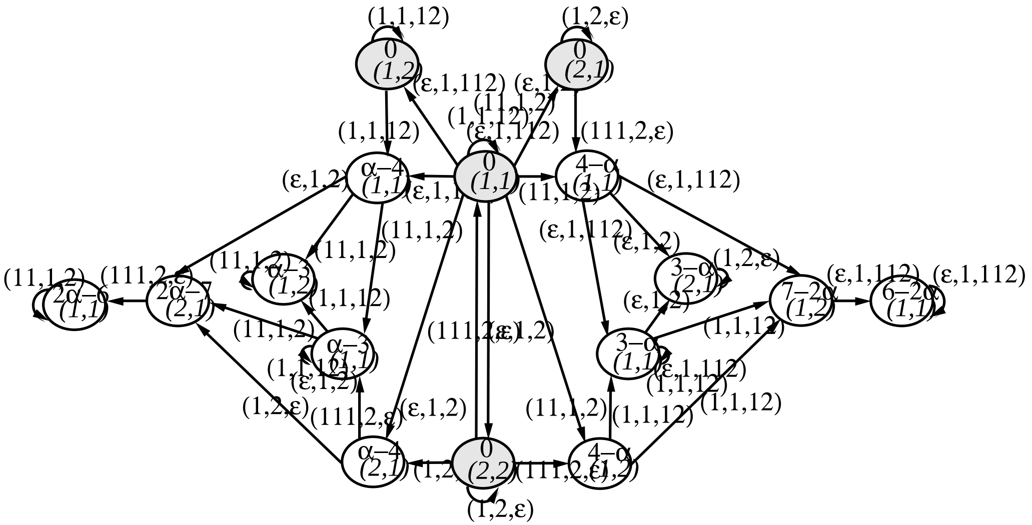


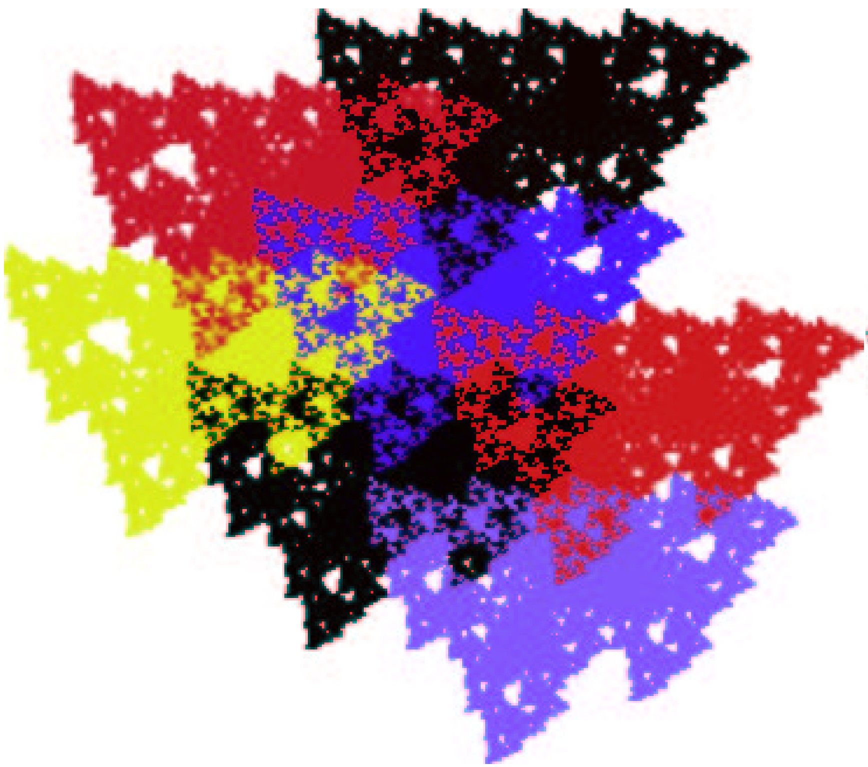


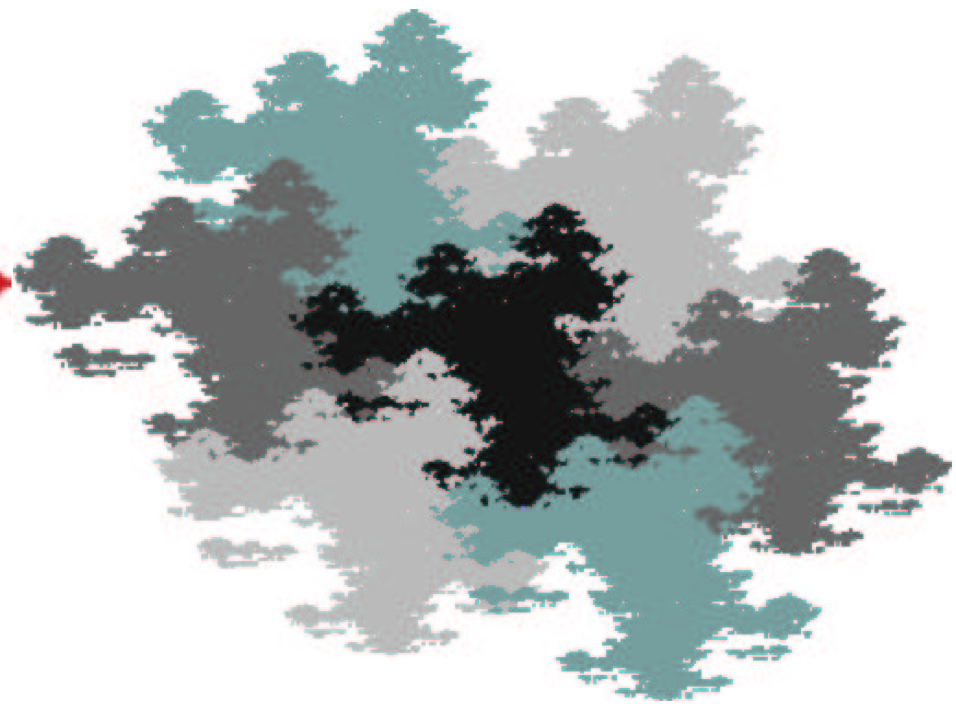
No counter-example is known : Each computed substitution of Pisot type has a pure discrete spectrum.

$d-1$-*dimensional periodic tilings*

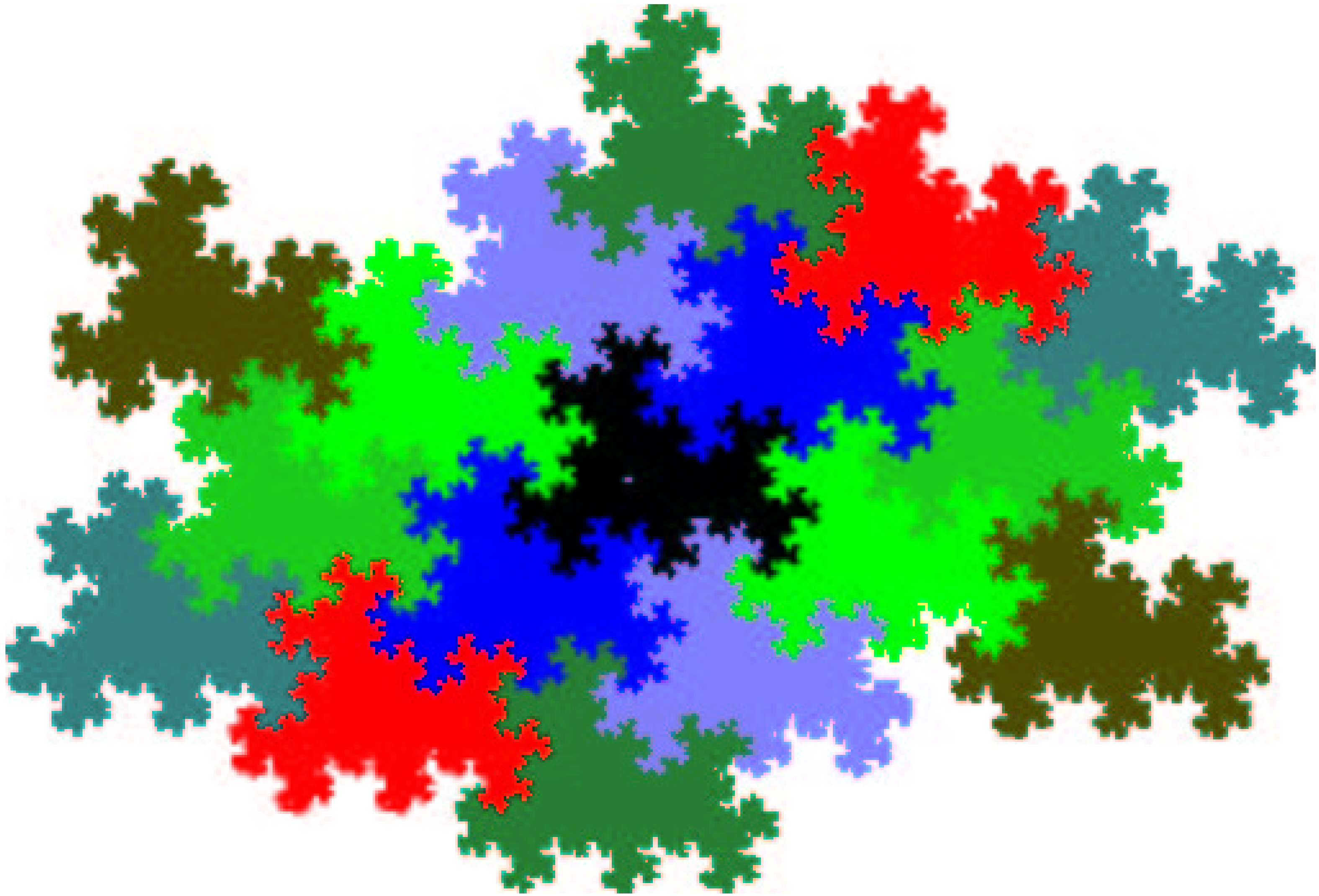
- Pisot unimodular : the abelian group translation is generated by the frequency of letters.
- Discrete spectrum is equivalent with the Rauzy fractal to be a fundamental domain for the associated lattice.

$d-1$-dimensional periodic tilings



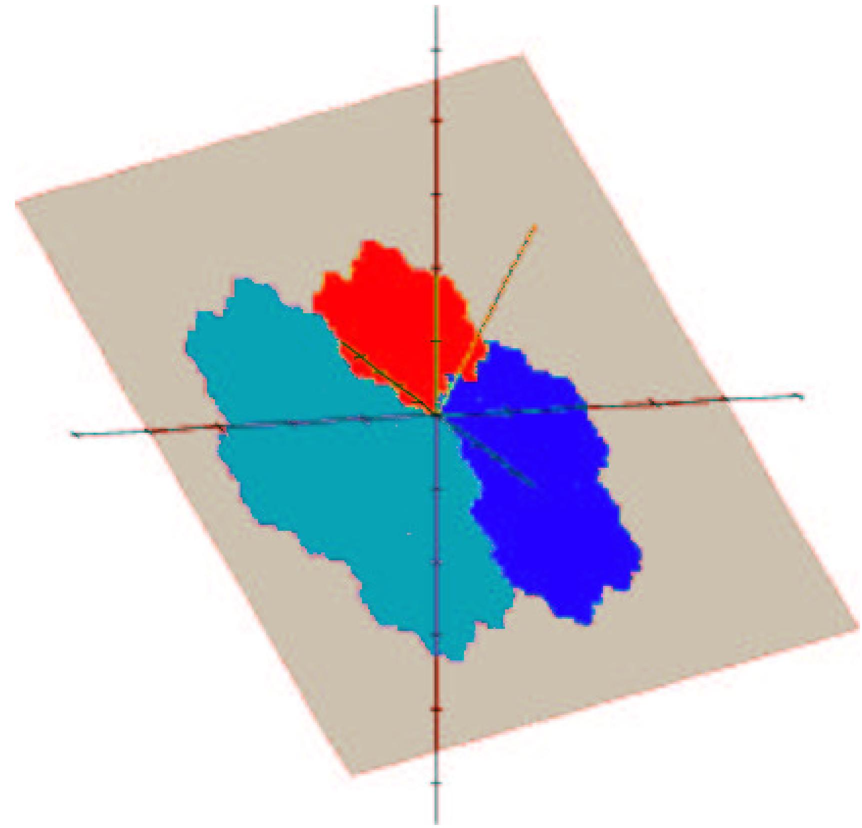


$d-1$-dimensional periodic tilings



d-dimensional tiling

- Embed the Rauzy fractal in the contracting plane.
- Add a transverse component along the expanding direction.
- Translate this $\mathbb{R}^d$ volume along $\mathbb{Z}^d$.



Prop. $d-1$-dimensional periodic tiling $\implies$ d-dimensional periodic tiling.

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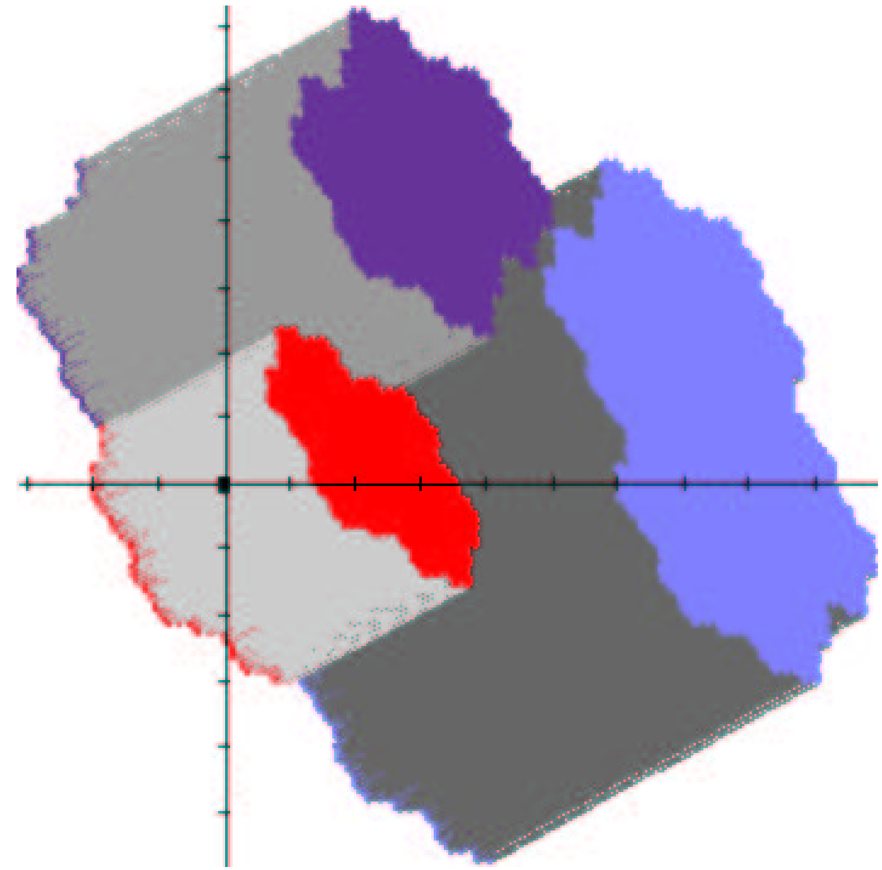

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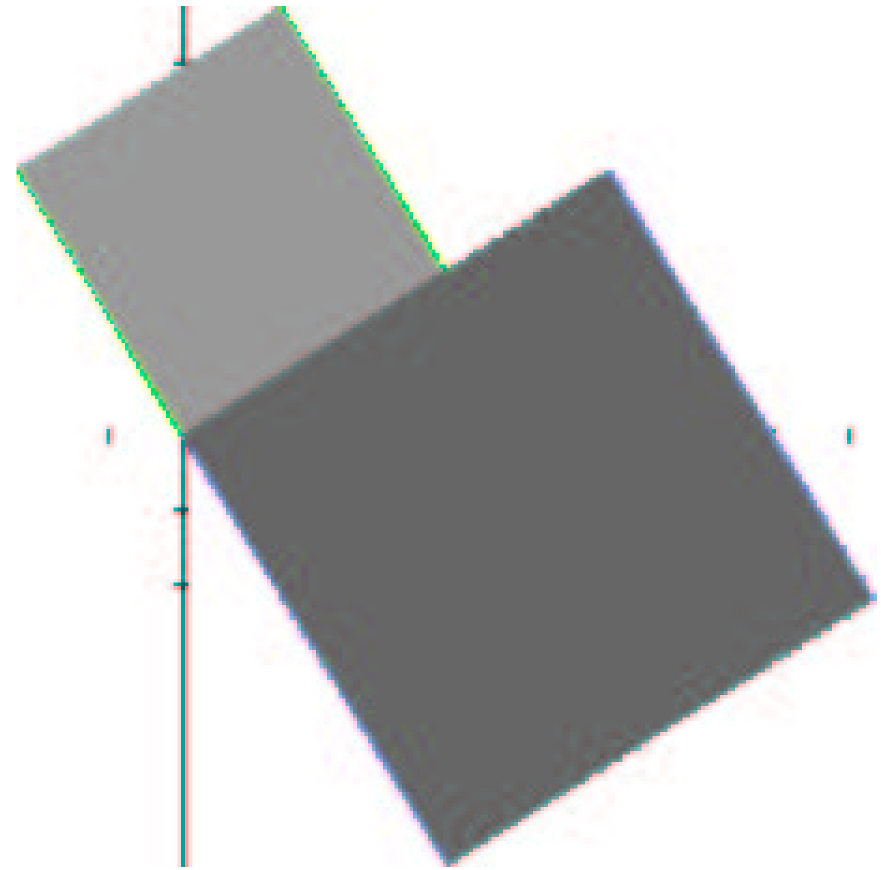

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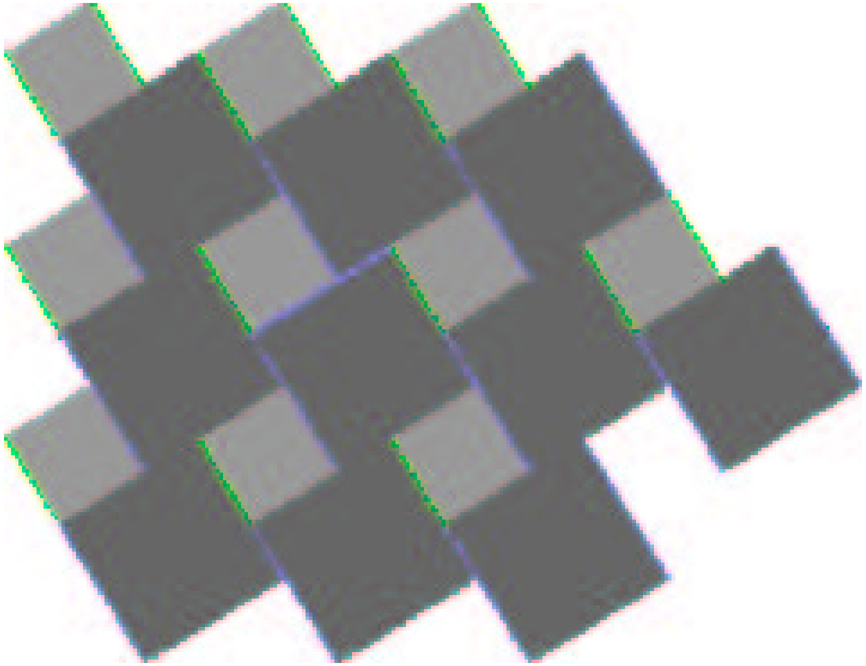

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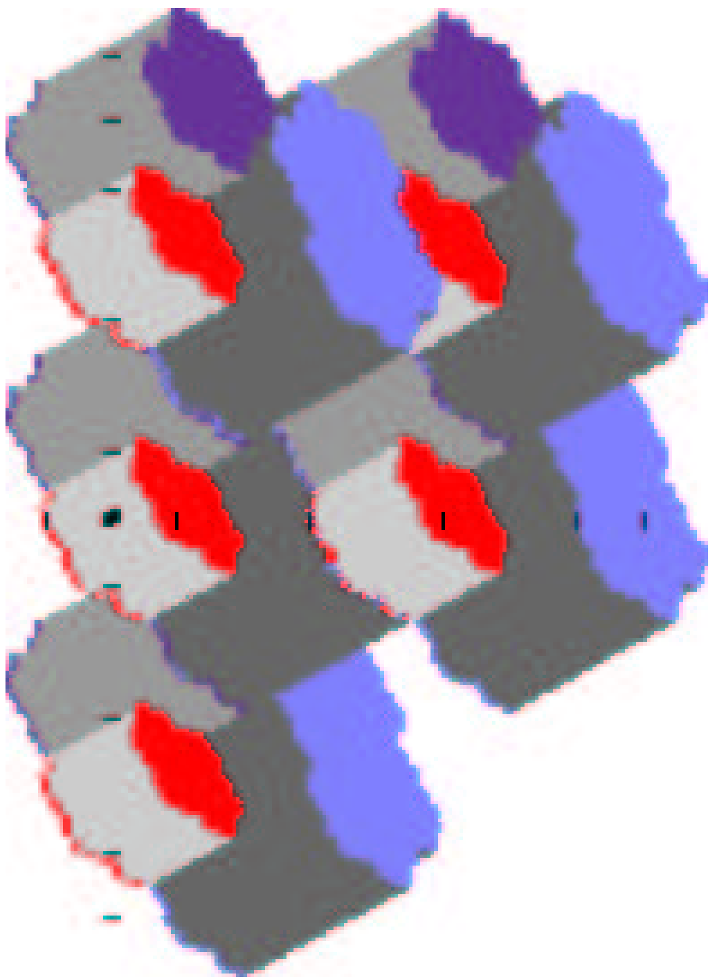
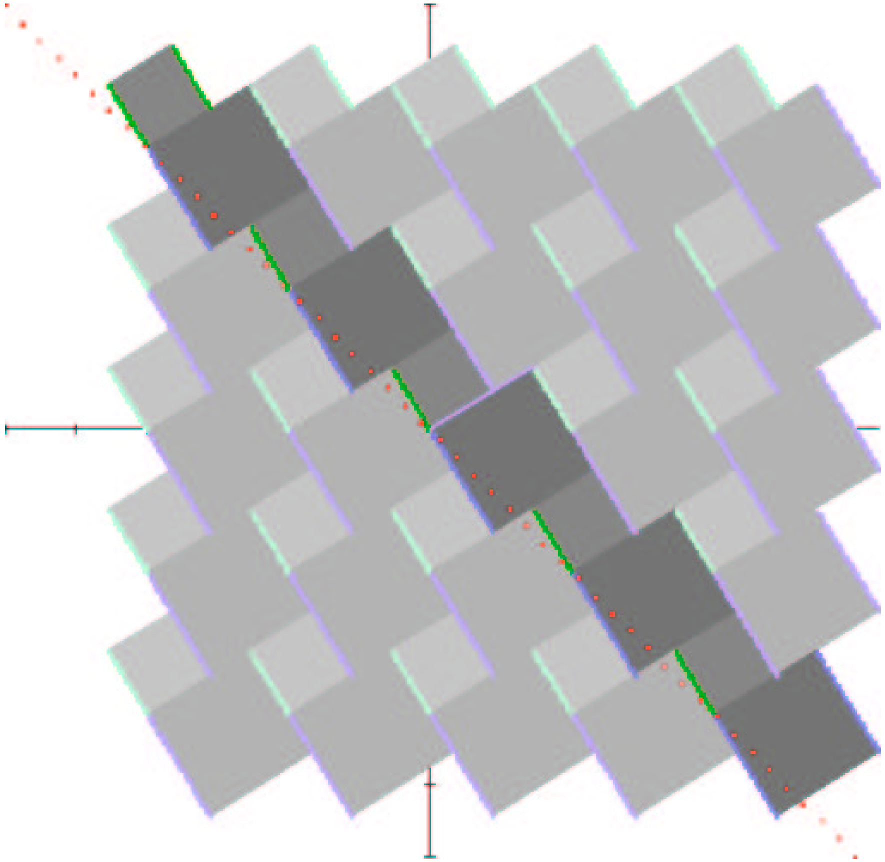


Equivalent tilings

Prop. The $d-1$-dim periodic tiling is obtained as the projection on the contracting plane of $\mathcal{R} + \{x \in \mathbb{Z}^d, \sum x_i = 0\}$ (copies of the d-dim Rauzy fractal on the discrete anti-diagonal plane).



Prop. $d-1$-dimensional periodic tiling $\iff$ d-dimensional periodic tiling.

$\iff$ Each piece of the Rauzy fractal has the measure of the projection of the corresponding unit cube face on the contracting plane.

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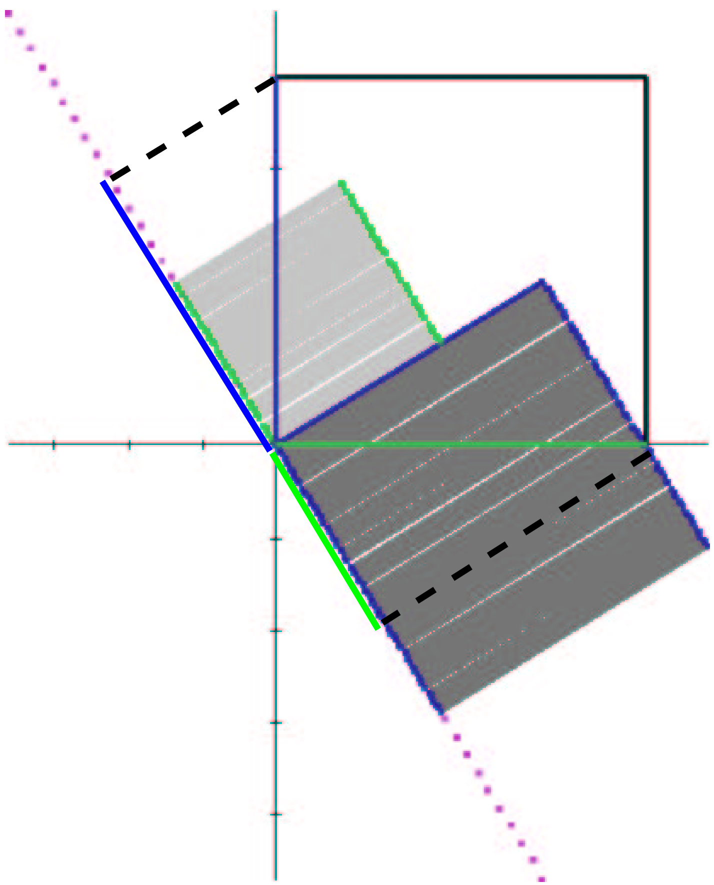
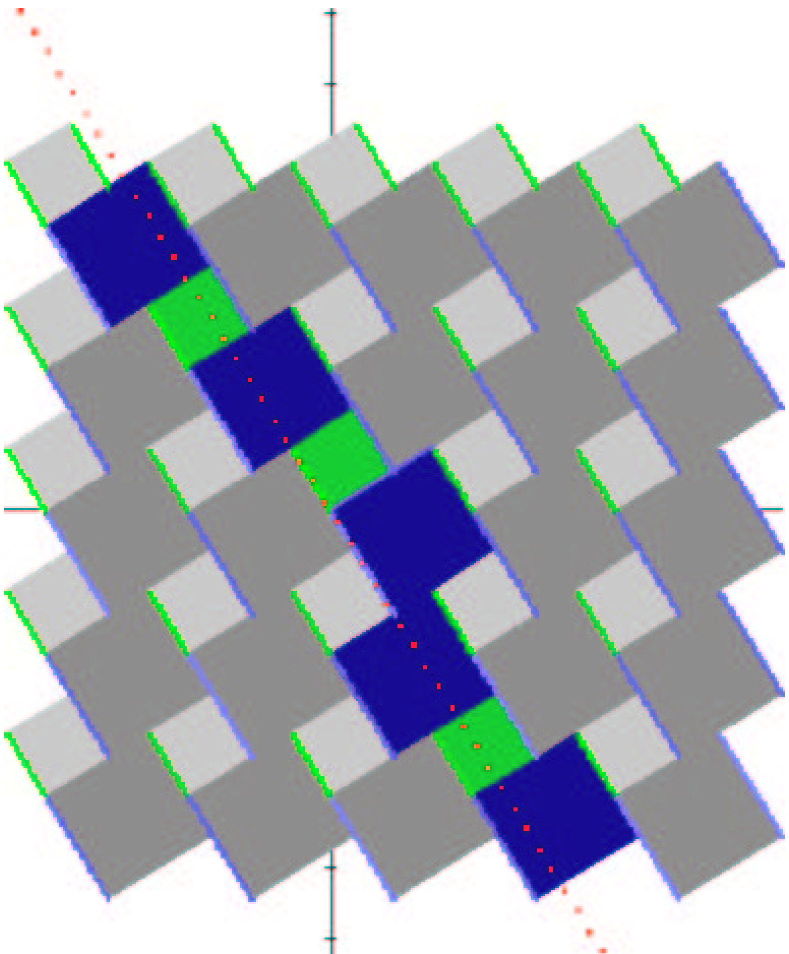


Prop. $d-1$-dimensional periodic tiling $\Longleftrightarrow$ d-dimensional periodic tiling.

$\Longleftrightarrow$ Each piece of the Rauzy fractal has the measure of the projection of the corresponding unit cube face on the contracting plane.

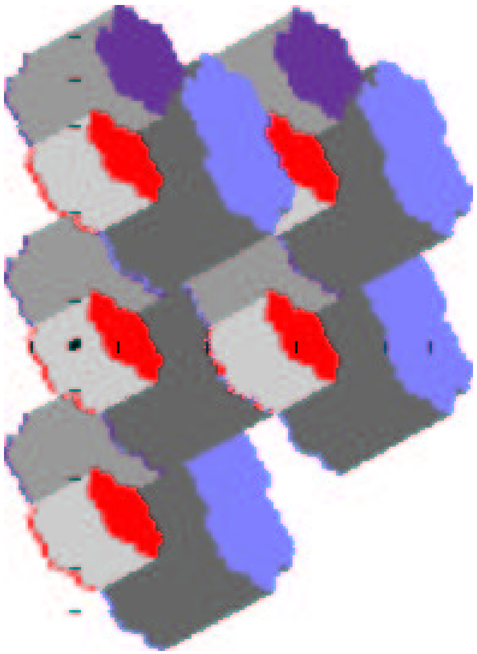
Self-similar aperiodic tiling

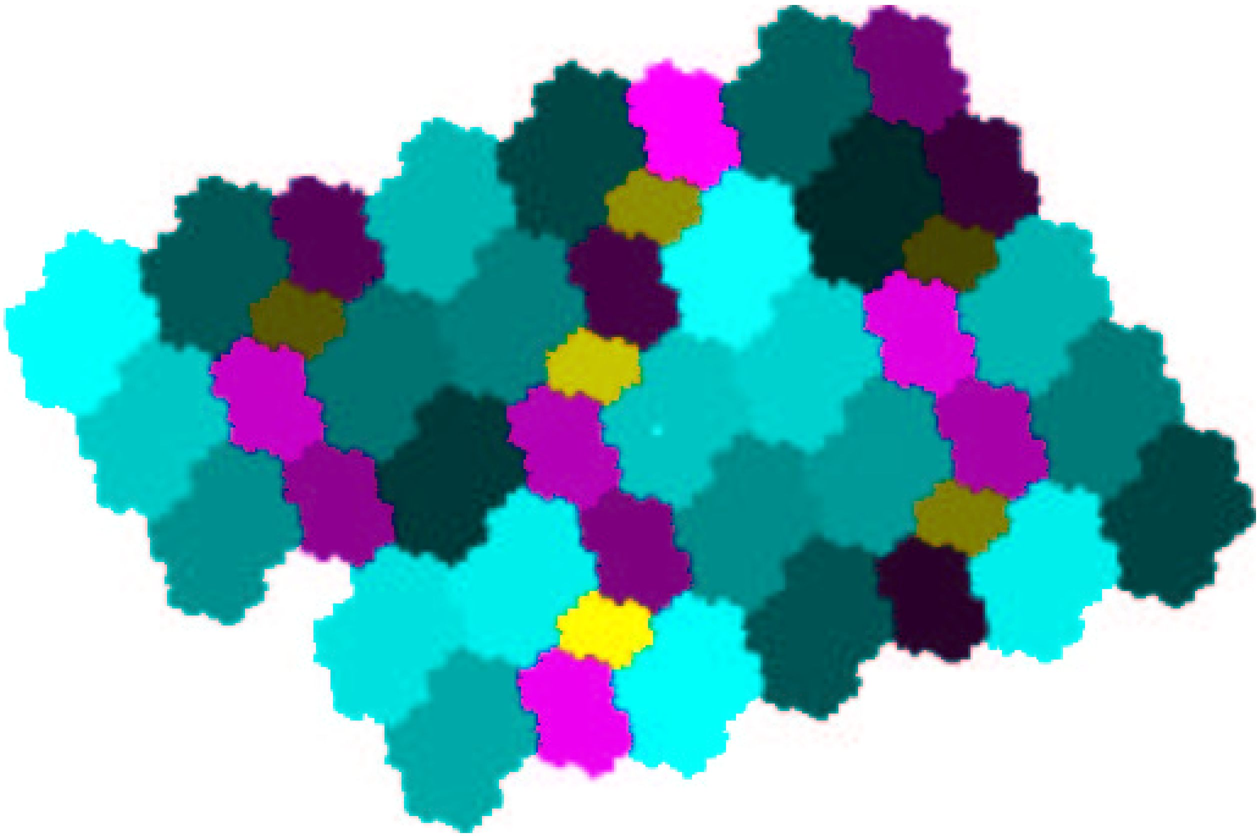
Prop. A self-similar aperiodic tiling is obtained as the intersection of the contracting hyperplane with the d-dimensional tiling.



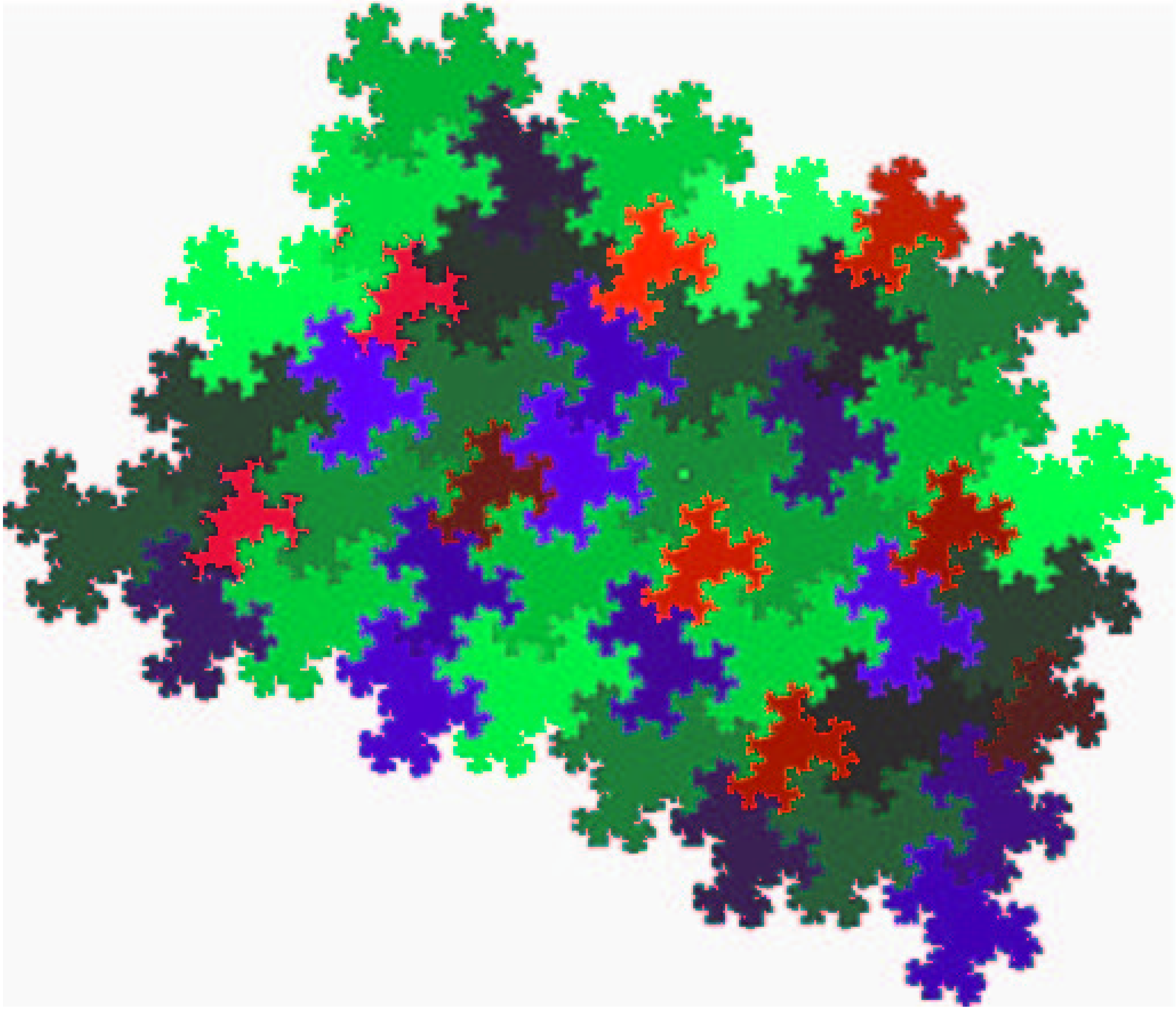
Th. (Ito-Rao) $d-1$-dimensional APERIODIC tiling $\Longleftrightarrow$ d-dimensional periodic tiling.

Self-similar aperiodic tiling



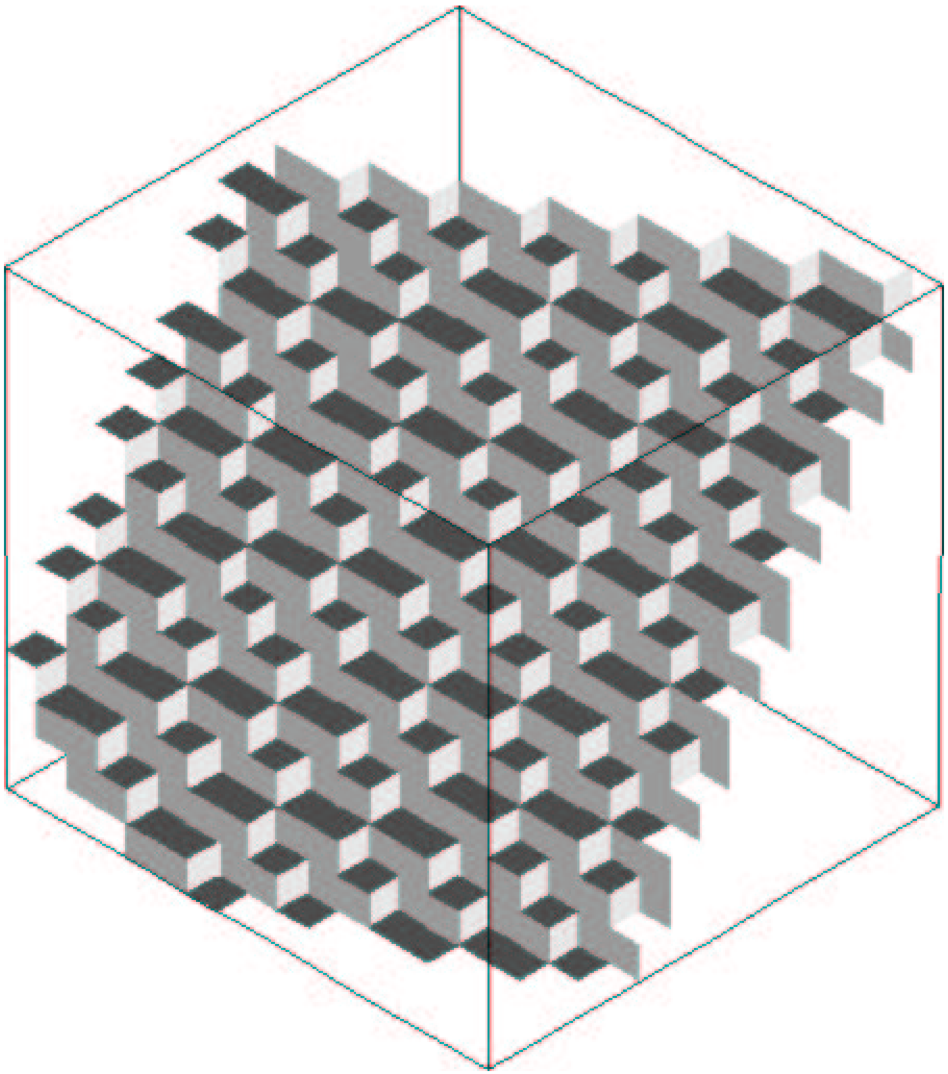


iling



Alternative explicit construction

- Discrete contracting plane.
- Regular lattice : vertices projected in $x + y + z = 0$.
- Replace each rhombus by a piece of the fractal.



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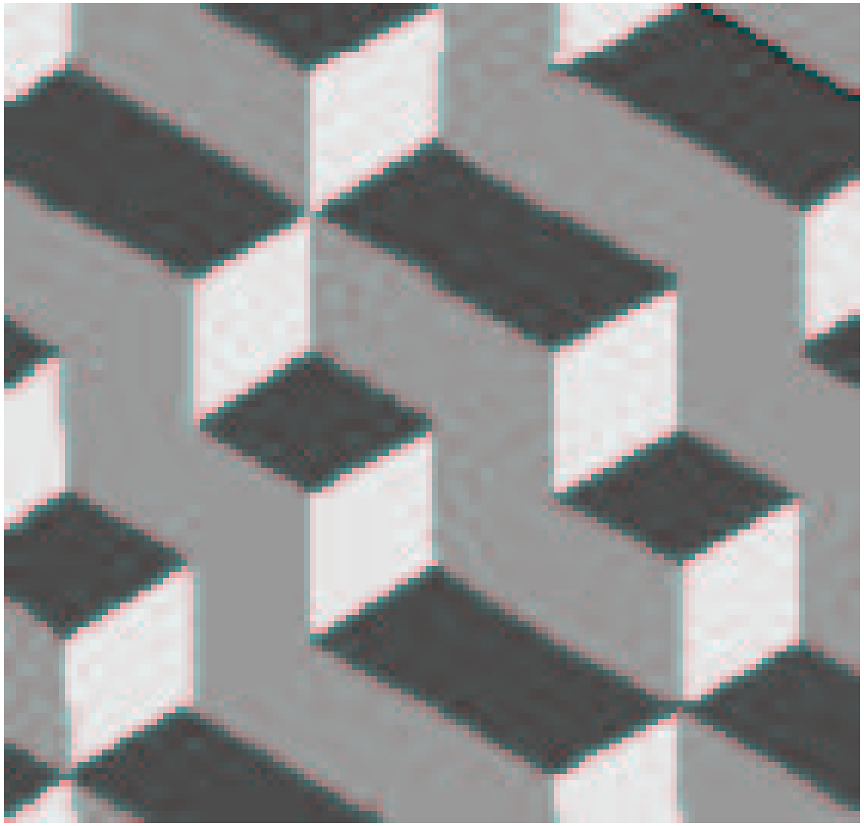

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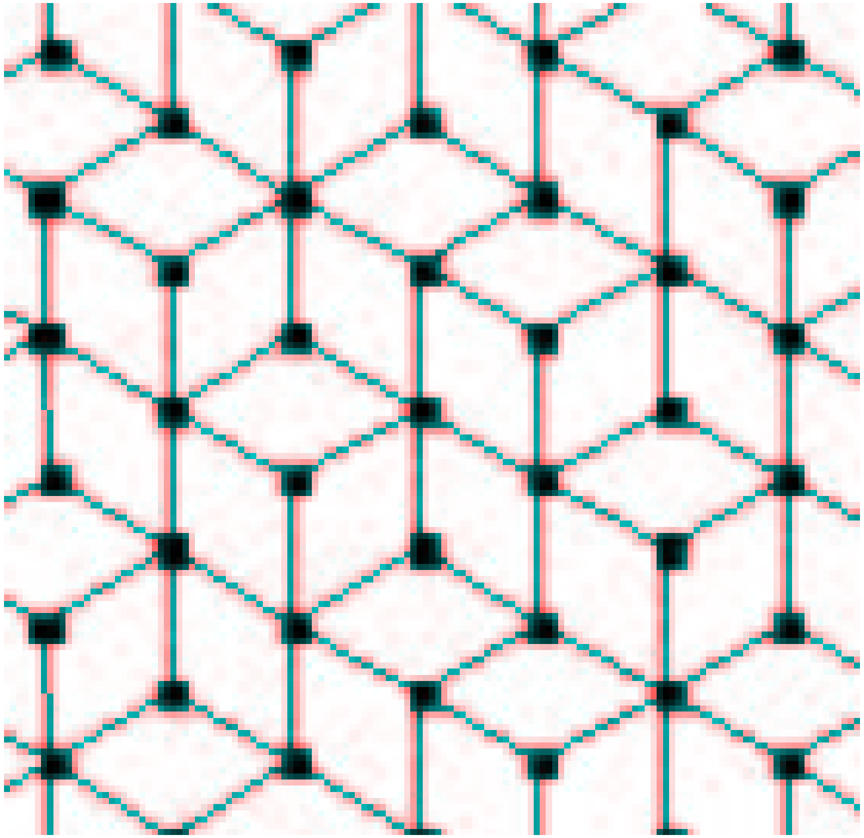

Alternative explicit construction

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- Replace each rhombus by a piece of the fractal.

1		3		2		1
	1		1		3	
3		2		1		3
	3		3		1	
1		1		3		2
	2		1		1	
1		3		2		1
	1		1		3	
3		2		1		1
	1		3		2	
2		1		3		3

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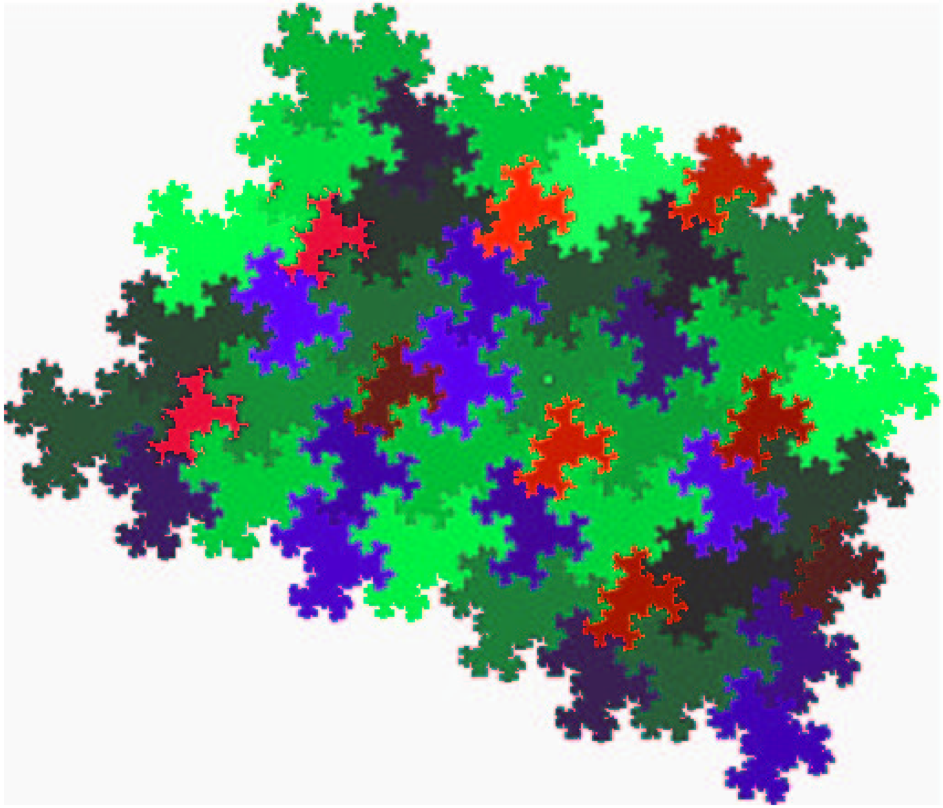
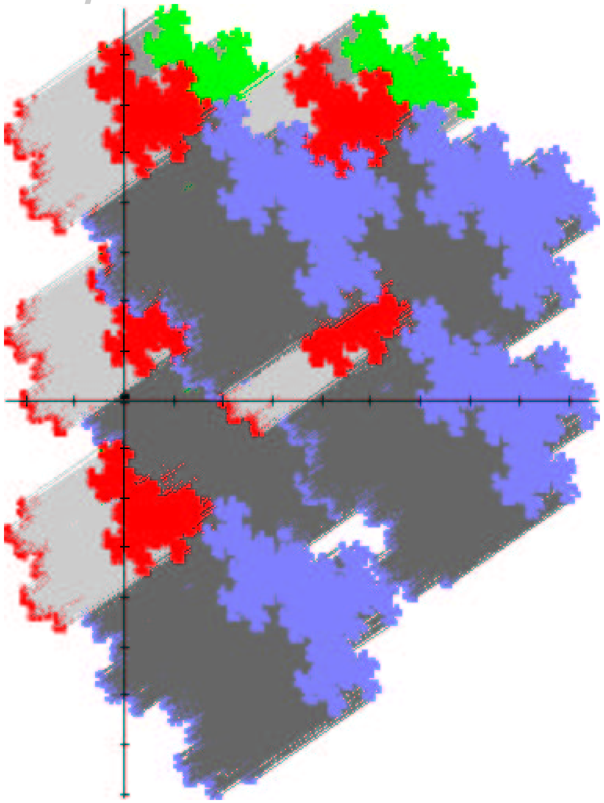
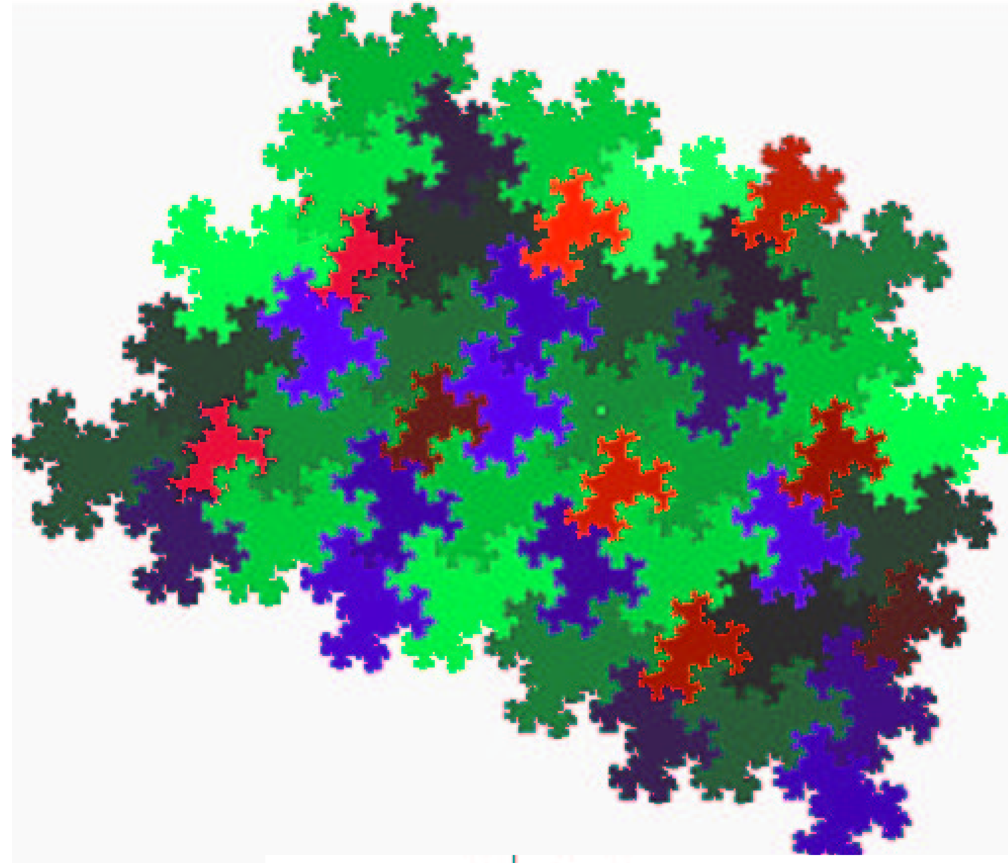


Then you get the aperiodic tiling if it exists.

Provides combinatorial conditions for tilings in terms of multi-dimensional substitutions and local rules





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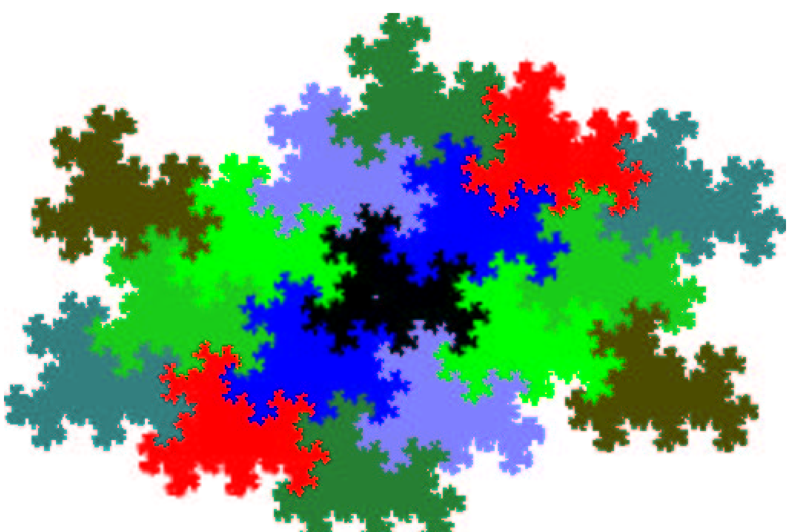

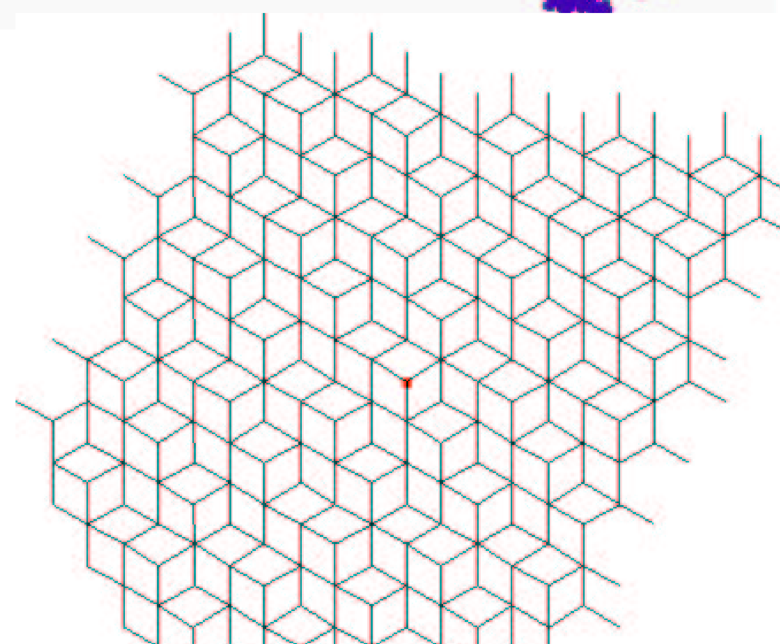

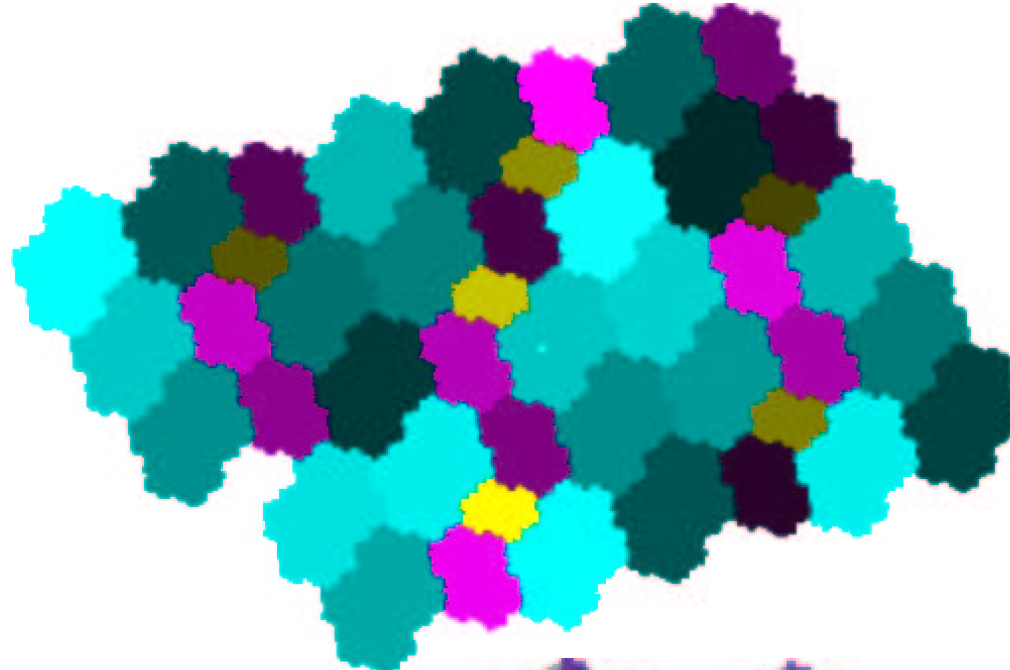

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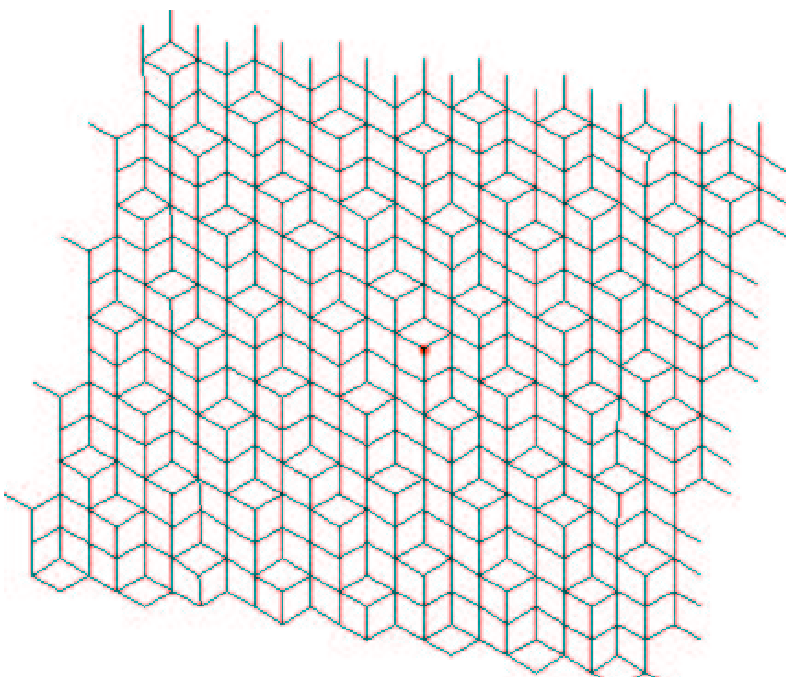

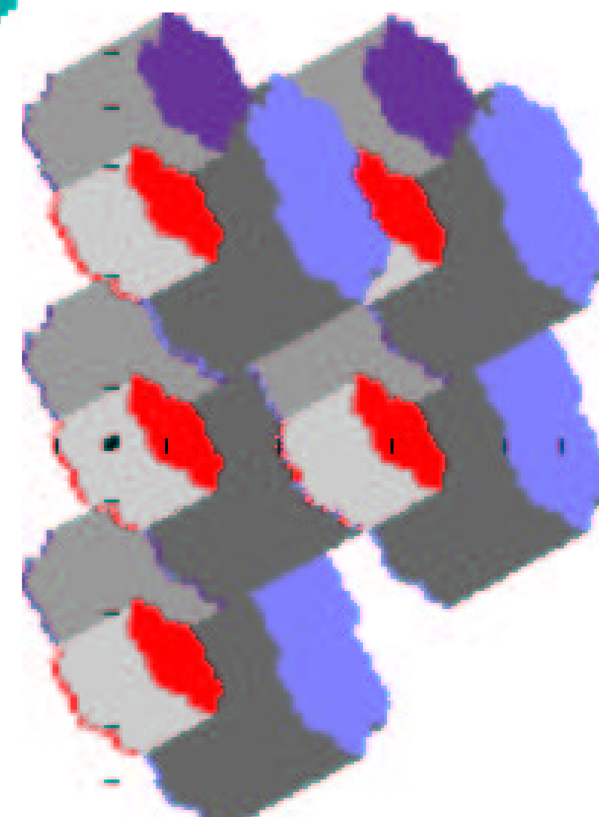

Conclusion

In the unimodular Pisot case, all the conditions are equivalent.

- Discrete spectrum.
- Periodic tiling.
- Aperiodic tiling.
- d-dimensional tiling.
- “Good” measures of pieces.
- Arithmetic automaton condition.
- Super coincidences.
- balanced pairs.

Extra sufficient conditions

- “Ring” condition on discrete plane.
- 0 inner point and coincidences.
- (F)-property for β-substitutions.

Which condition can we use to prove that every Pisot substitution satisfy it ?